Projection graphs of rings and near-rings

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Abstract. Association of graphs with algebraic structures facilitates the process of understanding the properties of algebraic structures through graphs. In this paper, projection graph $P(R)$ of a ring R is introduced as an undirected graph, whose vertices are the nonzero elements of R and any two distinct vertices x and y are adjacent if and only if their product is equal to either x or y. The projection graph $P(N)$ of a near-ring N is also defined in the same way. It is proved that $P(R)$ is a star graph if and only if R has no nonzero zero-divisors. A method of finding adjacent vertices with the help of annihilators is developed. The projection graphs of certain classes of rings are found to be bipartite and $P(R)$ is proved to be weakly pancyclic when R is a local ring with ascending chain condition on the annihilator ideals of its elements. $P(\mathbb{Z}_n)$ are constructed for certain values of n and their properties are studied. Moreover, $P(N)$ is shown as a complete graph when N is either a constant near-ring or an almost trivial near-ring.

Keywords: commutative rings, annihilator, near-ring, independent set, clique, planar graph.

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1. Introduction

There are many graphs associated to rings and the other algebraic structures such as groups, semigroups, semirings, near-rings, ternary rings, modules etc. to understand the properties of algebraic structures via graphs and vice versa.

The idea of associating a graph to a commutative ring R was introduced by Beck [11] in 1988. He defined a graph with the vertex set as the set of all elements of R and two distinct vertices x and y are adjacent if and only if $xy = 0$ and mainly studied about coloring of the graph. In 1993, Anderson and Naseer [5] determined all finite commutative rings with chromatic number 4. Anderson and Livingston [6] in 1999, redefined Beck's graph by taking $ZD^*(R)$, the set of nonzero zero-divisors of R, as the vertex set and named the graph of R as zero-divisor graph denoted by $\Gamma(R)$. They proved that the zero-divisor graph of a commutative ring R is complete if and only if either $R \cong \mathbb{Z}_2^2$ or $xy = 0$ for all $x, y \in ZD(R)$, the set of zero-divisors of R.

Afkhami and Khashyarmanesh [1] introduced cozero-divisor graph $\Gamma'(R)$ of a commutative ring R. The vertex set of $\Gamma'(R)$ is $W^*(R)$, the set of nonzero nonunits of R and $a, b \in W^*(R)$ are adjacent if and only if $a \notin bR$ and $b \notin aR$. They studied $\Gamma'(R)$ and its complement $\overline{\Gamma'(R)}$ in [2]. In particular, they characterized all commutative rings whose cozero-divisor graphs are double-star, unicyclic, a star, or a forest. Further, Akbari et al. [3] continued the study of cozero-divisor graphs of commutative rings and proved that if $\Gamma'(R)$ is a forest, then $\Gamma'(R)$ is a union of isolated vertices or a star.

The concept of annihilator graph was introduced in 2014 by Badawi [9]. The annihilator graph of a commutative ring R is the simple graph denoted by $AG(R)$, whose vertex set is $ZD^*(R)$ and two distinct vertices x and y are adjacent if and only if $Ann(xy) \neq Ann(x) \cup Ann(y)$, where $Ann(x) = \{y \in R \mid xy = 0\}$. If R is a commutative ring with more than 2 nonzero zero-divisors, then $AG(R)$ is proved to be connected and $diam(AG(R)) \leq 2$. More results on $AG(R)$ can be found in the survey article [10].

Teresa Arockiamary et al. [18] defined annihilator 3-uniform hypergraph $AH_3(N)$ of a right ternary near-ring (RTNR) N. Let $(N, +, [\ ])$ be an RTNR. Then, $AH_3(N)$ is defined as the 3-uniform hypergraph whose vertex set is the set of all elements of N having nontrivial annihilators and three distinct vertices x, y and z are adjacent whenever the intersection of their annihilators is not $\{0\}$, where the annihilator of x is given by $(0 : x) = \cap_{s \in N}(0 : x)_s$ and $(0 : x)_s = \{t \in N \mid [t\ s\ x] = 0\}$. $AH_3(N)$ is shown to be an empty hypergraph if N is a constant RTNR, and $AH_3(N)$ is trivial when N is a zero-symmetric integral RTNR.

Motivated by the results established in [6], [9], [10] and [18], the projection graphs of rings and near-rings are introduced in this article. Throughout, this article R is considered as a nonnil unital commutative ring unless otherwise mentioned. The induced subgraph of $P(R)$ on $R \setminus \{0, 1\}$ is denoted by $P_1(R)$. Also, $U(R)$ denotes the set of all units of R.

Let R be a commutative ring. Then, the vertex set of $P(R)$ is R^*, the set of all nonzero elements of R and $x, y \in R^*$ are adjacent if and only if the product xy in R equals either x or y. It is observed that $x, y \in W^*(R)$ are adjacent in $P(R)$ implies x, y are adjacent in $\overline{\Gamma'(R)}$ and therefore the induced subgraph of $P(R)$ on $W^*(R)$ is a subgraph of $\overline{\Gamma'(R)}$. It is proved that $P(R)$ is a connected graph with diameter at most 2. Let $|R| > 4$. Then, it is seen that $P_1(R)$ is nontrivial if and only if R has nonzero zero-divisors. Also $P(R)$ is a star if and only if R is a field. The girth of $P(R)$ is either 3 or ∞.

A method of finding adjacent vertices using concept of annihilators is given and it is illustrated for $R = \mathbb{Z} \times \mathbb{Z}$. $Reg(R)\backslash\{1\}$, $Nil(R)\backslash\{0\}$ are found independent sets, where $Reg(R)$ is the set of all regular elements of R and $Nil(R)$ is the set of all nilpotent elements of R. If R is presimplifiable ring which is not a domain, then it is proved that $P_1(R)$ is bipartite. $P(R)$ is shown to be weakly pancyclic when R is a local ring, which is not a domain, with ascending chain condition on the annihilator ideals of elements of R. The projection graphs of finite isomorphic rings are proved to be isomorphic. It is also shown that $P(R)$ is complete if and only if either $R \cong \mathbb{Z}_3$ or $R \cong \mathbb{Z}_4$. Some of the graph properties of $P(\mathbb{Z}_n)$ are verified for $n = 2q, 2^k$, q is prime and $k \geq 1$.

Let N be a near-ring. Then, the projection graph $P(N)$ of N is defined in the same way as that of a ring. It is shown that if N is either a constant near-ring or an almost trivial near-ring, then $P(N)$ is a complete graph. Also $P(N)$ is complete if N is a Boolean near-ring which is subdirectly irreducible.

2. Preliminaries

In this section the basic definitions along with the results relevant to this paper, related to rings ([8], [4], [14]), near-rings ([15], [16], [17]) and graphs ([12]) are given. Let R be a commutative ring with unity. Then, an element $x \in R$ is called *Von Neumann regular* if $x = ax^2$ for some $a \in R$. R is called (i) *Boolean* if every $x \in R$ is idempotent (ii) a *quasilocal ring* if R has finitely many maximal ideals. (iii) a *local ring* if R has a unique maximal ideal. (iv) [4] a *presimplifiable ring* if, for any $a, b \in R$, $a = ab$ implies either $a = 0$ or $b \in U(R)$. (v) a *domain-like ring* if $ZD(R) \subseteq Nil(R)$, where $Nil(R)$ equals the set of all nitpotent elements of R. (vi) a *nil ring* if every element in R is nilpotent. It is known that quasilocal rings are presimplifiable rings.

Lemma 2.1 ([14])**.** *If R is nil, then $xy \neq y$ for all $x, y \in R^*$.*

Lemma 2.2 ([4])**.** *If R is a commutative ring, then the following are equivalent:*

(i) *R is presimplifiable;*

(ii) *$ZD(R) \subseteq J(R)$;*

(iii) *$ZD(R) \subseteq \{1 - u \mid u \in U(R)\}$, where $J(R)$ denotes the Jacobson radical and $J(R)$ equals the intersection of all maximal ideals of R.*

Definition 2.1 ([15]). *A right near-ring N is an algebraic system with two binary operations $+$ and $\cdot$ satisfying the following conditions:*

(i) *$(N, +)$ is a group (not necessarily abelian);*

(ii) *$(N, \cdot)$ is a semigroup;*

(iii) *$(x + y)z = xz + yz$ for every $x, y, z \in N$.*

If $N = N_0 = \{x \in N | x0 = 0\}$, then N is called a *zero-symmetric near-ring.* If $N = N_c = \{x \in N | x0 = x\} = \{x \in N | xy = x$ for every $y \in N\}$, then N is called a *constant near-ring.* A *near-field* is a near-ring, in which there is a multiplicative identity and every non-zero element has a multiplicative inverse. Also by *Pierce Decomposition*, $(N, +) = N_0 + N_c$ and $N_0 \cap N_c = \{0\}$.

Definition 2.2 ([16]). *A near-ring N is called an almost trivial near-ring if for all $x, y \in N$, $xy = \begin{cases} x & \text{if } y \notin N_c \\ 0 & \text{if } y \in N_c \end{cases}$.*

Lemma 2.3 ([16]). *If N is a subdirectly irreducible Boolean near-ring, then N is an almost trivial near-ring.*

A pair $G = (V, E)$ is an *undirected graph* if V is the set of vertices and E is set of edges $\overline{xy}$, where $x, y \in V$ and $x \neq y$. If $x \in V$, then $N_G(x) = \{y \in V \mid \overline{xy} \in E, x \neq y\}$. The *girth* of G is the length of shortest cycle in G and if G has no cycles, then the girth of G is defined to be infinite. G is called *weakly pancyclic* if it contains cycles of all lengths between its girth and the longest cycle. The sequence of degrees of vertices in G arranged in a non decreasing order is called the *degree sequence* of G.

3. Projection graphs of rings

Definition 3.1. *Let $(R, +, \cdot)$ be a ring. Then, the projection graph of R, denoted by $P(R)$, is defined as an undirected graph whose vertex set is the set of all nonzero elements of R and two distinct vertices x and y are adjacent whenever the product $x \cdot y$ equals either x or y. That is, $P(R) = (V, E)$, where $V = R^*$ and $E = \{\overline{xy} \mid x \cdot y = x \text{ or } y, x \neq y\}$. For the sake of convenience, $x \cdot y$ is simply written as xy.*

Example 3.1. It is evident that the projection graph of $2\mathbb{Z}$ is an empty graph. The projection graphs of the rings $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_6$, $\mathbb{Z}_2^3$, $\mathbb{Z}_{12}$ and $\mathbb{Z}_3^2$ are shown in Figure 1, Figure 2, Figure 3, Figure 4, Figure 5 and Figure 6, respectively. Note that, $P(\mathbb{Z}_4)$ is a complete graph and $P(\mathbb{Z}_5)$ is a star. In $P(\mathbb{Z}_2^3)$, ijk stands for (i, j, k), where $i, j, k \in \mathbb{Z}_2$. In $P(\mathbb{Z}_3^2)$, ij stands for (i, j), where $i, j \in \mathbb{Z}_3$.

Proposition 3.1. *Let R be a commutative ring with nonzero identity. Then, $P(R)$ is a connected graph with diameter at most 2.*

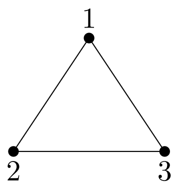

Figure 1: $P(\mathbb{Z}_4)$

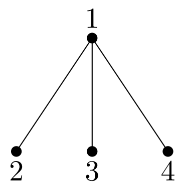

Figure 2: $P(\mathbb{Z}_5)$

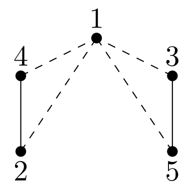

Figure 3: $P(\mathbb{Z}_6)$

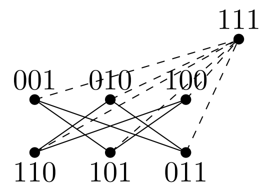

Figure 4: $P(\mathbb{Z}_2^3)$

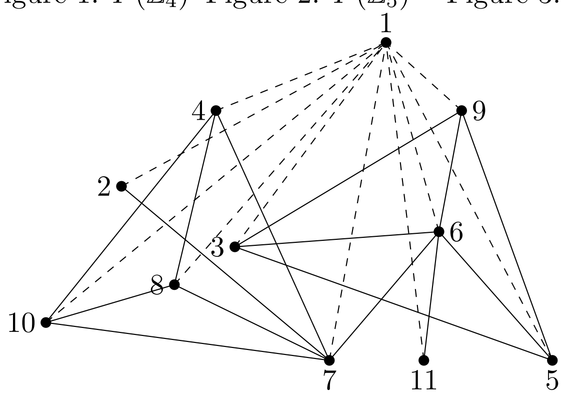

Figure 5: $P(\mathbb{Z}_{12})$

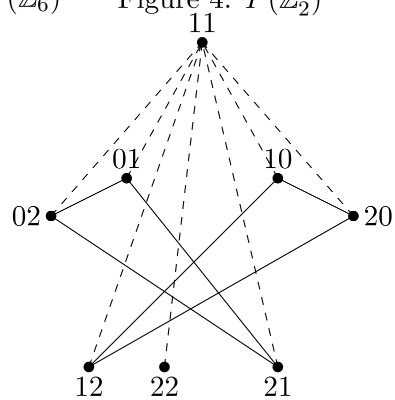

Figure 6: $P(\mathbb{Z}_3^2)$

Proof. Note that, $P(R)$ is nontrivial since $\overline{1x}$ is an edge for every $x \in R^*\backslash\{1\}$. Let x, $y \in R^*$. If $\overline{xy}$ is an edge, then the distance between x and y is 1. If $\overline{xy}$ is not an edge, then $x - 1 - y$ is a path between x and y. Thus, $P(R)$ is connected and the distance between x and y is at the most 2, which proves the proposition. □

Remark 3.1. Notice that the removal of 1 from the vertex set may result in disconnection of $P(R)$. For example, $P_1(\mathbb{Z}_5)$, $P_1(\mathbb{Z}_6)$ and $P_1(\mathbb{Z}_3^2)$ are disconnected. Also it is observed that $P_1(R)$ is disconnected for the Boolean ring $R = \mathbb{Z}_2^2$.

Let R be a commutative ring with nonzero identity. If $x, y \in ZD^*(R)$ are adjacent in $\Gamma(R)$, then x, y are not adjacent in $P(R)$. However, $P_1(R)$ is nontrivial if and only if R has nonzero zero-divisor, which is proved in this section.

Proposition 3.2. *If $x, y \in R^*\backslash\{1\}$ are distinct elements such that $x + y \neq 1$, then the following assertions hold in $P_1(R)$:*

(i) If $xy = 0$, then $1 - y \in N_{P_1(R)}(x)$ and $1 - x \in N_{P_1(R)}(y)$.

(ii) If x is adjacent to y, then $1 - x \in N_{P_1(R)}(1 - y)$.

Proof. (i) If $xy = 0$, then $x(1 - y) = x$ and $(1 - x)y = y$, where $1 - x, 1 - y$ are in $R^*\backslash\{1, x, y\}$, proving (i).

(ii) If x is adjacent to y, then either $xy = x$ or $xy = y$.

If $xy = x$, then $(1-x)(1-y) = 1-y$. Similarly, if $xy = y$, then $(1-x)(1-y) = 1-x$, where $1-x, 1-y \in R^*\backslash\{1, x, y\}$, proving (ii). □

Proposition 3.3. *If R is a Boolean ring with more than 4 elements and $x, y \in R^*\backslash\{1\}$, then the following assertions hold in $P_1(R)$:*

(i) If $xy = 0$ and $x + y \neq 1$, then $x - (x + y) - y$ is a path between x and y.

(ii) If $xy = 0$ and $x + y = 1$, then there is no $z \in R^*\backslash\{1\}$ such that $x - z - y$ is a path between x and y.

(iii) If x and y are adjacent and $x + y \neq 1$, then either $x + y \in N_{P_1(R)}(x)$ or $x + y \in N_{P_1(R)}(y)$, but not both.

(iv) If $xy \neq 0$ and x, y are not adjacent, then $x - xy - y$ is a path between x and y.

Proof. (i) If $xy = 0$ and $x + y \neq 1$, then $x(x + y) = x$ and $(x + y)y = y$, where $x + y \in R^*\backslash\{1, x, y\}$, proving (i).

(ii) Suppose $xy = 0$ and $x + y = 1$.

Let $z \in R^*\backslash\{1\}$ be adjacent to x. Then, either $xz = x$ or $xz = z$.

Case (a). Suppose $xz = x$. Then, zy is neither z nor y. For, if $zy = z$, then $x = xz = xzy = 0$, a contradiction to the choice of x. If $zy = y$, then $1 = x + y = xz + zy = z(x + y) = z$, a contradiction to the choice of z.

Case (b). Suppose $xz = z$. Then, zy is neither z nor y. For, if $zy = z$, then $z = (x+y)z = xz+yz = z+z = 0$, a contradiction to the choice of z. If $zy = y$, then $y = zy = xzy = 0$, a contradiction to the choice of y.

Hence, z is not adjacent to y in both the cases, which completes the proof of (ii).

(iii) Suppose x, y are adjacent and $x+y \neq 1$. Then, either $xy = x$ or $xy = y$. If $xy = x$, then $x(x+y) = x^2+xy = x+x = 0$, since R is of characteristic 2. Also $(x+y)y = xy+y^2 = x+y$. Hence, $x+y \notin N_{P_1(R)}(x)$, whereas $x+y \in N_{P_1(R)}(y)$.

Similarly, if $xy = y$, then it can be seen that $x + y \in N_{P_1(R)}(x)$ and $x + y \notin N_{P_1(R)}(y)$.

(iv) If $xy \neq 0$ and x, y are not adjacent, then $x(xy) = xy$ and $(xy)y = xy$, where $xy \in R^*\backslash\{1, x, y\}$, proving (vi). □

Proposition 3.4. *If $P_1(R)$ is nontrivial, then R has nonzero zero-divisor.*

Proof. Suppose x, $y \in R^*\backslash\{1\}$ and $\overline{xy}$ is an edge. Then, either $xy = x$ or $xy = y$. If $xy = x$, then $x(1 - y) = 0$, which shows that x is a nonzero zero-divisor. Similarly, if $xy = y$, then y is nonzero zero-divisor. □

Remark 3.2. If $e \in R$ is a nontrivial idempotent, then $1-e$ is also a nontrivial idempotent and the principal ideal generated by e has at least two elements, namely 0 and e. Also eR has more than 2 elements only if $|R| \geq 6$.

Proposition 3.5. *If $e \in R$ is a nontrivial idempotent, then* *(i) e is adjacent to every element in $eR\backslash\{0,e\}$.*

(ii) no element in $eR\backslash\{0\}$ is adjacent to an element in $(1-e)R\backslash\{0\}$.

Proof. Suppose $e \in R$ is a nontrivial idempotent.

(i) Let $x \in eR\backslash\{0,e\}$. Then, $x = er$ for some $r \in R^*\backslash\{1\}$ and hence $ex = e(er) = er = x$, which shows that e is adjacent to x.

(ii) Let $x \in eR\backslash\{0\}$ and $y \in (1-e)R\backslash\{0\}$. Then, $x = er$ and $y = (1-e)s$, for some r, s in R^* and therefore $xy = 0$ since $e(1-e) = 0$. Hence, x and y are not adjacent. □

Proposition 3.6. *Let $e \in R$ be a nontrivial idempotent. If the principal ideal generated by e is of size two, then either $\overline{ex} \in E$ or $\overline{(1-e)x} \in E$, for every $x \in R^*\backslash\{1, e, 1-e\}$.*

Proof. Suppose $|eR| = 2$. Then, er is either 0 or e for every r in R.

Let $A_1(e) = \{r \in R^* | er = e\}$ and $A_1'(e) = \{r \in R^* | er = 0\}$. Then, $R^* = A_1(e) \cup A_1'(e)$, where $1, e \in A_1(e)$ and $1-e \in A_1'(e)$.

Let $x \in R^*\backslash\{1, e, 1-e\}$. If $x \in A_1(e)$, then $ex = e$, which implies $\overline{ex} \in E$. If $x \in A_1'(e)$, then $(1-e)x = x$, which implies $\overline{(1-e)x} \in E$. □

Proposition 3.7. *Let R be a commutative ring with nonzero identity such that $|R| > 4$. Then, $P_1(R)$ is nontrivial if and only if R has a nonzero zero-divisor.*

Proof. By Proposition 3.4, it is enough to prove that $P_1(R)$ is nontrivial if R has nonzero zero-divisor.

Let $x \in R$ be nonzero zero-divisor. Then, there exists $y \in R^*$ such that $xy = 0$.

Suppose $1-y \neq x$. Then $x(1-y) = x - xy = x$ and so $\overline{x(1-y)}$ is an edge, where $x, 1-y \in R^*\backslash\{1\}$. Suppose $1-y = x$. Then, x is a nontrivial idempotent. Now, consider the cases:

(i) $|xR| = 2$ (ii) $|xR| > 2$.

If $|xR| = 2$, then $xR = \{0, x\}$ and therefore there exists $r \in R^*\backslash\{1\}$ such that $xr = x$, which implies $\overline{xr} \in E$, where $x, r \in R^*\backslash\{1\}$.

If $|xR| > 2$, then by Proposition 3.5(i), there exists $y \in xR\backslash\{0, x\}$ such that $\overline{xy} \in E$, where $x, y \in R^*\backslash\{1\}$. □

Corollary 3.1. *Let R be a ring with $|R| > 4$. Then, $P(R)$ is a star if and only if R satisfies any one of the following equivalent conditions:*

(i) $P_1(R)$ is trivial.

(ii) R has no nonzero zero-divisor.

(iii) Every element in R^* has trivial annihilator.

Proof. $P_1(R)$ is trivial if and only if $E = \{\overline{x1} | x \in R^*\backslash\{1\}\}$. Therefore, $P(R)$ is a star if and only if $P_1(R)$ is trivial.

(i) $\Leftrightarrow$ (ii) follows from the above proposition.

(ii) $\Leftrightarrow$ (iii) follows from the definition of annihilator. □

Corollary 3.2. *Let R be a ring with $|R| > 4$. Then, $P(R)$ is a star if and only if R is a field.*

Proposition 3.8. *Let R be a ring with $|R| > 4$. Then, the girth of $P(R)$ is either 3 or ∞.*

Proof. If R has no nonzero zero-divisors, then $P(R)$ is a star by Corollary 3.1 and hence the girth is ∞.

If R has nonzero zero-divisor, then $P_1(R)$ is nontrivial by Proposition 3.7.

Let $\overline{xy} \in E$, where $x, y \in R^*\backslash\{1\}$. Then, $1 - x - y - 1$ forms a cycle and hence the girth is 3. □

For any ring R, write $V = R^* = \{1\} \cup (Reg(R)\backslash\{1\}) \cup (ZD(R)\backslash\{0\})$, where $Reg(R) = \{x \in R^* | x \notin ZD(R)\}$. Then, $N_{P(R)}(1) = R^*\backslash\{1\}$ and for every $x \in R^*\backslash\{1\}$, $N_{P(R)}(x) = \{y \in R^* | xy = x \text{ or } xy = y, y \neq x\}$. Now, for every $x \in R^*\backslash\{1\}$, write $A_1(x) = \{y \in R^* | xy = x\}$ and $A_2(x) = \{y \in R^* | xy = y\}$. Then, it is observed that $x = xy = xy^2 = \ldots = xy^k = \ldots$ holds if $y \in A_1(x)$ and $y = xy = x^2y = \ldots = x^ky = \ldots$ holds if $y \in A_2(x)$. Thus, $N_{P(R)}(x)$ contains an infinite number of elements if any one of the above sequences does not terminate.

Proposition 3.9. *Let $x \in R^*\backslash\{1\}$. Then, the following assertions hold:*

(i) $A_1(x) \cap A_2(x) = \{x\}$ *if and only if x is an idempotent.*

(ii) $A_1(x) = Ann(x) + 1$; $A_2(x) = Ann(1 - x)\backslash\{0\}$.

Proof. (i) Suppose $x \in R^*\backslash\{1\}$ is an idempotent element. Then, $x^2 = x$ and so $x \in A_1(x) \cap A_2(x)$. Also, $y \in A_1(x) \cap A_2(x)$ implies $y = xy = x$ and hence $A_1(x) \cap A_2(x) = \{x\}$.

Conversely, suppose $A_1(x) \cap A_2(x) = \{x\}$. Then, $xx = x$, which proves (i).

(ii) By the definition of $A_1(x)$, $y \in A_1(x) \Leftrightarrow xy = x \Leftrightarrow x(y - 1) = 0 \Leftrightarrow y - 1 \in Ann(x)$.

Now, $y - 1 \in Ann(x) \Leftrightarrow y \in Ann(x) + 1$. For, if $y - 1 \in Ann(x)$, then $y = (y - 1) + 1 \in Ann(x) + 1$. Also if $y \in Ann(x) + 1$, then $y = z + 1$, for some $z \in Ann(x)$, which implies $y - 1 = z \in Ann(x)$. Hence, $A_1(x) = Ann(x) + 1$. By the definition of $A_2(x)$, $y \in A_2(x) \Leftrightarrow y \neq 0$ and $xy = y \Leftrightarrow y \neq 0$ and $y(1 - x) = 0 \Leftrightarrow y \in Ann(1 - x)\backslash\{0\}$ and hence $A_2(x) = Ann(1 - x)\backslash\{0\}$. □

Proposition 3.10. *If $x \in Reg(R)\backslash\{1\}$, then $N_{P(R)}(x) \subseteq (ZD(R)\backslash\{0\}) \cup \{1\}$.*

Proof. Let $x \in Reg(R)\backslash\{1\}$ and $y \in N_{P(R)}(x)$. Then, $xy = x$ or $xy = y$.

If $xy = x$, then $x(y-1) = 0$, which implies $y = 1$ by the hypothesis.

If $xy = y$, then $(x-1)y = 0$, which implies $y \in ZD(R)\backslash\{0\}$, completing the proof. □

Corollary 3.3. *$Reg(R)\backslash\{1\}$ is an independent set.*

Proof. Let $x \in Reg(R)\backslash\{1\}$ and $y \in N_{P(R)}(x)$. Then, $y \notin Reg(R)\backslash\{1\}$ from the above proposition. Hence, $Reg(R)\backslash\{1\}$ is independent. □

Remark 3.3. If R is finite, then $V = R^* = \{1\} \cup (U(R)\backslash\{1\}) \cup (ZD(R)\backslash\{0\})$. Hence, $U(R)\backslash\{1\}$ is independent by the above corollary.

Theorem 3.1. *For any $x \in R^*\backslash\{1\}$, the following assertions hold, in which $\mathbb{E}$ denotes the set of all nontrivial idempotents in R:*

(i) $N_{P(R)}(x) = \{1\} \cup (Ann(1-x)\backslash\{0\})$ *if* $x \in Reg(R)\backslash\{1\}$.

(ii) $N_{P(R)}(x) = ((Ann(x)+1) \cup Ann(1-x))\backslash\{0\}$ *if* $x \in ZD(R)\backslash\{0\}$ *and* $x \notin \mathbb{E}$.

(iii) $N_{P(R)}(x) = ((Ann(x)+1) \cup Ann(1-x))\backslash\{0, x\}$ *if* $x \in ZD(R)\backslash\{0\}$ *and* $x \in \mathbb{E}$.

Proof. Let $x \in R^*\backslash\{1\}$. Then, by the definitions of $A_1(x)$ and $A_2(x)$ and Proposition 3.9(ii), $N_{P(R)}(x) = A_1(x) \cup A_2(x) = (Ann(x)+1) \cup (Ann(1-x)\backslash\{0\})$.

(i) If $x \in Reg(R)\backslash\{1\}$, then $Ann(x) = \{0\}$. Hence, $N_{P(R)}(x) = \{1\} \cup (Ann(1-x)\backslash\{0\})$.

(ii) If $x \in ZD(R)\backslash\{0\}$ and $x \notin \mathbb{E}$, then $N_{P(R)}(x) = (Ann(x)+1) \cup (Ann(1-x)\backslash\{0\})$.

(iii) If $x \in ZD(R)\backslash\{0\}$ and $x \in \mathbb{E}$, then $N_{P(R)}(x) = ((Ann(x)+1) \cup Ann(1-x))\backslash\{0, x\}$, by Proposition 3.9(i). □

Proposition 3.11. *If $x \in R^*\backslash\{1\}$ is not a zero-divisor, then $N_{P(R)}(x)\backslash\{1\}$ together with 0 forms an ideal.*

Proof. If x is not a zero-divisor, then by Theorem 3.1(i), $(N_{P(R)}(x)\backslash\{1\}) \cup \{0\} = Ann(1-x)$, which is an ideal. □

Illustration 3.1. Consider $R = \mathbb{Z} \times \mathbb{Z}$, where $ZD(R) = (\mathbb{Z} \times \{0\}) \cup (\{0\} \times \mathbb{Z})$ and $Reg(R) = \{(m,n) \in \mathbb{Z} \times \mathbb{Z} | m, n \neq 0\}$.

If $x = (1,1)$, then $N_{P(R)}(x) = R^*\backslash\{(1,1)\}$.

If $x = (m,n) \in Reg(R)\backslash\{(1,1)\}$, then $N_{P(R)}(x) = (\{0\} \times \mathbb{Z}^*) \cup \{(1,1)\}$ if $m \neq 1, n = 1$, $N_{P(R)}(x) = (\mathbb{Z}^* \times \{0\}) \cup \{(1,1)\}$ if $m = 1, n \neq 1$, $N_{P(R)}(x) = \{(1,1)\}$ if $m, n \neq 1$. Thus, $Reg(R)\backslash\{(1,1)\}$ is independent.

If $x = (m,n) \in ZD(R)\backslash\{(0,0)\}$, then $N_{P(R)}(0,1) = (\mathbb{Z} \times \{1\}) \cup (\{0\} \times \mathbb{Z}^*)\backslash\{(0,1)\}$, $N_{P(R)}(1,0) = (\{1\} \times \mathbb{Z}) \cup (\mathbb{Z}^* \times \{0\})\backslash\{(1,0)\}$.

$N_{P(R)}(x) = \mathbb{Z} \times \{1\}$ if $m = 0, n \neq 1$, $N_{P(R)}(x) = \{1\} \times \mathbb{Z}$ if $m \neq 1, n = 0$.

Note that, $(0,1)$ and $(1,0)$ are the nontrivial idempotents in R.

Proposition 3.12. *Let $e \in R$ be a nontrivial idempotent. Then*

(i) $N_{P(R)}(e) = (((1-e)R+1) \cup eR)\backslash\{0,e\}$.

(ii) *Every element in $eR\backslash\{0\}$ is adjacent to every element in $(1-e)R+1$.*

(iii) *For every $x \in eR\backslash\{0,e\}$ and $y \in ((1-e)R+1)\backslash\{e\}$, $e-x-y-e$ forms a cycle.*

Proof. (i) If $e \in R$ is a nontrivial idempotent, then by Theorem 3.1(iii), $N_{P(R)}(e) = ((Ann(e)+1) \cup Ann(1-e))\backslash\{0,e\}$.

Now, if $r \in Ann(e)$, then $re = 0$, which implies $r = r1 = r((1-e)+e) = r(1-e) \in (1-e)R$. Also, $r \in (1-e)R$ implies $r \in Ann(e)$. Hence, $Ann(e) = (1-e)R$.

Similarly, it can be proved that $Ann(1-e) = eR$. Thus, $N_{P(R)}(e) = (((1-e)R+1) \cup eR)\backslash\{0,e\}$.

(ii) Let $x \in eR\backslash\{0\}$ and $y \in (1-e)R+1$. Then, $x \in Ann(1-e)\backslash\{0\}$, which implies $xe = x$ and there exists $z \in Ann(e)$ such that $y = z+1$.

Now, $xy = x(z+1) = xe(z+1) = x$. Hence, $\overline{xy} \in E$, proving (ii).

(iii) Let $x \in eR\backslash\{0,e\}$ and $y \in ((1-e)R+1)\backslash\{e\}$. Then, $\overline{ex}, \overline{ye} \in E$ by (i) and $\overline{xy} \in E$ by (ii). Hence, $e-x-y-e$ forms a cycle. □

Proposition 3.13. *Let $e \in R$ be a nontrivial idempotent such that both of eR and $(1-e)R+1$ contain more than 2 elements. Then, the following assertions hold in $P_1(R)$:*

(i) *$P_1(R)$ contains $K_{i,j}$, where $i = |eR|-2$ and $j = |(1-e)R+1|-2$.*

(ii) *$P_1(R)$ is not planar if both of eR and $(1-e)R+1$ contain more than 5 elements.*

Proof. (i) Let $V_1 = eR\backslash\{0,e\}$ and $V_2 = ((1-e)R+1)\backslash\{1,e\}$. Then, for any $x \in V_1$ and $y \in V_2$, $\overline{xy} \in E$ by Proposition 3.12(ii), proving (i).

(ii) Clearly, $P_1(R)$ contains $K_{3,3}$ if both of eR and $(1-e)R+1$ have more than 5 elements by (i). Hence, $P_1(R)$ is not a planar graph. □

Proposition 3.14. *The following assertions hold in $P(R)$:*

(i) *If $x \in R^*$ is a nilpotent element, then there exists an integer $k \geq 2$ such that x^i is adjacent to $1-x^{k-i}$ for every $1 \leq i \leq k-1$.*

(ii) *If $x \in R^*$ is a nilpotent element, then $N_{P(R)}(x)$ is a multiplicatively closed set of the form $I+1$ for an ideal I of R.*

(iii) *$Nil(R)\backslash\{0\}$ is an independent set.*

Proof. (i) If $x \in R^*$ is a nilpotent element, then there exists an integer $k \geq 2$ such that $x^k = 0$ and $x^i \neq 0$ for $1 \leq i \leq k-1$. Hence, $x^i(1 - x^{k-i}) = x^i$, which implies that x^i is adjacent to $1 - x^{k-i}$ for all $1 \leq i \leq k - i$.

(ii) Let $x \in R^*$ be a nilpotent element and k be the least positive integer such that $x^k = 0$. Then, it can be seen that $(1-x)(1+x+x^2+\ldots+x^{k-1}) = 1$ and so $1-x$ is a unit. Hence, by Theorem 3.1(ii), $N_{P(R)}(x) = Ann(x)+1$. Thus, by taking $I = Ann(x)$, $N_{P(R)}(x) = I + 1$, which is a multiplicatively closed set.

(iii) Let $x, y \in Nil(R)\backslash\{0\}$ and k and l be the least positive integers such that $x^k = 0 = y^l$.

Suppose, $\overline{xy} \in E$. Then, either $xy = x$ or $xy = y$.

If $xy = x$, then $x = xy = xy^2 = \ldots = xy^k$, a contradiction to the choice of x.

Similarly, $xy = y$ implies $y = x^l y$, a contradiction to the choice of y. Hence, $\overline{xy} \notin E$.

□

Example 3.2. In $R = \frac{\mathbb{Z}_2[x]}{(x^3)}, Nil(R)\backslash\{0\} = \{[x], [x^2], [x^2 + x]\}$, which is an independent set.

Remark 3.4. If R is a domainlike ring, then every zero-divisor is a nilpotent and hence the set of nonzero zero-divisors in R is independent.

Proposition 3.15. *If R is not a domain, then $P_1(R)$ is bipartite when R has any one of the following equivalent conditions:*

(i) *Every nonunit is a nilpotent.*

(ii) *R has a unique prime ideal.*

(iii) $\frac{R}{Nil(R)}$ *is a field.*

Proof. Suppose that every nonunit in R is a nilpotent. Then, $R^*\backslash\{1\} = (Nil(R)\backslash\{0\}) \cup (U(R)\backslash\{1\})$, in which $Nil(R)\backslash\{0\}$ and $U(R)\backslash\{1\}$ are independet sets. Hence, any edge $\overline{xy}$ with $x, y \in R^*\backslash\{1\}$ has one end in $Nil(R)\backslash\{1\}$ and the other end in $U(R)\backslash\{1\}$. Thus, $Nil(R)\backslash\{1\}$ and $U(R)\backslash\{1\}$ form a bipartition for $P_1(R)$, as required.

As it is known that (i) ⇔ (ii) ⇔ (iii), the proposition follows. □

Proposition 3.16. *If R is a ring which is not domain, then $P_1(R)$ is bipartite when R has any one of the following equivalent conditions:*

(i) *R is presimplifiable.*

(ii) $ZD(R) \subseteq J(R)$.

(iii) $ZD(R) \subseteq \{1 - u | u \in U(R)\}$.

Proof. By Lemma 2.2, (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii).

Suppose that R is presimplifiable.

Let $\overline{xy}$ be any edge with $x, y \in R^*\backslash\{1\}$. Then, $xy = x$ or $xy = y$. Now, consider the following cases:

(i) $x, y \in U(R)\backslash\{1\}$ (ii) $x, y \in W^*(R)$ (iii) $x \in U(R)\backslash\{1\}$ and $y \in W^*(R)$.

Since $U(R)\backslash\{1\}$ is independent case (i) is not possible. Also, since R is presimplifiable and x, y are nonzero elements, if $xy = x$, then $y \in U(R)$. Similarly, if $xy = y$, then $x \in U(R)$, which shows that case (ii) is also not possible.

Hence, the only possible choice is case (iii). That is, $x \in U(R)\backslash\{1\}$, $y \in W^*(R)$. Thus, $U(R)\backslash\{1\}$ and $W^*(R)$ form a bipartition for $P_1(R)$, as desired. □

Corollary 3.4. *If R is a local ring, which is not a domain, then $P_1(R)$ is bipartite.*

Proof. As R is local, it is presimplifiable and hence the proof follows from Proposition 3.16. □

Proposition 3.17. *Let R be a local ring, which is not a domain.*

If $x, y \in R^*\backslash\{1\}$ and $Ann(x) \cap Ann(y) \neq \{0\}$, then there exists a path $x - u - y$ with $u \in U(R)\backslash\{1\}$.

Proof. Since R is local, it has a unique maximal ideal $\mathcal{M}$, say.

Let $x, y \in R^*\backslash\{1\}$ and $t(\neq 0) \in Ann(x) \cap Ann(y)$. Then, $tx = ty = 0$, which implies $(1 - t)x = x$ and $(1 - t)y = y$.

Hence, as $1 - t \in R^*\backslash\{1, x, y\}$, $x - (1 - t) - y$ is a path between x and y. Now, it is claimed that $1 - t$ is a unit. Suppose $1 - t$ is not a unit. Then, it must be in a maximal ideal. Now, both $t, 1 - t \in \mathcal{M}$, which is closed under addition.

Hence, $1 \in \mathcal{M}$, showing that $\mathcal{M} = R$, a contradiction to the fact that $\mathcal{M}$ is a proper ideal. Thus, the claim is proved. □

Proposition 3.18. *Let R be a local ring, which is not a domain, and R has ascending chain condition(ACC) on ideals of the form $Ann(x)$, $x \in R$. Then, the following assertions hold:*

- *(i) $P(R)$ contains cycles of lengths j, $3 \leq j \leq 2k + 1$, where k is the number of nontrivial annihilators in R.*
- *(ii) $P(R)$ is weakly pancyclic.*

Proof. Since the ideals $Ann(x), x \in R$ satisfy ACC, there exist $x_1, \ldots, x_k, x_{k+1} \ldots$ in R such that $Ann(x_1) \subset Ann(x_2) \subset \ldots \subset Ann(x_k) = Ann(x_{k+1}) = \ldots$ for some positive integer k.

(i) Let $y_i \in Ann(x_i)\backslash Ann(x_{i-1})$ for every $1 \leq i \leq k$. Then, $x_i y_i = x_{i+1} y_i = 0$, which implies $x_i(1 - y_i) = x_i$ and $x_{i+1}(1 - y_i) = x_{i+1}$, where $1 - y_i \in R^*\backslash\{1, x_i, x_{i+1}\}$. Hence, $x_i - (1 - y_i) - x_{i+1}$ is a path as in Proposition 3.17.

Thus, each one of the following is a cycle: $1-x_1-(1-y_1)-1$, (a cycle of length 3), $1-x_1-(1-y_1)-x_2-1$, (a cycle of length 4), $1-x_1-(1-y_1)-x_2-(1-y_2)-1$, (a cycle of length 5) and so on, proving (i).

(ii) $P(R)$ is weakly pancyclic by (i) and the definition of weakly pancyclic graph. □

The proof of the following proposition is omitted as it is trivial from the natural product defined in a quotient ring.

Proposition 3.19. *Let I be a nontrivial ideal in R. If x, y are adjacent in $P(R)$, then $x+I$ and $y+I$ are adjacent in $P(\frac{R}{I})$, where $\frac{R}{I}$ denotes the quotient ring.*

The following proposition shows that the projection graphs of finite isomorphic rings are isomorphic.

Proposition 3.20. *Let R and S be finite rings such that $R \cong S$. Then, $P(R) \cong P(S)$.*

Proof. By the hypothesis, there exists a one-one, onto ring homomorphism ϕ between R and S. Let ϕ^* be the restriction of ϕ to R^*. Then, ϕ^* is a one-one, onto function. As $|R^*| = |S^*|$, $|V(P(R))| = |V(P(S))|$, where $V(P(R))$ and $V(P(S))$ denote the sets of vertices of R and S respectively.

Let $x, y \in V(P(R))$ such that x and y are adjacent. Then, $xy = x$ or $xy = y$. If $xy = x$, then $\phi^*(xy) = \phi^*(x)$, which implies $\phi^*(x)\phi^*(y) = \phi^*(x)$. Therefore, $\phi^*(x)$ is adjacent to $\phi^*(y)$ in $P(S)$.

A similar argument holds for the case, where $xy = y$, proving that ϕ^* preserves the adjacency between vertices. Thus, $P(R) \cong P(S)$. □

Example 3.3. Let $R = \frac{\mathbb{Z}_2[x]}{(x^2)}$; $S = \frac{\mathbb{Z}_2[x]}{(x^2+1)}$. Then, $R \cong S$ and $P(R) \cong P(S)$.

Remark 3.5. The converse of the above proposition need not be true. For, if $R = \mathbb{Z}_4$ and $S = \frac{\mathbb{Z}_2[x]}{(x^2)}$, then $P(R) \cong P(S)$ and $R \ncong S$.

Proposition 3.21. *P(R) is not complete in each of the following cases:*

(*i*) *R has nontrivial idempotent elements.*

(*ii*) $|(U(R)| \geq 3$.

Proof. (i) If R has nontrivial idempotent element e, then $P(R)$ is not complete since e and $1-e$ are not adjacent.

(ii) If there are more than three units, then $P(R)$ is not complete since $U(R)\backslash\{1\}$ is independent. □

Proposition 3.22. *Let R be finite. Then, $P(R)$ is complete if and only if either $R \cong \mathbb{Z}_3$ or $R \cong \mathbb{Z}_4$.*

Proof. It is known that $P(\mathbb{Z}_3)$ and $P(\mathbb{Z}_4)$ are complete. Hence, if $R \cong \mathbb{Z}_3$ or $R \cong \mathbb{Z}_4$, then $P(R)$ is complete by Proposition 3.20.

Conversely, suppose that $P(R)$ is complete. Then, $|U(R)| \leq 2$ and R has no nontrivial idempotents by the above proposition.

Let $R = \{0, 1, u\} \cup ZD(R)$, where $u \neq 1$ is a unit. Then, it is claimed that $|ZD(R)| \leq 1$.

Suppose $x, y \in ZD(R)$ be distinct nonzero zero-divisors. Then, $xy = x$ or $xy = y$ by the hypothesis.

If $xy = x$, then $(x+u)y = xy+uy = x+y$ since $xu = x$ by the completeness. But, $(x+u)y = x+u$ or $(x+u)y = y$ since $P(R)$ is complete.

If $(x+u)y = x+u$, then from the previous step, $x+u = x+y$ which implies $y = u$, a contradiction to the choice of y. Therefore, $(x+u)y = y$, which implies $x = 0$.

By a similar argument, it can be shown that if $xy = y$, then $y = 0$. Hence, there can be at the most one nonzero zero-divisor. Thus, $|R| \leq 4$.

If $|R| = 3$, then $R \cong \mathbb{Z}_3$.

If $|R| = 4$, then $R \cong \mathbb{Z}_4$, since R is the unital commutative ring of cardinality 4 with no nontrivial idempotents, which completes the proof. □

Proposition 3.23. *If $P(R)$ is not a star, then there exists $x \in R^*\backslash\{1\}$ such that either xR or $(1-x)R$ has a nonzero annihilating ideal.*

Proof. If $P(R)$ is not a star, then there exists $\overline{xy} \in E$, for some $x, y \in R^*\backslash\{1\}$, which implies that either $y \in (Ann(x)+1)\backslash\{1\}$ or $y \in Ann(1-x)\backslash\{0\}$ by Theorem 3.1.

If $y \in (Ann(x)+1)\backslash\{1\}$, then there exists a nozero $z \in Ann(x)$ such that $y = z+1$ and $(y-1)xr = zxr = 0$ for every r in R, showing that $Ann(xR) \neq \{0\}$.

If $y \in Ann(1-x)\backslash\{0\}$, then $(1-x)y = 0$ and therefore $(1-x)yr = 0$ for every $r \in R$. Hence, $Ann((1-x)R) \neq \{0\}$. This completes the proof. □

Proposition 3.24. *If $x, y \in R^*$ are adjacent, then either $xR \subseteq yR$ or $yR \subseteq xR$.*

Proof. Suppose $x, y \in R^*$ and $\overline{xy} \in E$. Then, either $xy = x$ or $xy = y$.

Consider the following possible cases:

(i) $x, y \in U(R)$ (ii) $x \in U(R)$ and $y \notin U(R)$ (iii) $x, y \notin U(R)$.

Case (i) If $x, y \in U(R)$, then $xR = yR = R$.

Case (ii) If $x \in U(R)$ and $y \notin U(R)$, then $xR = R$ and so $yR \subseteq xR$.

Case (iii) Let $x, y \notin U(R)$. If $xy = x$, then $z \in xR$ implies $z = xr$ for some $r \in R$. Therefore, $z = (xy)r = y(xr) \in yR$ and so $xR \subseteq yR$.

Similarly, if $xy = y$, then it can be shown that $yR \subseteq xR$, which completes the proof. □

4. Projection graphs of $\mathbb{Z}_n$

In this section, $\mathbb{Z}_n$, $n \geq 3$, is considered and $P(\mathbb{Z}_n)$ is studied. It is observed that the vertex set V of $P(\mathbb{Z}_n)$ is given by $V = \mathbb{Z}_n^* = U(\mathbb{Z}_n) \cup (ZD(\mathbb{Z}_n)\backslash\{0\})$ and $|V| = n - 1$.

Proposition 4.1. *Let $n \geq 3$. Then:*

(*i*) *$P(\mathbb{Z}_n)$ is complete if and only if $n = 3, 4$.*

(*ii*) *$P(\mathbb{Z}_n)$ is a star if and only if n is a prime.*

Proof. (i) The proof follows from Proposition 3.22.

(ii) $\mathbb{Z}_n$ has no zero-divisors if and only if n is a prime. Hence, (ii) follows from Corollary 3.1. □

Proposition 4.2. $diam(P(\mathbb{Z}_n)) = \begin{cases} 1, & if\, n = 3, 4 \\ 2, & otherwise. \end{cases}$

Proof. By Proposition 4.1(i), it is clear that the diameter of $P(\mathbb{Z}_n)$ is 1 if and only if $n = 3, 4$. Hence, by Proposition 3.1, the diameter of $P(\mathbb{Z}_n)$ is 2 if $n \geq 5$. □

Proposition 4.3. $girth(P(\mathbb{Z}_n)) = \begin{cases} \infty, & if\, n\ \ is\ \ prime \\ 3, & otherwise. \end{cases}$

Proof. By Proposition 4.1(ii), it is clear that the girth of $P(\mathbb{Z}_n)$ is ∞ if and only if n is a prime. Hence, if n is not a prime, then the girth of $P(\mathbb{Z}_n)$ is 3 by Proposition 3.8. □

Remark 4.1. Note that, $\mathbb{Z}_n$ has nontrivial idempotent, if and only if $x^2 \equiv x \ mod\ n$ for some $1 < x < n$ if and only if n divides $x(1 - x)$ if and only if n has at least two nontrivial divisors.

Proposition 4.4. *Let $x, y \in \mathbb{Z}_n^*$. Then:*

(*i*) *$Ann(x) = Ann(c)$ if $(x, n) = c$.*

(*ii*) *$Ann(x) = \{0\}$ if and only if $x \in U(\mathbb{Z}_n)$.*

(*iii*) *$Ann(x) = Ann(y)$ if and only if $(x, n) = (y, n)$.*

(*iv*) *If $(x, n) = x$, then $Ann(x) = k\mathbb{Z}_n$, where $k = \frac{n}{x}$ and $|k\mathbb{Z}_n| = x$.*

(*v*) *$Ann(e) = (1 - e)\mathbb{Z}_n$ and $Ann(1 - e) = e\mathbb{Z}_n$, where e is a nontrivial idempotent.*

Proof. (i) Suppose $(x,n)=c$. Then, there exist integers k and l,m such that $x=kc$ and $c=lx+mn$.

Now, $Ann(x)\subseteq Ann(c)$. For, $t\in Ann(x)\Rightarrow tx=0\Rightarrow tlx=0\Rightarrow tc=0\Rightarrow t\in Ann(c)$.

Also, $Ann(c)\subseteq Ann(x)$, since $t\in Ann(c)\Rightarrow tc=0\Rightarrow tkc=0\Rightarrow tx=0\Rightarrow t\in Ann(x)$, which proves (i).

(ii) If $x\in U(\mathbb{Z}_n)$, then $Ann(x)=\{t\in\mathbb{Z}_n|tx=0\}=\{0\}$. Conversely, suppose $x\notin U(\mathbb{Z}_n)$. If $x=0$, then $Ann(x)=\mathbb{Z}_n$.

If $x\neq 0$, then there exists $y\in\mathbb{Z}_n^*$ such that $xy=0$, which implies $Ann(x)\neq\{0\}$.

(iii) The proof of (iii) follows from (i).

(iv) As $Ann(x)$ is an ideal and every ideal in $\mathbb{Z}_n$ is principal, $Ann(x)=a\mathbb{Z}_n$ for some $a\in\mathbb{Z}_n$.

If $(x,n)=x$, then there exists an integer k such that $kx=n$, which implies $k\in Ann(x)$ and hence $k\mathbb{Z}_n\subseteq Ann(x)$. Also, $t\in Ann(x)\Rightarrow tx=0\Rightarrow tx=ln$, for some $l\in\mathbb{Z}_n\Rightarrow t=kl\in k\mathbb{Z}_n$. Hence, $Ann(x)\subseteq k\mathbb{Z}_n$ and $|k\mathbb{Z}_n|=x$, proving (iv).

(v) Assertion (v) follows from the proof of Proposition 3.12 (i). □

Proposition 4.5. *Let s,t be two distinct factors of n. Then:*

(i) $Ann(s)\neq Ann(t)$

(ii) $Ann(s)\subset Ann(t)$*, whenever* $s\mid t$.

(iii) $Ann(s)\cap Ann(t)=\{0\}$ *if and only if* $(s,t)=1$.

Proof. (i) Note that, $(s,n)=s$ and $(t,n)=t$. Therefore, from Proposition 4.4(iv), $Ann(s)=k\mathbb{Z}_n$ and $Ann(t)=l\mathbb{Z}_n$, where $k=\frac{n}{s}$, $l=\frac{n}{t}$. Hence, $Ann(s)\neq Ann(t)$, since $k\neq l$.

(ii) If $s\mid t$, then $sk=t$ for some integer k and therefore $r\in Ann(s)\Rightarrow rs=0\Rightarrow krs=0\Rightarrow tr=0\Rightarrow r\in Ann(t)$. Hence, $Ann(s)\subset Ann(t)$, since $|Ann(s)|=s<t=|Ann(t)|$.

(iii) Suppose $(s,t)=1$. Then, there exist integers k and l such that $ks+lt=1$. Hence, if $r\in Ann(s)\cap Ann(t)$, then $r=rks+rlt$ and so $r=0$.

Conversely, suppose $(s,t)=r\neq 1$. Then, $r\mid s$ and $r\mid t$ and hence by (ii), $Ann(s)\cap Ann(t)\supset Ann(r)\neq\{0\}$. □

Definition 4.1. *Define a relation $\sim$ on $\mathbb{Z}_n^*$ by $x\sim y$ if and only if $Ann(x)=Ann(y)$ for every $x,y\in\mathbb{Z}_n^*$.*

Remark 4.2. The relation $\sim$ defined above on $\mathbb{Z}_n^*$ is an equivalence relation. Hence, if $x\in\mathbb{Z}_n^*$ and $[x]_\sim$ denotes the equivalence class of x, then by Proposition 4.4(iii), $[x]_\sim=\{y\in\mathbb{Z}_n^*|Ann(y)=Ann(x)\}=\{y\in\mathbb{Z}_n^*|(y,n)=(x,n)\}$.

Proposition 4.6. *Using the above notations, the following statements are true:*

(i) $[1]_\sim = U(\mathbb{Z}_n)$; $|[1]_\sim| = \phi(n)$.

(ii) $[1]_\sim \backslash \{1\}$ *is an independent set of size* $\phi(n) - 1$.

(iii) *If* $d \mid n$, *then* $[d]_\sim = \{y \in \mathbb{Z}_n^* | (y, n) = d\}$.

(iv) $ZD(\mathbb{Z}_n)\backslash\{0\} = \cup_{(x,n)\neq 1}[x]_\sim = \cup_{d|n, d\neq 1}[d]_\sim$.

Proof. (i) By using Remark 4.2, $[1]_\sim = \{y \in \mathbb{Z}_n^* | Ann(y) = \{0\}\} = \{y \in \mathbb{Z}_n^* | (y, n) = 1\} = U(\mathbb{Z}_n)$ and hence $|[1]_\sim| = \phi(n)$.

(ii) The proof follows from Corollary 3.3 using (i).

(iii) Let $d \mid n$. Then, $(d, n) = d$ and hence $[d]_\sim = \{y \in \mathbb{Z}_n^* | Ann(y) = Ann(d)\} = \{y \in \mathbb{Z}_n^* | (y, n) = d\}$.

(iv) From Remark 4.2, $\mathbb{Z}_n^* = [1]_\sim \cup (\cup_{x \in \mathbb{Z}_n^* \backslash \{1\}}[x]_\sim)$ and hence $ZD(\mathbb{Z}_n)\backslash\{0\} = \cup_{(x,n)\neq 1}[x]_\sim = \cup_{d|n,d\neq 1}[d]_\sim$, by (iii). □

Proposition 4.7. *Let* $n = p^k$, *for some* $k \geq 2$. *Then, the following assertions hold:*

(i) $ZD(\mathbb{Z}_n)\backslash\{0\}$ *is an independent set.*

(ii) $P_1(\mathbb{Z}_n)$ *is bipartite.*

(iii) $P(\mathbb{Z}_n)$ *is weakly pancyclic.*

Proof. If $n = p^k$, then $ZD(\mathbb{Z}_n)\backslash\{0\} = \cup_{i=1}^{k-1}[p^i]_\sim$, where $[p^i]_\sim = \{y \in \mathbb{Z}_n^* | (y, n) = p^i\}$, by Proposition 4.6(iv).

(i) It is claimed that $ZD(\mathbb{Z}_n) = Nil(\mathbb{Z}_n)$. For, if $x \in ZD(\mathbb{Z}_n)\backslash\{0\}$, then $x \in [p^i]_\sim$, for some i, which implies $x = tp^i$ for some integer t. Hence, $x^{k-i} = 0$ and thus x is a nilpotent element, proving the claim.

Hence, $ZD(\mathbb{Z}_n)\backslash\{0\} = Nil(\mathbb{Z}_n)\backslash\{0\}$, which is independent by 3.14(iii).

(ii) From the proof of (i), it is noted that the set of all nonunits is equal to $Nil(\mathbb{Z}_n)$, which is the unique maximal ideal. Hence, $\mathbb{Z}_n$ is local and thus $P_1(\mathbb{Z}_n)$ is bipartite by Corollary 3.4.

(iii) It is claimed that the ideals of the form $Ann(x)$, $x \in \mathbb{Z}_n$, have ACC.

If $x \in \mathbb{Z}_n^*$, then either $x \in U(\mathbb{Z}_n)$ or $x \in [p^i]_\sim = \{t \in \mathbb{Z}_n^* | Ann(t) = Ann(p^i)\}$, for some i. If $x \in U(\mathbb{Z}_n)$, then $Ann(x) = \{0\}$.

Also, by Proposition 4.5 (ii), $Ann(p) \subset Ann(p^2) \subset \ldots \subset Ann(p^{k-1})$, proving the claim. Thus, $P(\mathbb{Z}_n)$ is weakly pancyclic by Proposition 3.18. □

Proposition 4.8. *If* $n = 2^k$, *for some* $k \geq 2$, *then the following assertions hold:*

(i) $|U(\mathbb{Z}_n)\backslash\{1\}| = |ZD(\mathbb{Z}_n)\backslash\{0\}| = \frac{n}{2} - 1$, $U(\mathbb{Z}_n) = [1]_\sim = \{2j + 1 \in \mathbb{Z}_n^* | j \in \mathbb{Z}_n\}$.

(ii) *If* $x \in [2^i]_\sim$ *and* $x + u = 1$, *then* $deg(x) = deg(u) = 2^i$, *for* $1 \leq i \leq k - 1$.

(iii) *The degree sequence is given by* $(2^{(a_1)}, 2^{2^{(a_2)}}, \ldots, 2^{k-1^{(a_{k-1})}}, n - 2^{(1)})$, *where* (a_i) *denotes the multiplicity and* $(a_i) = 2|[2^i]_\sim|$ *for* $1 \leq i \leq k - 1$.

Proof. (i) $|U(\mathbb{Z}_n)| = \phi(n) = 2^k - 2^{k-1} = n - \frac{n}{2} = \frac{n}{2}$.

Hence, $|U(\mathbb{Z}_n)\backslash\{1\}| = |ZD(\mathbb{Z}_n)\backslash\{0\}| = \frac{n}{2} - 1$. Also, $U(\mathbb{Z}_n) = [1]_\sim = \{y \in \mathbb{Z}_n^* | (y, 2^k) = 1\} = \{2j + 1 \in \mathbb{Z}_n^* | j \in \mathbb{Z}_n\}$.

(ii) Let $x \in [2^i]_\sim$ and $x + u = 1$. Then, $u = 1 - x \in U(\mathbb{Z}_n)\backslash\{1\}$ since x is nilpotent from 4.7(i). Therefore, by Theorem 3.1(i), $N_{P(R)}(u) = \{1\} \cup (Ann(1-u)\backslash\{0\}) = \{1\} \cup (Ann(x)\backslash\{0\}) = \{1\} \cup (Ann(2^i)\backslash\{0\}) = \{1\} \cup (2^{k-i}\mathbb{Z}_n\backslash\{0\})$ and so $|N_{P(R)}(u)| = 2^i$. Thus, $deg(u) = 2^i$. Also, $N_{P(\mathbb{Z}_n)}(x) = Ann(2^i) + 1 = 2^{k-i}\mathbb{Z}_n + 1$ and so $|N_{P(\mathbb{Z}_n)}(x)| = |2^{k-i}\mathbb{Z}_n| = 2^i$. Thus, $deg(x) = 2^i$. From the above discussion, it is clear that $deg(u) = deg(x) = 2^i$.

(iii) Note that, $\mathbb{Z}_n^* = \{1\} \cup (U(\mathbb{Z}_n)\backslash\{1\}) \cup (ZD(\mathbb{Z}_n)\backslash\{0\})$, where $ZD(\mathbb{Z}_n)\backslash\{0\} = \cup_{i=1}^{k-1}[2^i]_\sim$.

As the degree of 1 is $n-2$ and for every $x \in ZD(\mathbb{Z}_n)\backslash\{0\}$, there is a unique $u \in U(\mathbb{Z}_n)\backslash\{1\}$ such that $x + u = 1$, (iii) follows from (ii). □

Proposition 4.7 and Proposition 4.8 are illustrated in Figure 7 and Table 1 for $n = 32$.

Illustration 4.1. Consider $\mathbb{Z}_{32}$, where $ZD(\mathbb{Z}_n)\backslash\{0\} = \cup_{i=1}^{4}[2^i]_\sim$ and $U(\mathbb{Z}_n) = [1]_\sim = \{y \in \mathbb{Z}_n^* | (y, 2^5) = 1\} = \{1, 3, 5, \ldots, 31\}$.

i	$\{x \| x \in [2^i]_\sim\}$	$Ann(2^i) = k\mathbb{Z}_n, k = \frac{n}{2^i}$	$u = 1 - x$	$deg(x) = deg(u)$
1	$\{2, 6, \ldots, 30\}$	$\{0, 16\}$	$\{31, 27, \ldots, 3\}$	2
2	$\{4, 12, 20, 28\}$	$\{0, 8, 16, 24\}$	{29,21,13,5}	4
3	$\{8, 24\}$	$\{0.4, 8, \ldots, 28\}$	$\{25, 9\}$	8
4	$\{16\}$	$\{0.2, 4, \ldots, 30\}$	$\{17\}$	16

Table 1: $\mathbb{Z}_{32}$

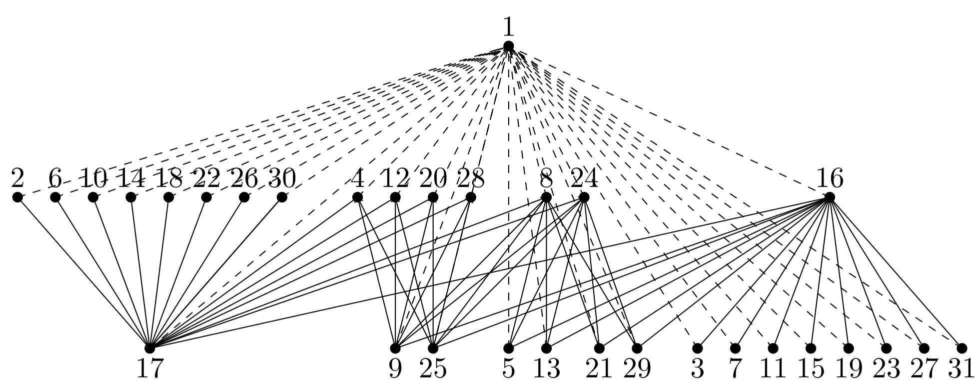

Figure 7: $P(\mathbb{Z}_{32})$

Proposition 4.9. *Let $n = 2q$. Then, the following assertions hold:*

(*i*) $ZD(\mathbb{Z}_n) = \{2, 4, \ldots, 2q-2\} \cup \{q\}$, $U(\mathbb{Z}_n) = \{1, 3, 5, \ldots, 2q-1\} \backslash \{q\}$.

(*ii*) $q, q+1$ *are the nontrivial idempotents.*

(*iii*) $N_{P(\mathbb{Z}_n)}(q) = \{1, 3, \ldots, 2q-1\} \backslash \{q\}$, $N_{P(\mathbb{Z}_n)}(q+1) = \{2, 4, \ldots, 2q-2\} \backslash \{q+1\}$.

(*iv*) $N_{P(\mathbb{Z}_n)}(x) = \{1, q\}$ *if* $x \in [2]_\sim \backslash \{q+1\}$, $N_{P(\mathbb{Z}_n)}(x) = \{1, q+1\}$ *if* $x \in U(\mathbb{Z}_n) \backslash \{1\}$.

(*v*) $deg(x) = \begin{cases} n-2, & \text{if } x = 1 \\ q-1, & \text{if } x = q, q+1 \\ 2, & \text{otherwise.} \end{cases}$

(*vi*) *The number of triangles in* $P(\mathbb{Z}_n)$ *is* $2q-4$.

(*vii*) $P(\mathbb{Z}_n)$ *is the union of two copies of triangular book*

(*viii*) $|E| = 4q-6$.

(*ix*) $P(\mathbb{Z}_n)$ *is planar.*

(*x*) $P_1(\mathbb{Z}_n)$ *is disconnected.*

Proof. (i) By Proposition 4.6(iv), $ZD(\mathbb{Z}_n) \backslash \{0\} = [2]_\sim \cup [q]_\sim$, where $[2]_\sim = \{y \in \mathbb{Z}_n^* | (y, n) = 2\} = \{2, 4, \ldots, 2q-2\}$ and $[q]_\sim = \{y \in \mathbb{Z}_n^* | (y, n) = q\} = \{q\}$. Hence, $U(\mathbb{Z}_n) = \mathbb{Z}_n \backslash ZD(\mathbb{Z}_n) = \{1, 3, 5, \ldots, 2q-1\} \backslash \{q\}$.

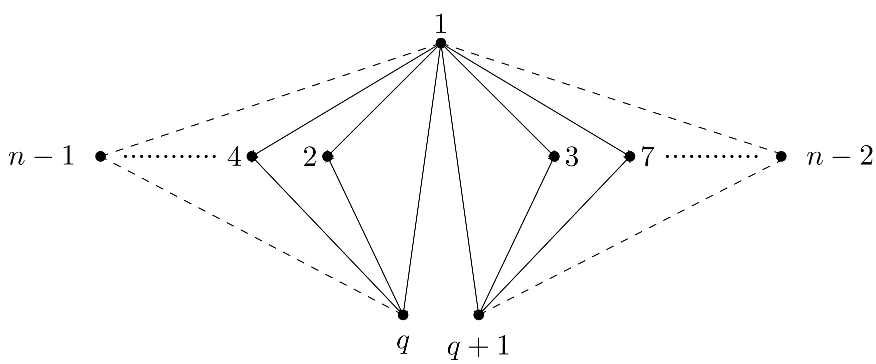

Figure 8: $P(\mathbb{Z}_{2q})$

(ii) Since q is odd, $q(q+1) \equiv 0 \ mod\ 2q$ and hence q and $q+1$ are the idempotents.

(iii) Note that, $1 - q = q + 1$ and as in the proof of Proposition 3.12, $Ann(1-q) = Ann(q+1) = q\mathbb{Z}_n = \{0, q\}$. Also, $Ann(q) = 2\mathbb{Z}_n$ by Proposition 4.4(iv). Hence, by using Theorem 3.1(iii), $N_{P(\mathbb{Z}_n)}(q) = ((Ann(q) + 1) \cup Ann(1-q))\backslash\{0, q\} = ((2\mathbb{Z}_n + 1) \cup q\mathbb{Z}_n)\backslash\{0, q\} = \{1, 3, 5, \ldots, 2q-1\}\backslash\{q\}$.

Similarly, $N_{P(\mathbb{Z}_n)}(q+1) = ((Ann(q+1)+1) \cup Ann(q))\backslash\{0, q+1\} = ((q\mathbb{Z}_n + 1) \cup 2\mathbb{Z}_n)\backslash\{0, q+1\} = \{1, 2, 4, \ldots, 2q-2\}\backslash\{q+1\}$.

(iv) If $x \in [2]_\sim\backslash\{q+1\}$, then $N_{P(\mathbb{Z}_n)}(x) = ((Ann(x)+1) \cup Ann(1-x))\backslash\{0\}$ by Theorem 3.1(ii) $= (Ann(2)+1)$ by the definition of $\sim = q\mathbb{Z}_n + 1$ by Proposition 3.1(iv) $= \{1, q+1\}$. Also, since $|q\mathbb{Z}_n| = 2$, by Proposition 3.6, either $\overline{qx} \in E$ or $\overline{(1-q)x} \in E$, for every $x \in \mathbb{Z}_n^*\backslash\{1, q, 1-q\}$. But, $\mathbb{Z}_n^*\backslash\{1, q, 1-q\} = ([2]_\sim\backslash\{q+1\}) \cup ([1]_\sim\backslash\{1\})$, where $[1]_\sim\backslash\{1\} = U(\mathbb{Z}_n)\backslash\{1\}$.

Hence, for $x \in U(\mathbb{Z}_n)\backslash\{1\}$, $N_{P(\mathbb{Z}_n)}(x) = \{1, q\}$.

(v) The proof of (v) follows from (iii) and (iv).

(vi) From (iv), it can be seen that $1 - x - (q+1) - 1$ form triangles, which share $\overline{(q+1)1}$ in common for every $x \in U(\mathbb{Z}_n)\backslash\{1\}$.

Similarly, $1 - q - y - 1$ form triangles, which share $\overline{1q}$ in common for every $y \in [2]_\sim\backslash\{q+1\}$, as drawn in Figure 8.

Hence, the number of triangles $= |U(\mathbb{Z}_n)\backslash\{1\}| + |[2]_\sim\backslash\{q+1\}| = 2(q-1) = 2q - 4$. (vii) From Figure, it is clear that $P(\mathbb{Z}_n)$ is the union of two copies of triangular book.

(viii) As each triangle in one page of the triangular book counts two edges excluding the common edge, $|E| = (2(2q-4)) + 2 = 4q - 6$.

(ix) Obviously, $P(\mathbb{Z}_n)$ is planar.

(x) $P(\mathbb{Z}_n)$ is disconnected if 1 is removed. Hence, $P_1(\mathbb{Z}_n)$ is disconnected. □

5. Projection graphs of near-rings

In this section, the projection graph $P(N)$ of a near-ring N is defined as the same as that of a ring and the properties of $P(N)$ are discussed. Throughout, this section N denotes a right near-ring with at least 3 elements.

Proposition 5.1. *If N is a near-field, then $P(N)$ is a star.*

Proof. Let N be a near-field and 1 be the multiplicative identity. Then, $\overline{x1} \in E$ since the equation $x1 = x$ holds in N, for every $x \in N^*$. If $\overline{xy} \in E$, then either $xy = x$ or $xy = y$, which implies $x = 1$ or $y = 1$ as every nonzero element in N has multiplicative inverse. Hence, $E = \{\overline{x1} \mid x \in N^*\}$. Thus, $P(N)$ is a star. □

Proposition 5.2. *If N is a near-ring, then the following hold in $P(N)$:*

(*i*) *Every nonzero element in N is adjacent to every element in its constant part.*

(*ii*) *The subgraph induced on the constant part forms a clique.*

Proof. The proof follows from the definition of constant part of N. □

Corollary 5.1. *If N is a constant near-ring, then $P(N)$ is complete.*

Proof. If N is a constant near-ring, then $N = N_c$ and hence $P(N)$ is complete, by Proposition 5.2(ii). □

Remark 5.1. The converse of the above proposition need not be true. For, consider $N = (D_8, +, \cdot)$, where $(D_8, +)$ is the dihedral group and $\cdot$ is defined by $x \cdot y = \begin{cases} x, & \text{if } y \neq 0 \\ 0, & \text{if } y = 0. \end{cases}$ Clearly, N is a near-ring, which is not constant and $P(N)$ is complete.

Theorem 5.1. *If N is an almost trivial near-ring, then $P(N)$ is complete.*

Proof. Suppose N is an almost trivial near-ring, then $xy = \begin{cases} x, & \text{if } y \notin N_c \\ 0, & \text{if } y \in N_c \end{cases}$, for every x, $y \in N$.

Let $x, y \in N^*$. Then, by Pierce decomposition, $x = x_0 + x_c$ and $y = y_0 + y_c$, where x_0 and y_0 are the zero-symmetric parts and x_c and y_c are the constant parts of x and y, respectively.

Now, consider the following possible cases:

(i) $x, y \in N_0$ (ii) $x, y \in N_c$ (iii) $x \in N_0$ and $y \in N_c$ (iv) $x, y \notin N_0 \cup N_c$.

It is claimed that $\overline{xy} \in E$. For,

(i) If $x, y \in N_0$, then $x = x_0$ and $x_c = 0$. Therefore, $xy = x$.

(ii) If $x, y \in N_c$, then $x = x_c$ and $x_0 = 0$. Therefore, $xy = x_c = x$.

(iii) If $x \in N_0$ and $y \in N_c$, then $y = y_c$ and $y_0 = 0$. So, $yx = y$.

(iv) If $x, y \notin N_0 \cup N_c$, then $x = x_0 + x_c$, $y = y_0 + y_c$, where $x_0, y_0 \in N_0 \backslash \{0\}$ and x_c, $y_c \in N_c \backslash \{0\}$. Hence, $xy = (x_0 + x_c)(y_0 + y_c) = x_0(y_0 + y_c) + x_c(y_0 + y_c) = x_0 + x_c = x$.

Hence, the claim is proved. □

Proposition 5.3. *If N is a Boolean near-ring, which is subdirectly irreducible, then $P(N)$ is complete.*

Proof. The proof follows from Lemma 2.3 and Theorem 5.1. □

6. Conclusion

In this paper, the projection graphs $P(R)$ of a ring R and $P(N)$ of a near-ring N are introduced and their graph properties are studied. A method of finding adjacent vertices in $P(R)$, using annihilators is provided. Certain algebraic properties of rings are observed through their projection graphs. This paper may be extended by considering substructures of rings and near-rings and more algebraic properties can be obtained through their projection graphs.

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