Subspace diskcyclic tuples of operators on Banach spaces

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Abstract. In this paper, we study subspace diskcyclic and subspace-disk transitive tuples of operators. We give some characterizations of these tuples. Also, we give a set of sufficient conditions for a tuple to be subspace-diskcyclic. We find a relation between the subspace-diskcyclicity of a tuple of operators and the tuple of the direct sum of those operators. Finally, we show that if a tuple of operators is subspace-diskcyclic, then not every operator in the tuple has to be subspace-diskcyclic.

Keywords: subspace-diskcyclic operators, tuple of operators.

1. Introduction

A bounded linear operator T on a separable Banach space X is hypercyclic if there is a vector $x \in X$ such that $Orb(T, x) = \{T^n x : n \geq 0\}$ is dense in X, such a vector x is called hypercyclic for T. The first example of a hypercyclic operator on a Banach space was constructed by Rolewicz in [12]. He showed that if B is the backward shift on $\ell^p(\mathbb{N})$ then λB is hypercyclic if and only if $|\lambda| > 1$.

In 1974, Hilden and Wallen [6] defined the supercyclicity concept. An operator T is called supercyclic if there is a vector x such that the scaled orbit $\mathbb{C}Orb(T, x)$ is dense in X. The notion of a diskcyclic operator was introduced by Zeana [17]. An operator T is called diskcyclic if there is a vector $x \in X$ such that the disk orbit $\mathbb{D}Orb(T, x) = \{\alpha T^n x : \alpha \in \mathbb{C}, |\alpha| \leq 1, n \in \mathbb{N}\}$ is dense in X, such a vector x is called diskcyclic for T. For more information about diskcyclic operators, the reader may refer to [3] [1] [17].

In 2011, Madore and Martínez-Avendaño [9] considered the density of the orbit in a non-trivial subspace instead of the whole space, this phenomenon is called the subspace-hypercyclicity. An operator is called $\mathcal{M}$-hypercyclic or subspace-hypercyclic for a subspace $\mathcal{M}$ of X if there exists a vector such that the intersection of its orbit and $\mathcal{M}$ is dense in $\mathcal{M}$. For more information on subspace-hypercyclicity, one may refer to [7], [8] and [11].

In [14] Xian-Feng et al. defined the subspace-supercyclic operator as follows: An operator is called $\mathcal{M}$-supercyclic or subspace-supercyclic for a subspace $\mathcal{M}$

of X if there exists a vector such that the intersection of its scaled orbit and $\mathcal{M}$ is dense in $\mathcal{M}$.

Also, Bamerni and Kılıçman [15] introduced the subspace-diskcyclicity concept in a Banach space X that is the disk orbit of an operator T is dense in a subspace of X.

Let $\mathcal{T} = (T_1, \ldots, T_n)$ be a commuting n-tuple of continuous linear operators on a Banach space X and $\mathcal{F} = \left\{T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} : k_i \geq 0, 1 \leq i \leq n\right\}$ be the semigroup generated by $\mathcal{T}$, then $\mathcal{T}$ is called hypercyclic if there is $x \in X$ such that $Orb(\mathcal{T}, x) = \{Tx : T \in \mathcal{F}\}$ is dense in X ([5]).

A tuple $\mathcal{T}$ is supercyclic if there exists $x \in X$ such that $\mathbb{C}Orb(\mathcal{T}, x) = \{\alpha Tx : T \in \mathcal{F}, \alpha \in \mathbb{C}\}$ is dense in X ([13]).

For subspaces, Moosapoor [10] defined subspace-hypercyclic tuples of operators as follows: A tuple $\mathcal{T}$ is subspace-hypercyclic for a subspace $\mathcal{M}$ if there exists a vector $x \in X$ such that $Orb(\mathcal{T}, x) \cap \mathcal{M}$ is dense in $\mathcal{M}$. By the same way, a tuple $\mathcal{T}$ is subspace-suercyclic for a subspace $\mathcal{M}$ if there exists a vector $x \in X$ such that $\mathbb{C}Orb(\mathcal{T}, x) \cap \mathcal{M}$ is dense in $\mathcal{M}$ ([16]).

Both subspace-hypercyclic and subspace-suercyclic tuples were studied in details; therefore, in this paper, we study some properties of subspace-diskcyclic tuples. In particular, we give an equivalent assertion to subspace- diskcyclic tuple which is called subspace-disk transitive tuple. Also, we give some sufficient conditions for a tuple to be subspace-diskcyclic which we call subspace-diskcyclic tuple criterion. We find a relation between the subspace-diskcyclicity of a tuple of operators and the tuple of the direct sum of those operators. Finally, we show that if a tuple of operators is subspace-diskcyclic, then not every operator in the tuple has to be subspace-diskcyclic.

2. Main results

In this section, we characterize the equivalent conditions for a tuple of operators to be subspace-disk transitive. We provide some sufficient conditions for a tuple to be subspace-diskcyclic which is called subspace-diskcyclic tuple criterion. Also, we study the diskcyclicity of tuples of direct sum of operators.

In what follows, we let $\mathbb{U} = \{\alpha \in \mathbb{C} : |\alpha| < 1\}$ and $\mathbb{D}C(\mathcal{T}, \mathcal{M})$ be the set of all $\mathcal{M}$-diskcyclic vectors for the tuple $\mathcal{T}$, that is

$$\mathbb{D}C(\mathcal{T}, \mathcal{M}) = \{x \in X : \mathbb{D}Orb(\mathcal{T}, x) \cap \mathcal{M} \text{ is dense in } \mathcal{M}\}.$$

Definition 2.1. *If $\mathcal{T} = (T_1, \ldots, T_n)$ is a tuple on a Banach space X, $\mathcal{F} = \{T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} : k_i \geq 0, 1 \leq i \leq n\}$ and $\mathcal{M}$ be a closed subspace of X then the tuple $\mathcal{T}$ is called subspace-diskcyclic for $\mathcal{M}$ (or $\mathcal{M}$-diskcyclic) if there exists a vector $x \in X$ such that*

$$\mathbb{D}Orb(\mathcal{T}, x) \cap \mathcal{M} = \{\alpha Tx : T \in \mathcal{F}, \alpha \in \mathbb{D}\} \cap \mathcal{M}$$

is dense in $\mathcal{M}$.

It is clear from the above definition, that every subspace-hypercyclic tuple is subspace-diskcyclic which in turn is subspace-supercyclic.

Definition 2.2. *If $\mathcal{T} = (T_1, \dots, T_n)$ is a tuple on a Banach space X, $\mathcal{F} = \{T_1^{k_1} T_2^{k_2} \dots T_n^{k_n} : k_i \geq 0, 1 \leq i \leq n\}$ and $\mathcal{M}$ be a closed subspace of X then the tuple $\mathcal{T}$ is called subspace-disk transitive (or $\mathcal{M}$-disk transitive) if for any two nonempty sets U and V in $\mathcal{M}$, there exists $\alpha \in \mathbb{U}$ and some positive integers k_i, $1 \leq i \leq n$ such that $T_n^{-k_n} T_{n-1}^{-k_{n-1}} \dots T_1^{-k_1}(\alpha U) \cap V$ contains a relatively open nonempty subset G of $\mathcal{M}$.*

We give the following example of a subspace-diskcyclic tuple.

Example 2.1. Suppose that T is a diskcyclic operator on a Banach space X and I is the identity operator. Then, it is easy to show that the tuple $\mathcal{T} = (T \oplus I, I \oplus T)$ is both $\mathcal{M}$-diskcyclic and $\mathcal{N}$-diskcyclic where $\mathcal{M} = X \oplus \{0\}$ and $\mathcal{N} = \{0\} \oplus X$ since both $T \oplus I$ and $I \oplus T$ are subspace-diskcyclic operators [15, Example 2.2.].

The following example shows that not every subspace-diskcyclic tuples is diskcyclic.

Example 2.2. Let $\mathcal{T} = (\alpha B \oplus I, \beta B \oplus I)$ be a 2-tuple where α, β are complex numbers with modulus greater than 1, I is the identity operator and B is the backward shift on the sequence space $\ell^2(\mathbb{N})$. Since αB is diskcyclic [3] then it has a diskcyclic vector, say x. Therefore, the tuple $\mathcal{T}$ has an $\mathcal{M}$-diskcyclic vector $(x, 0)$ for the subspace $\mathcal{M} = \ell^2(\mathbb{N}) \oplus \{0\}$. However, the tuple $\mathcal{T}$ is not diskcyclic since $\alpha B \oplus I$ is not diskcyclic operator.

The following proposition gives an equivalent assertion to subspace- disk transitive tuple.

Proposition 2.1. *Let M be a subspace of a Banach space X and $\mathcal{T} = (T_1, T_2, \dots, T_n)$ be a tuple of operators. Then, the following statements are equivalent.*

1. *The tuple $\mathcal{T}$ is $\mathcal{M}$-disk transitive,*
2. *For any two relatively open subsets U and V of $\mathcal{M}$ there exist $\alpha \in \mathbb{U}^C$ and some positive integers k_i, $1 \leq i \leq n$ such that $T_n^{-k_n} T_{n-1}^{-k_{n-1}} \dots T_1^{-k_1}(\alpha U) \cap V \neq \phi$ and $T_1^{k_1} T_2^{k_2} \dots T_n^{k_n}(\mathcal{M}) \subset \mathcal{M}$.*
3. *For any two relatively open subsets U and V of $\mathcal{M}$ there exists $\alpha \in \mathbb{U}^C$ and some positive integers k_i, $1 \leq i \leq n$ such that $T_n^{-k_n} T_{n-1}^{-k_{n-1}} \dots T_1^{-k_1}(\alpha U) \cap V$ is non-empty open set in $\mathcal{M}$.*

Proof. (1) $\Rightarrow$ (2): Let U and V be two relatively open subsets of $\mathcal{M}$. By the statement (1), there exist $\alpha \in \mathbb{U}^C$, some positive integers k_i, $1 \leq i \leq n$ and an open set G in $\mathcal{M}$ such that $G \subset T_n^{-k_n} T_{n-1}^{-k_{n-1}} \dots T_1^{-k_1}(\alpha U) \cap V$. It follows that

$$T_n^{-k_n} T_{n-1}^{-k_{n-1}} \dots T_1^{-k_1}(\alpha U) \cap V \neq \phi. \tag{1}$$

Since $G \subset T_n^{-k_n} T_{n-1}^{-k_{n-1}} \ldots T_1^{-k_1}(\alpha U)$, it follows that $\frac{1}{\alpha} T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n}(G) \subset U \subset \mathcal{M}$. Let $x \in \mathcal{M}$ and $x_0 \in G$. Then, there exists $r \in \mathbb{N}$ such that $(x_0 + rx) \in G$. Then, we get

$$\begin{aligned}\frac{1}{\alpha} T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} x_0 + \frac{1}{\alpha} T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} rx &= \frac{1}{\alpha} T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n}(x_0 + rx) \\ &\in \frac{1}{\alpha} T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n}(G) \subset \mathcal{M}.\end{aligned}$$

Since $x_0 \in G$ then $\frac{1}{\alpha} T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} x_0 \in \frac{1}{\alpha} T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n}(G) \subset \mathcal{M}$, it follows that $\frac{r}{\alpha} T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} x \in \mathcal{M}$ and so $T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n} x \subset \mathcal{M}$, i.e,

$$T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n}(\mathcal{M}) \subset \mathcal{M}. \tag{2}$$

The proof follows by (1) and (2).

(2)⇒ (3): The restriction function $T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n}|_{\mathcal{M}} \in \mathcal{B}(\mathcal{M})$, then $T_n^{-k_n} T_{n-1}^{-k_{n-1}} \ldots T_1^{-k_1}(\alpha U) \cap \mathcal{M}$ is open in $\mathcal{M}$ for any open set U of $\mathcal{M}$. Since $V \subset \mathcal{M}$ is open, it follows that $T_1^{-k_1} T_2^{-k_2} \ldots T_n^{-k_n}(\alpha U) \cap V$ is an open set in $\mathcal{M}$.

(3)⇒ (1) is trivial. □

The next theorem shows that every subspace-disk transitive tuple is subspace-diskcyclic for the same subspace. First, we need the following lemma.

Lemma 2.1. *Let $\mathcal{T} = (T_1, T_2, \ldots, T_n)$ be $\mathcal{M}$-diskcyclic tuple. Then, there exists $k_j \in \mathbb{N}$, $1 \leq j \leq n$ such that*

$$\mathbb{D}C(\mathcal{T}, \mathcal{M}) = \bigcap_{i \in \mathbb{N}} \Big(\bigcup_{\alpha \in \mathbb{U}^C} T_n^{-k_n} T_{n-1}^{-k_{n-1}} \ldots T_1^{-k_1}(\alpha B_i)\Big).$$

where $\{B_i : i \in \mathbb{N}\}$ is a countable open basis for the relative topology of a subspace $\mathcal{M}$.

Proof. We have $x \in \mathbb{D}C(\mathcal{T}, \mathcal{M})$ if and only if

$$\mathbb{D}Orb(\mathcal{T}, x) \cap \mathcal{M} = \{\alpha T x : T \in \mathcal{F}, \alpha \in \mathbb{D}\} \cap \mathcal{M}$$

is dense in $\mathcal{M}$ if and only if for each $i > 0$, there exist $\alpha \in \mathbb{D} \setminus \{0\}$ and $k_j \in \mathbb{N}$, $1 \leq j \leq n$ such that $\alpha T_1^{k_1} T_2^{k_2} \ldots T_n^{k_n x} \in B_i$ if and only if

$$x \in \bigcap_{i \in \mathbb{N}} \Big(\bigcup_{\alpha \in \mathbb{U}^C} T_n^{-k_n} T_{n-1}^{-k_{n-1}} \ldots T_1^{-k_1}(\alpha B_i)\Big).$$

Theorem 2.1. *Let $\mathcal{M}$ be a subspace of a Banach space X and $\mathcal{T}=(T_1, T_2, \ldots, T_n)$ be a tuple of operators. Suppose that $\mathcal{T}$ is $\mathcal{M}$-disk transitive tuple. Then,*

$$\bigcap_{i \in \mathbb{N}} \Big(\bigcup_{\alpha \in \mathbb{U}^C} T_n^{-k_n} T_{n-1}^{-k_{n-1}} \ldots T_1^{-k_1}(\alpha B_i)\Big).$$

is dense in $\mathcal{M}$.

Proof. Since T is $\mathcal{M}$-transitive, then by Proposition 2.1, for each $i, j \in \mathbb{N}$, there exist $k_{i,j}^{(r)} \in \mathbb{N}$, $1 \leq r \leq n$ and $\alpha_{i,j} \in \mathbb{U}^C$ such that

$$T_n^{-k_{i,j}^{(n)}} T_{n-1}^{-k_{i,j}^{(n-1)}} \dots T_1^{-k_{i,j}^{(1)}} (\alpha_{i,j} B_i) \cap B_j$$

is nonempty open in $\mathcal{M}$. Suppose that

$$A_i = \bigcup_{j \in \mathbb{N}} \left(T_n^{-k_{i,j}^{(n)}} T_{n-1}^{-k_{i,j}^{(n-1)}} \dots T_1^{-k_{i,j}^{(1)}} (\alpha_{i,j} B_i) \cap B_j \right),$$

for all $i \in \mathbb{N}$. Then, A_i is nonempty and open in $\mathcal{M}$ since it is a countable union of open sets in $\mathcal{M}$. Furthermore, each A_i is dense in $\mathcal{M}$ since it intersects each B_j. By the Baire category theorem, we get

$$\bigcap_{i \in \mathbb{N}} A_i = \bigcap_{i \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} \left(T_n^{-k_{i,j}^{(n)}} T_{n-1}^{-k_{i,j}^{(n-1)}} \dots T_1^{-k_{i,j}^{(1)}} (\alpha_{i,j} B_i) \cap B_j \right)$$

is a dense set in $\mathcal{M}$. Clearly,

$$\begin{aligned} &\bigcap_{i \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} \left(T_n^{-k_{i,j}^{(n)}} T_{n-1}^{-k_{i,j}^{(n-1)}} \dots T_1^{-k_{i,j}^{(1)}} (\alpha_{i,j} B_i) \cap B_j \right) \\ &\subset \bigcap_{i \in \mathbb{N}} (\bigcup_{\alpha \in \mathbb{U}^C} T_n^{-k_n} T_{n-1}^{-k_{n-1}} \dots T_1^{-k_1} (\alpha B_i)) \cap \mathcal{M}. \end{aligned}$$

It follows that $\bigcap_{i \in \mathbb{N}} \left(\bigcup_{\alpha \in \mathbb{U}^C} T_n^{-k_n} T_{n-1}^{-k_{n-1}} \dots T_1^{-k_1} (\alpha B_i) \right) \cap \mathcal{M}$ is desne in $\mathcal{M}$. The proof is completed. □

Corollary 2.1. *If $\mathcal{T}$ is an $\mathcal{M}$-disk transitive tuple, then $\mathcal{T}$ is $\mathcal{M}$-diskcyclic.*

Proof. The proof follows by Proposition 2.1 and Theorem 2.1. □

Theorem 2.2 ($\mathcal{M}$-Diskcyclic Tuple Criterion)**.** *Let M be a subspace of a Banach space X and $\mathcal{T} = (T_1, T_2, \dots, T_n)$ be a tuple of operators. Suppose that for each $1 \leq i \leq n$, $\left\langle r_k^{(i)} \right\rangle_{k \in \mathbb{N}}$ is an increasing sequence of positive integers and $D_1, D_2 \in \mathcal{M}$ are two dense sets in $\mathcal{M}$ such that*

1. *For every $y \in D_2$, there is a sequence $\langle x_k \rangle_{k \in \mathbb{N}}$ in $\mathcal{M}$ such that $\|x_k\| \to 0$ and $T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} x_k \to y$ as $k \to \infty$,*
2. *$\|T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} x\| \|x_k\| \to 0$, for all $x \in D_1$ as $k \to \infty$,*
3. *$T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} \mathcal{M} \subseteq \mathcal{M}$, for all $k \in \mathbb{N}$.*

Then, $\mathcal{T}$ is said to be satisfied $\mathcal{M}$-diskcyclic criterion, and $\mathcal{T}$ is an $\mathcal{M}$-diskcyclic tuple.

Proof. Let U_1 and U_2 be two relatively open sets in $\mathcal{M}$. Then, we can find $x \in D_1 \cap U_1$ and $y \in D_2 \cap U_2$ since both D_1 and D_2 are dense in $\mathcal{M}$. It follows from the condition (2) that there exists a sequence of non-zero scalars $\langle \lambda_k \rangle_{k \in \mathbb{N}}$ such that $\lambda_k T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} x \to 0$ and $\lambda_k^{-1} x_k \to 0$. Suppose that $\|T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} x\|$ and $\|x_k\|$ are not both zero. Then, we have the following cases:

1. if $\|T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} x\| = 0$, set $\lambda_k = 2^k \|x_k\|$.

Then, $\mathcal{T}$ turns to be $\mathcal{M}$-hypercyclic tuple [4, Theorem 2.4.] and thus $\mathcal{M}$-diskcyclic.

2. if $\|T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} x\| \|x_k\| \neq 0$, set $\lambda_k = \|x_k\|^{\frac{1}{2}} \|T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} x\|^{-\frac{1}{2}}$,

3. if $\|x_k\| = 0$, set $\lambda_k = 2^{-k} \|T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} x\|^{-1}$.

For these two cases if $\|T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} x\| \to 0$, then $\mathcal{T}$ is $\mathcal{M}$-hypercyclic tuple and so $\mathcal{M}$-diskcyclic. Otherwise, it follows easily that $|\lambda_k| \leq 1$, for all $k \in \mathbb{N}$. Set $z = x + {\lambda_k}^{-1} x_k$ for a large enough k. Since $x \in U_1 \subset \mathcal{M}$ and ${\lambda_k}^{-1} x_k \in \mathcal{M}$, then $z \in \mathcal{M}$. Since

$$\|z - x\| \to 0,$$

it follows that $z \in U_1$.

Now, since

$$\begin{aligned} &\lambda_k T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} z \\ &= \lambda_k T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} x + T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} x_k \end{aligned}$$

then, by using the condition (3), we get

$$\lambda_k T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} z \text{ and } T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} x_k$$

belong to $\mathcal{M}$ and so $\lambda_k T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} x \in \mathcal{M}$.

Moreover, since $T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} x_k \to y$ for a large enough k, then

$$\left\| \lambda_k T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} z - y \right\| \to 0.$$

Thus, $\lambda_k T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \dots T_n^{r_k^{(n)}} z \in U_2$. It follows that there exists $k \in \mathbb{N}$ such that

$$U_1 \cap T_n^{-r_k^{(n)}} T_{n-1}^{-r_k^{(n-1)}} \dots T_1^{-r_k^{(1)}} \left(\lambda_k^{-1} U_2 \right) \neq \phi.$$

By Proposition 2.1 and Corollary 2.1, T is an $\mathcal{M}$-diskcyclic tuple. □

The following theorem gives the relation between the subspace-diskcyclicity of a tuple of operators and the tuple of the direct sum of those operators.

Proposition 2.2. *Let $\mathcal{M}$ be a subspace of a Banach space X and $\mathcal{T} = (T_1, T_2, \ldots, T_n)$ be a tuple. Then, $\mathcal{T}$ satisfies subspace-diskcyclic criterion if and only if the tuple $\mathcal{S} = (T_1 \oplus T_1, T_2 \oplus T_2, \ldots, T_n \oplus T_n)$ satisfies subspace-diskcyclic criterion.*

With out loss of generality, we suppose that $\mathcal{S} = (T_1 \oplus T_1, T_2 \oplus T_2)$ and then the general case follows by the same way.

For the "if" part, let $\mathcal{M}$ be a closed subspace of X such that $\mathcal{S}$ satisfies $\mathcal{M} \oplus \mathcal{M}$-diskcyclic criterion. Let D_1 and D_2 be dense sets in $\mathcal{M}$ then $W = D_1 \oplus D_2$ is dense in $\mathcal{M} \oplus \mathcal{M}$. Let $x \in D_1$ and $y \in D_2$, then $(x, y) \in W$. By hypothesis, there exist two increasing sequence of positive integers $\left\langle r_k^{(i)} \right\rangle_{k \in \mathbb{N}}$ for $i = 1, 2$ and a sequence $\langle (x_k, y_k) \rangle_{k \in \mathbb{N}}$ in $\mathcal{M} \oplus \mathcal{M}$ such that $\|(x_k\, y_k)\| \to (0, 0)$ and $(T_1 \oplus T_1)^{r_k^{(1)}} (T_2 \oplus T_2)^{r_k^{(2)}} (x_k, y_k) \to (x, y)$ as $k \to \infty$. which means that $(T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} x_k, T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} y_k) \to (x, y)$. It follows that for each $y \in D_2$ there is a sequence $\langle y_k \rangle_{k \in \mathbb{N}} \to 0$ in $\mathcal{M}$ such that

$$T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} y_k \to y. \tag{3}$$

By hypothesis, we have $\|(T_1 \oplus T_1)^{r_k^{(1)}} (T_2 \oplus T_2)^{r_k^{(2)}} (x, y)\| \, \|(x_k, y_k)\| \to (0, 0)$. Then, for all $x \in D_1$ it easy follows that

$$\left\| T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} x \right\| \|y_k\| \to 0. \tag{4}$$

Also, since $(T_1 \oplus T_1)^{r_k^{(1)}} (T_2 \oplus T_2)^{r_k^{(2)}} (\mathcal{M} \oplus \mathcal{M}) \subseteq (\mathcal{M} \oplus \mathcal{M})$, then

$$T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} (\mathcal{M}) \subseteq \mathcal{M}. \tag{5}$$

From (3), (4) and (5), the tuple $\mathcal{T} = (T_1, T_2)$ satisfies diskcyclic criterion.

For the "only if" part, since $T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \mathcal{M} \subseteq \mathcal{M}$, for all $k \in \mathbb{N}$, then,

$$T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \mathcal{M} \oplus T_1^{r_k^{(1)}} T_2^{r_k^{(2)}} \mathcal{M} \subseteq \mathcal{M} \oplus \mathcal{M}.$$

So,

$$(T_1 \oplus T_1)^{r_k^{(1)}} (T_2 \oplus T_2)^{r_k^{(2)}} (\mathcal{M} \oplus \mathcal{M}) \subseteq \mathcal{M} \oplus \mathcal{M}.$$

The remainder of the proof follows easily from [13, Corollary 1]. □

Proposition 2.3. *Let $\mathcal{M}$ be a subspace of a Banach space X and $\mathcal{T} = (T_1, T_2, \ldots, T_n)$ be a tuple of operators. If the semigroup $\mathcal{F}$ contains an $\mathcal{M}$-diskcyclic operator, then $\mathcal{T}$ is $\mathcal{M}$-diskcyclic tuple.*

Proof. Suppose that T is an $\mathcal{M}$-diskcyclic operator in $\mathcal{F}$, then

$$\mathcal{M} = \overline{\mathbb{D}Orb(T, x) \cap \mathcal{M}} \subseteq \overline{\mathbb{D}Orb(\mathcal{T}, x) \cap \mathcal{M}} \subseteq \mathcal{M}.$$

It follows that $\overline{\mathbb{D}Orb(\mathcal{T}, x) \cap \mathcal{M}} = \mathcal{M}$ and so $\mathcal{T}$ is $\mathcal{M}$-diskcyclic tuple. □

The following example gives a tuple of operators which is $\mathcal{M}$-diskcyclic, however, not every operator in the tuple is $\mathcal{M}$-diskcyclic.

Example 2.3. Let $T_1, T_2 \in B(\ell^2(\mathbb{Z}))$ be bilateral forward weighted shifts with the weight sequences w_n, k_n respectively, where

$$w_n = \begin{cases} \frac{1}{3} & \text{if } n \geq 0 \\ \frac{1}{2} & \text{if } n < 0 \end{cases} \quad \text{and} \quad k_n = \begin{cases} 4 & \text{if } n \geq 0 \\ 5 & \text{if } n < 0 \end{cases}$$

and let $\mathcal{M}$ be the subspace of $\ell^2(\mathbb{Z})$ consisting of all sequences with zeroes on the even entries; that is,

$$\mathcal{M} = \left\{ \{a_n\}_{n=-\infty}^{\infty} \in \ell^2(\mathbb{Z}) : a_{2n} = 0, n \in \mathbb{Z} \right\},$$

then by [2, Theorem 3.6] T_1 is not $\mathcal{M}$-diskcyclic but T_2 is $\mathcal{M}$-diskcyclic. However, the tuple $\mathcal{T} = (T_1, T_2)$ is $\mathcal{M}$-diskcyclic by Proposition 2.3.

3. Conclusion

We studied both subspace-diskcyclic and subspace-disk transitive tuples. We provided some sufficient conditions for a tuple to be subspace-diskcyclic which is called subspace-diskcyclic tuple criterion. Then, we found a relation between the subspace-diskcyclicity of a tuple of operators and the tuple of the direct sum of those operators. By giving an example, we showed that if a tuple is subspace-diskcyclic, then there may be a non-diskcyclic operator in that tuple.

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