Nonlinear mappings preserving the kernel or range of skew product of operators

H. Benbouziane*
Faculty of Sciences DharMahraz Fes
University Sidi Mohammed Ben Abdellah
Fez
Morocco
hassane.benbouziane@usmba.ac.ma

M. Ech-Chérif El Kettani
Faculty of Sciences DharMahraz Fes
University Sidi Mohammed Ben Abdellah
Fez
Morocco
mostapha.echcherifelkettani@usmba.ac.ma

Ahmedou Mohamed Vadel
Faculty of Sciences DharMahraz Fes
University Sidi Mohammed Ben Abdellah
Fez
Morocco
fadel.abdellahi@yahoo.fr

Abstract. Let $\mathcal{H}$ be a Hilbert space and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operator on H. We characterise surjective maps $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$, such that $F(\phi(A) \diamond \phi(B)) = F(A \diamond B)$, for all $A, B \in \mathcal{B}(H)$, where $F(A)$ denotes any of $R(A)$ or $N(A)$ and $A \diamond B$ denotes any binary operations A^*B, AB^*A for all $A, B \in \mathcal{B}(\mathcal{H})$.

Keywords: nonlinear preservers problem, kernel range operator, skew product.

1. Introduction and preliminaries

Throughout this note, $\mathcal{H}$ will denote a Hilbert space over the complex field $\mathbb{C}$ and $\mathcal{B}(\mathcal{H})$ will denote the algebra of all bounded linear operators on $\mathcal{H}$ with unit I. For $A \in \mathcal{B}(\mathcal{H})$ denoted by $R(A)$ the range of A, $N(A)$ its kernel and A^* its adjoint. The hyper-range of $A \in \mathcal{B}(X)$ is defined by $\mathcal{R}^\infty(A) := \bigcap_{n\in\mathbb{N}} R(A^n)$.

For any $x, f \in \mathcal{H}$, as usual, we denote $x \otimes f$ the rank at most one operator defined by $(x \otimes f)(y) = f(y)x =< y, f > x$, for every $y \in \mathcal{H}$. The set of all rank one operators is denoted by $\mathcal{F}_1(\mathcal{H})$. Fix an arbitrary orthogonal basis $\{e_i\}_{i\in\Gamma}$ of $\mathcal{H}$. For $x \in \mathcal{H}$, write $x = \sum_{i\in\Gamma} \lambda_i e_i$, and define the conjugate operator $J : \mathcal{H} \to \mathcal{H}$ by $Jx = \overline{x} = \sum_{i\in\Gamma} \overline{\lambda_i} e_i$.

*. Corresponding author

The study of maps on operator algebras preserving certain properties is a topic which attracts much attention of many authors, see for example [3, 6, 7, 9, 10, 12, 13], and the references therein. In this direction, in the last decades, a great activity has occurred in characterising maps preserving a certain property of the product or triple product (see [1, 2, 4, 6, 11]). In [2], the authors determine the form of surjective maps on $\mathcal{B}(\mathcal{H})$ which satisfies $F(\phi(A) \diamond \phi(B)) = F(A \diamond B)$ for all $A, B \in \mathcal{B}(H)$ where $F(A)$ denotes any of $R(A)$ or $N(A)$ and $A \diamond B$ denotes any binary operations: the usual product AB and triple product ABA for all $A, B \in \mathcal{B}(\mathcal{X})$. They also cover the main results of [12] by characterizing the maps that satisfy $N(\phi(A) - \phi(B)) = N(A - B)$ (or $R(\phi(A) - \phi(B)) = R(A - B)$).

As a continuation, in this direction, we propose to determine the forms of all surjective maps $\phi : \mathcal{B}(H) \to \mathcal{B}(\mathcal{H})$ which satisfy one of the following preserving properties:

- $N(\phi(A)\phi(B)^*\phi(A)) = N(AB^*A)$;
- $N(\phi(A)^*\phi(B)) = N(A^*B)$;
- $R(\phi(A)\phi(B)^*\phi(A)) = R(AB^*A)$;
- $R(\phi(A)^*\phi(B)) = R(A^*B)$,

for all $A, B \in \mathcal{B}(H)$.

2. Preliminaries

In this section, we collect some lemmas that will be used in the proof of our main results. The first one gives the range and kernel of rank one operators.

Lemma 2.1. *Let $x, f \in \mathcal{H}$ nonzeros vectors. We have*

1. $R(x \otimes f) = span\{x\}$ *and* $N(x \otimes f) = \{f\}^{\perp}$.
2. *If* $f(x) = 1$, *then* $N(I - x \otimes f) = R(x \otimes f) = span\{x\}$ *and* $R(I - x \otimes f) = N(x \otimes f) = \{f\}^{\perp}$.
3. *If* $f(x) \neq 0$ *then* $\mathcal{R}^{\infty}(x \otimes f) = R(x \otimes f) = span\{x\}$.

Proof. See, for example, [8, Lemma 2.1]. □

The second, quoted from [4], characterizes maps preserving zero skew products of operators in both directions.

Lemma 2.2. *Let $\mathcal{H}$ be a complex Hilbert space with dim $\mathcal{H} \geq 3$. Suppose $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a surjective map such that*

(1) $$A^*B = 0 \Leftrightarrow \phi(A)^*\phi(B) = 0 \ for\ all\ A, B \in \mathcal{B}(\mathcal{H}).$$

Then, ϕ preserves rank one operators in both directions and $\phi(0) = 0$. Moreover, there exist unitary $U \in \mathcal{B}(\mathcal{H})$ and a map $h : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ such that:

$$\phi(x \otimes f) = Ux \otimes h(x, f), \text{ for all } x, f \in \mathcal{H},$$

or

$$\phi(x \otimes f) = UJx \otimes h(x, f), \text{ for all } x, f \in \mathcal{H}.$$

Proof. See, [4, Theorem 2.1]. □

The following lemma determines the structure of surjective maps preserving the zero skew triple product of operators.

Lemma 2.3. *Let $\mathcal{H}$ be a complex Hilbert space with $\dim \mathcal{H} \geq 3$. Suppose that $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is a surjective map. Then, ϕ satisfies*

$$AB^*A = 0 \Leftrightarrow \phi(A)\phi(B)^*\phi(A) = 0, \text{ for all } A,\ B \in \mathcal{B}(\mathcal{H}), \tag{2}$$

if and only if there exist unitary linear or conjugate linear operators U, V on $\mathcal{H}$ and functional $h : \mathcal{H} \to \mathbb{C} \setminus \{0\}$ such that ϕ is of one of the forms:

$$\phi(A) = h(A)UAV, \text{ for all } A \in \mathcal{B}(\mathcal{H}),$$

or

$$\phi(A) = h(A)UA^*V, \text{ for all } A \in \mathcal{B}(\mathcal{H}).$$

Proof. See, [11, Corollary 3.5]. □

We end this section by stating and proving the following lemma which will be used later.

Lemma 2.4. *Let $A, B \in \mathcal{B}(\mathcal{H})$. The following statements are equivalent.*

1. *$N(R^*A) = N(R^*B)$ for all rank one operators R.*
2. *$R(A^*R) = R(B^*R)$ for all rank one operators R.*
3. *$A = cB$ for a nonzero scalar $c \in \mathbb{C}$.*

Proof. It's easy to check that (3) implies (1) and (3) implies (2).

$1 \Rightarrow 3$): Assume that $N(R^*A) = N(R^*B)$ for all rank one operators R. Let $R = x \otimes f$ be a rank one operator where $x, f \in \mathcal{H}$. By hypothesis we have

$$\begin{aligned} N(R^*A) = N(R^*B) \quad &\Longleftrightarrow \quad N((A^*R)^*) = N((B^*R)^*) \\ &\Longleftrightarrow \quad R(A^*R)^{\perp} = R(B^*R)^{\perp}. \end{aligned}$$

Which implies that $span\{A^*x\}^{\perp} = span\{B^*x\}^{\perp}$.

Since $span\{A^*x\}$ and $span\{B^*x\}$ are closed subspaces, we deduce that $span\{A^*x\} = span\{B^*x\}$. Therefore, $A^*x = c_xB^*x$, where $c_x \in \mathbb{C}$ is a scalar depending to x.

Now, to complete the proof, it is suffice to show that $N(A^*) = N(B^*)$. Indeed, suppose that there is $g \in \mathcal{H}$ such that $A^*g = 0$ and $B^*g \neq 0$. Then, there is a non zero vector $x \in \mathcal{H}$ such that $< x.B^*g >= 1$.

Note that, $(x \otimes g)B(x) = x \otimes B^*g(x) =< x, B^*g > x = x \neq 0$. Then, $x \notin N((x \otimes g)B)$. But $x \in N((x \otimes g)A)$ because $(x \otimes g)A(x) = (x \otimes A^*g)(x) = 0$. Which contradict the hypothesis.

$2 \Rightarrow 3$) let x be a non zero vector in $\mathcal{H}$. By hypothesis, we have

$$R(A^*x \otimes x) = R(B^*x \otimes x).$$

Which implies that $span\{A^*x\} = span\{B^*x\}$. We can show, by the same method as above, that $N(A^*) = N(B^*)$. Therefore, A^* and B^* are linearly dependent. Thus, A and B are linearly dependent, as desired. $\square$

3. Nonlinear maps preserving the kernel

We begin this section with the following result which characterizes surjective maps that preserve the kernel of triple skew product of operators.

Theorem 3.1. *Let $\mathcal{H}$ be a complex Hilbert space with dimension ≥ 3. A surjective map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ satisfies*

$$N(\phi(A)\phi(B)^*\phi(A)) = N(AB^*A), \text{ for all } A,\ B \in \mathcal{B}(\mathcal{H}), \tag{3}$$

if and only if there exist $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{K} \setminus \{0\}$ and U unitairy operator in $\mathcal{B}(\mathcal{H})$ such that $\phi(A) = \varphi(A)UA$ for all $A \in \mathcal{B}(\mathcal{H})$.

Proof. The necessarily condition is easily verified. Conversely, assume that ϕ satisfies the equation (3). In particular,

$$N(\phi(A)\phi(B)^*\phi(A)) = \mathcal{H} \iff N(AB^*A) = \mathcal{H}, \text{ for all } A,\ B \in \mathcal{B}(\mathcal{H}).$$

Then, ϕ satisfies the equation (2). Since ϕ is surjective, by Lemma 2.3, there exist unitary linear or conjugate linear operators U, V on $\mathcal{H}$ and functional $h : \mathcal{H} \to \mathbb{C} \setminus \{0\}$ such that ϕ is of one of the forms:

$$\phi(A) = h(A)UAV, \text{ for all } A \in \mathcal{B}(\mathcal{H}), \tag{4}$$

or

$$\phi(A) = h(A)UA^*V, \text{ for all } A \in \mathcal{B}(\mathcal{H}). \tag{5}$$

We shall show that ϕ can not take the form (5). Assume for the sak of contradiction that ϕ takes a such form, and let us first show that V is a scalar operator. It suffices to prove that V^* is a scalar operator. To do that, assume, on the contrary, that there exists a non zero vector $x \in \mathcal{H}$ such that V^*x and

x are linearly independent. We could find $f \in \mathcal{N}$ such that $< x, f >= 1$ and $< V^*x.f >= 0$. For any $B \in \mathcal{B}(\mathcal{H})$, we have $\phi(I) = h(I)UV$. Then

$$N(B^*) = N(\phi(I)\phi(B^*)^*\phi(I)) = N(UBV), \text{ for all } B \in \mathcal{B}(\mathcal{H}). \tag{6}$$

According to the lemma 2.1 and applying (6) to $B = I - x \otimes f$, we obtain

$$\begin{aligned} span\{f\} &= N((I - x \otimes f)^*) \\ &= N(U(I - x \otimes f)V) \\ &= N(UV - Ux \otimes V^*f) \\ &= N(I - V^*x \otimes V^*f) \end{aligned}$$

Since $< V^*f.V^*x >=< f.x >= 1$, then by Lemma 2.1, $span\{f\} = span\{V^*x\}$. Therefore, $f = \lambda V^*x$, for some non zero $\lambda \in \mathcal{H}$.

This shows that $< V^*x.f >= \lambda \parallel f \parallel^2 \neq 0$, which is a contradiction. Hence, V is a scalar operator and $\phi(A) = \varphi(A)UA$, where φ is a scalar function $\mathcal{B}(\mathcal{H}) \to \mathbb{K}^*$. Since U is injective, (6) becomes

$$N(B^*) = N(B), \text{ for all } B \in \mathcal{B}(\mathcal{H}). \tag{7}$$

On the other hand, we can find $z_1, z_2 \in \mathcal{H}$ such that z_1, z_2 are linearly independent and $< z_1, z_2 >= 1$. Applying (7) to $B = I - z_1 \otimes z_2$ we obtain

$$\begin{aligned} span\{z_1\} &= N(I - z_1 \otimes z_2) \\ &= N((I - z_1 \otimes z_2)^*) = N(I - z_2 \otimes z_1) \\ &= span\{z_2\}. \end{aligned}$$

This contadiction shows that ϕ takes the formes (4).

Now, let $x, f \in \mathcal{H}$ such that $< x, f >= 1$. For $B = I - x \otimes f$, from (3) and Lemma 2.1, we have

$$\begin{aligned} span\{f\} &= N((I - x \otimes f)^*) \\ &= N(B^*) = N(UB^*V) \\ &= N(U(I - f \otimes x)V) \\ &= N(UV - Uf \otimes V^*x) \\ &= N(UV(I - V^*f \otimes V^*x)) \\ &= N((I - V^*f \otimes V^*x)) \\ &= span\{V^*f\}. \end{aligned}$$

Therefore, V^* is a scalar operator and V is also. Which proves that $\phi(A) = \varphi(A)UA$, for all $A \in \mathcal{B}(\mathcal{H})$, with $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{K}^*$ is a scalar function. This completes the proof. $\square$

The following theorem characterizes surjective maps that preserve the kernel of skew product of operators.

Theorem 3.2. *Let $\mathcal{H}$ be a complex Hilbert space with dimension ≥ 3. A surjective map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ satisfies*

$$N(\phi(A)^*\phi(B)) = N(A^*B), \text{ for all } A,\ B\ \in \mathcal{B}(\mathcal{H}), \tag{8}$$

if and only if there exists $c \in \mathbb{K} \setminus \{0\}$ and and unitary $U \in \mathcal{B}(\mathcal{H})$ such that

$$\phi(A) = cUA, \quad \forall A \in \mathcal{B}(\mathcal{H}). \tag{9}$$

Proof. The "if" part is easily verified. We, therefore, will only deal with the "only if" part. So, assume that ϕ is a surjective map from $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})$ satisfying (8). In particular,

$$N(\phi(A)^*\phi(B)) = \mathcal{H} \iff N(A^*B) = \mathcal{H}, \text{ for all } A,\ B\ \in \mathcal{B}(\mathcal{H}).$$

This entails that ϕ satisfies the equation (1). since ϕ is surjective, by Lemma 2.2, there exist unitary operator $U \in \mathcal{B}(\mathcal{H})$ and a map $h : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ such that:

$$\phi(x \otimes f) = Ux \otimes h(x, f), \text{ for all } x, f \in \mathcal{H}, \tag{10}$$

or

$$\phi(x \otimes f) = UJx \otimes h(x, f), \text{ for all } x, f \in \mathcal{H}. \tag{11}$$

Let $f, x \in \mathcal{H}$ and put $g = h(x, f)$. If (10) holds, then

$$\begin{aligned} \{f\}^\perp &= N((x \otimes f)^*(x \otimes f)) \\ &= N((\phi(x \otimes f))^*(\phi(x \otimes f))) \\ &= N((Ux \otimes g)^*(Ux \otimes g)) \\ &= \{g\}^\perp. \end{aligned}$$

So, there exists $\lambda \in \mathcal{H}$ such that $g = \lambda f$.

If (11) holds, with no extra effort, we get the same result. Therefore, for every $R \in \mathcal{F}_1$ we obtain

$$\phi(R) = \lambda VR, \tag{12}$$

or

$$\phi(R) = \lambda VJR. \tag{13}$$

Let $A \in \mathcal{B}(\mathcal{H})$ and $R \in \mathcal{F}_1$. If (12) holds, then

$$N(R^*A) = N(\phi(R)^*\phi(A)) = N(R^*V^*\phi(A)).$$

Therefore, by Lemma 2.4, there exists non zero scalar $c \in \mathcal{H}$ such that $\phi(A) = cVA$ or

$$N(R^*A) = N(\phi(R)^*\phi(A)) = N(R^*J^*V^*\phi(A)).$$

Then, $\phi(A) = cVJA$, for some non zero scalar $c \in \mathcal{H}$.

Now, assume that V is unitary. Take an orthonormal basis $\{e_i\}_{i\in\Gamma}$ of $\mathcal{H}$ and define the conjugate operator $J : \mathcal{H} \to \mathcal{H}$ by $Jx = \overline{x} = \sum_{i\in\Gamma} \overline{\lambda_i} e_i$. Then, J is conjugate unitary. Let $U = VJ$ then U is unitary (see, [5, Claim 3 in Theorem 5.1]). We conclude that $\phi(A) = cUA$, for all $A \in \mathcal{B}(\mathcal{H})$ with U is unitary, the proof is complete. □

4. Nonlinear maps preserving the range

The first theorem in this section characterizes surjective maps that preserve the Range of triple skew product of operators.

Theorem 4.1. *Let $\mathcal{H}$ be a real or complex Hilbert space of dimension ≥ 3. A surjective map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ satisfies*

$$R(\phi(A)\phi(B)^*\phi(A)) = R(AB^*A), \text{ for all } A,\ B \in \mathcal{B}(\mathcal{H}), \tag{14}$$

if and only if there exists $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{K} \setminus \{0\}$ and V unitairy in $\mathcal{B}(\mathcal{H})$ such that $\phi(A) = \varphi(A)AV$, for all $A \in \mathcal{B}(\mathcal{H})$.

Proof. The necessary condition is easily verified since the operator V is surjective. Conversely, assume that ϕ is a surjective map satisfying (14). Then

$$R(\phi(A)\phi(B)^*\phi(A)) = \mathcal{H} \iff R(AB^*A) = \mathcal{H}, \text{ for all } A,\ B \in \mathcal{B}(\mathcal{H}).$$

Which shows that ϕ satisfying the equation (1). It follows, by Lemma 2.3, that there exist unitary linear or conjugate linear operators U, V on $\mathcal{H}$ and functional $h : \mathcal{H} \to \mathbb{C} \setminus \{0\}$ such that ϕ is of one of the forms:

$$\phi(A) = h(A)UAV, \text{ for all } A \in \mathcal{B}(\mathcal{H}), \tag{15}$$

or

$$\phi(A) = h(A)UA^*V, \text{ for all } A \in \mathcal{B}(\mathcal{H}). \tag{16}$$

Similarly to the proof of Theorem 2.1, let us first show that ϕ can not take the second form. Assume, to the contrary, that ϕ takes a such form. Let x be a non zero vector in $\mathcal{H}$. By (14) and Lemma 2.3, we have

$$\begin{aligned} span\{x\} &= R((x \otimes x)^*) \\ &= R(U(x \otimes x)U^*UV) \\ &= R(Ux \otimes x)U^*) \\ &= R(Ux \otimes Ux) \\ &= span\{Ux\}. \end{aligned}$$

Which proves that U is a scalar operator. Thus,

$$\phi(A) = h(A)A^*V, \text{ for all } A \in \mathcal{B}(\mathcal{H}).$$

In particular, for $A = x \otimes y$ where x and y are linearly independent, we obtain

$$span\{y\} = R(B^*) = R(\phi(I)\phi(B)^*\phi(I)) = R(BV) = R(B) = span\{x\},$$

which is a contradiction. We conclude that ϕ takes the form (15).

To finish the proof, it remains to show that U is a scalar operator. Indeed, for any nonzero vector $x \in \mathcal{H}$ we have

$$\begin{aligned} span\{x\} &= R((x \otimes x)^*) \\ &= R(\phi(I)\phi((x \otimes x))^*\phi(I)) \\ &= R(U(x \otimes x)^*V) \\ &= R(Ux \otimes x) \\ &= span\{Ux\}. \end{aligned}$$

This proves that Ux and x are linearly dependent, for all $x \in \mathcal{H}$. Therefore, there is a non zero scalar C such that $U = cI$. The proof is complete. □

By replacing the range of operator by the hyper-range of operator in the previous theorem we get the following result.

Theorem 4.2. *Let $\mathcal{H}$ be a real or complex Hilbert space of dimension ≥ 3.*
A surjective map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ satisfies

$$\mathcal{R}^\infty(\phi(A)\phi(B)^*\phi(A)) = \mathcal{R}^\infty(AB^*A), \text{ for all } A,\ B\ \in \mathcal{B}(\mathcal{H}), \tag{17}$$

if and only if, there exists $\varphi : \mathcal{B}(\mathcal{H}) \to \mathbb{K} \setminus \{0\}$ and V unitary in $\mathcal{B}(\mathcal{H})$ such that $\phi(A) = \varphi(A)AV$, for all $A \in \mathcal{B}(\mathcal{H})$.

We end this paper by the following result which characterizes surjective maps that preserve the Range of skew product of operators.

Theorem 4.3. *Let $\mathcal{H}$ be a complex Hilbert space of dimension ≥ 3. A surjective map $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ satisfies*

$$R(\phi(A)^*\phi(B)) = R(A^*B), \text{ for all } A,\ B\ \in \mathcal{B}(\mathcal{H}), \tag{18}$$

if and only if there exists $c \in \mathbb{K} \setminus \{0\}$ and and unitary $U \in \mathcal{B}(\mathcal{H})$ such that

$$\phi(A) = cUA, \quad \text{for all } A \in \mathfrak{B}(H). \tag{19}$$

Proof. The necessarily condition is easily verified since the operators U is surjective. Conversely, assume that ϕ is a surjective additive map from $\mathcal{B}(\mathcal{H})$ into $\mathcal{B}(\mathcal{H})$ satisfying (18). In particular,

$$R(\phi(A)^*\phi(B)) = \{0\} \iff R(A^*B) = \{0\}, \text{ for all } A,\ B\ \in \mathcal{B}(\mathcal{H}).$$

This implies that ϕ satisfies the equation (1). By following the same approach of the proof of Theorem 3.2, we obtain

$$\phi(R) = \lambda UR \text{ or } \phi(R) = \lambda UJR, \quad \text{for} \quad \text{every } R \in \mathcal{F}_1.$$

By the same reasoning and by applying Lemma 2.4, the map ϕ has the desired form. □

References

[1] H. Benbouziane, Y. Bouramdane, M. Ech-Cherif El Kettani, A. Lahssaini, *Nonlinear maps preserving condition spectrum of jordan skew triple product of opertors*, Opertors and Matrices, 12 (2018), 933-9242.

[2] H. Benbouziane, Y. Bouramdane, M. Ech-Cherif El Kettani, *Nonlinear maps preserving certain subspaces*, Proyecciones Journal of Mathematics, 38 (2019), 163-174.

[3] J. Cui, J. Hou, *Additive maps on standard operator algebras preserving invertibilities or zero divisors*, Linear Algebra Appl., 359 (2003), 219-233.

[4] J. Cui, C.K. Li, N.S. Sze, *Unitary similarity invariant function preservers of skew products of operators*, J. Math. Anal. Appl., 454 (2017), 716-729.

[5] J. Cui, C.K. Li, Y.-T. Poon, *Pseudospectra of special operators and pseudospectrum preservers*, J. Math. Anal. Appl., 419 (2014), 1261–1273.

[6] G. Dobovišek, B. Kuzma, G. Lešnjak, C.K. Li, T. Petek, *Mappings that preserve pairs of operators with zero triple Jordan product*, Linear Algebra Appl., 426 (2007), 255-279.

[7] G. Dolinar, S.P. Du, J.C. Hou, P. Legiša, *General preservers of invariant subspace lattices*, Linear Algebra Appl., 429 (2008), 100-109.

[8] M. Elhodaibi, A. Jaatit, *On additive maps preserving the hyper-range or hper-kernel of operators*, Int. Math. Forum, 7 (2012), 1223-1231.

[9] L. Fang, G. Ji, Y. Pang, *Maps preserving the idempotency of products of operators*, Linear Algebra Appl., 426 (2007), 40-52.

[10] G. Frobenius, *Über die Darstellung der endlichen Gruppen durch lineaire Substitusionen*, Sitzungsber, Deutsch. Akad. Wiss. Berling, (1897), 171-172.

[11] J. Hou, K. He, X. Zhang, *Nonlinear maps preserving numerical radius of indefinite skew products of operators*, Linear Algebra Appl., 430 (2009), 2240-2253.

[12] M. Oudghiri, *Additive mappings preserving the kernel or the range of operators*, Extracta Math., 24 (2009), 251-258.

[13] P. Semrl, *Two characterizations of automorphisms on* $\mathcal{B}(X)$, Studia Math., 150 (1993), 143-149.

Accepted: May 6, 2022