Chain dot product graph of a commutative ring

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Abstract. In this article, we generalized the concepts of total dot product graph (the chain zero-divisor dot product), which were investigated in 2015 by A. Badawi, to what we call chain total dot product graph $CTD(R)$ (the chain zero-divisor dot product graph $CZD(R)$). We give some basic graph properties for the graphs $CTD(R)$ and $CZD(R)$ such as connectedness, diameter and the girth.

Keywords: zero-divisor graph, dot product zero-divisor graph, diameter, girth.

1. Introduction

Graph theory has recently become a significant tool for studying the structure of rings, in addition to being a beautiful and sophisticated theory in its own right. As a result, several writers explore the relationship between rings and graph theory. see for example [3, 5, 4].

Throughout this article, let A be a commutative ring with nonzero identity 1, for the natural number n, let $R = A \times A \times \cdots \times A(n - times)$. Badawi in [2] presented the total and the zero-divisor dot product graphs associated to the ring A, where the total dot product graph, denoted by $TD(R)$, is the graph with vertex set $R^* = R \backslash \{(0, 0, \cdots, 0)\}$, and two vertices x, y are adjacent if $x.y = 0 \in A$ (the normal dot product between x and y is zero). Also the zero-divisor dot product graph, denoted by $ZD(R)$, is the induced subgraph of the total dot product graph $TD(R)$ with vertex set $Z(R)^* = Z(R) \backslash \{(0, 0, \cdots, 0)\}$.

In this article, we generalized these concepts by developing the concept of the dot product. Let $A_1, A_2, \ldots, A_n$ be commutative rings with nonzero identity 1, such that $A_1 \subseteq A_2 \subseteq \ldots \subseteq A_n$. Let $R = A_1 \times A_2 \times \ldots \times A_n$, then the generalized dot product between $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$ is $x.y = x_1y_1 + x_2y_2 + \cdots + x_ny_n \in A_n$.

Now, we introduce our generalization. Let A be a commutative ring with nonzero identity 1, $R = A \times A[\alpha_1] \times A[\alpha_1, \alpha_2] \cdots \times A[\alpha_1, \alpha_2, \ldots \alpha_n]$, where $A[\alpha_1, \alpha_2, \ldots \alpha_k]$ is a ring with elements of the form $x = x_{k1} + x_{k2}\alpha_1 + x_{k3}\alpha_2 + \cdots + x_{kk}\alpha_k$ such that $\alpha_i\alpha_j = 0$ for $1 \leq i, j \leq k$, with the operations

Addition: $(x_{k1}+x_{k2}\alpha_1+x_{k3}\alpha_2+\cdots+x_{kk}\alpha_k)+(y_{k1}+y_{k2}\alpha_1+y_{k3}\alpha_2+\cdots+y_{kk}\alpha_k)=(x_{k1}+y_{k1})+(x_{k2}+y_{k2})\alpha_1+(x_{k3}+y_{k3})\alpha_2+\cdots+(x_{kk}+y_{kk})\alpha_k$, and

Multiplication: $(x_{k1}+x_{k2}\alpha_1+x_{k3}\alpha_2+\cdots+x_{kk}\alpha_k)(y_{k1}+y_{k2}\alpha_1+y_{k3}\alpha_2+\cdots+y_{kk}\alpha_k)=x_{k1}y_{k1}+(x_{k1}y_{k2}+x_{k2}y_{k1})\alpha_1+(x_{k1}y_{k3}+x_{k3}y_{k1})\alpha_2+\cdots+(x_{k1}y_{kk}+x_{kk}y_{k1})\alpha_k$.

The chain dot product graph, denoted by $CTD(R)$ is a graph with a vertex set $R^*=R\backslash\{(0,0,\cdots,0)\}$, and two vertices x,y are adjacent if $x.y=0\in A$ (the generalized dot product between x and y is 0). Similarly, as above, the chain zero-divisor dot product graph, denoted by $CZD(R)$, is the induced subgraph of the chain total dot product graph $CTD(R)$ with a vertex set $Z(R)^*=Z(R)\backslash\{(0,0,\cdots,0)\}$ (the nonzero zero-divisors of R).

For undefined notation or terminology consult [6] for graph theory and [7] for ring theory.

2. Some basic properties of $CTD(R)$ and $CZD(R)$

In this section, we will study some properties of $CTD(R)$ and $CZD(R)$, such as connectedness, diameter and girth.

We start by defining the $k-th$ neighborhood for the vertex x.

Definition 2.1. *Let G be a finite simple graph, and x be any vertex in G and let k be any nonnegative integer. Then, the $k-th$ neighborhood for the vertex x, denoted by $N^k(x)$, is defined as*

$$
\begin{aligned}
N^0(x) &= \{x\},\\
N^1(x) &= N(x), \textit{the usual neighborhood of } x.\\
&\ \vdots\\
\textit{for } k &\geq 1\\
N^k(x) &= \left\{ y\in V(G)\backslash\bigcup_{j=1}^{k-1}N^j(x) : z \quad \textit{is adjacent to } y\textit{, for any } z\in N^{k-1}(x)\right\}\\
&\quad \textit{where } V(G) \textit{ is the vertex set of the graph } G.
\end{aligned}
$$

The definition of $N^k(x)$ makes it obvious that there is a path of length k, between the vertex x and any vertex in $N^k(x)$.

Lemma 2.1. *Let G be a finite simple graph, and x,y be two distinct vertices. Then, there is a path between x and y if and only if there exist two non negative integers n,m such that $N^n(x)$ and $N^m(x)$ are not disjoint sets.*

Proof. Suppose that $x-a_1-a_2-\cdots-a_t-y$ is a path between x and y. Then, $a_1\in N^1(x)\cap N^t(y)$. Conversely, assume that $N^n(x)$ and $N^m(x)$ are not disjoint sets, for some non negative integers n,m. Hence, $N^n(x)$ and $N^m(x)$ have at least one vertex in common , say z. Thus, and since $z\in N^n(x)$, there is a path between the vertex x and z, say $x - c_1 - c_2 - \cdots - c_n - z$. Similarly, and since $z \in N^m(y)$, there is a path between the vertex y and z, say $z - d_1 - d_2 - \cdots - d_m - y$. Therefore, $x - c_1 - c_2 - \cdots - c_n - z - d_1 - d_2 - \cdots - d_m - y$. $\square$

The following theorem describes when $CTD(R)$ is disconnected.

Theorem 2.1. *If A is an integral domain and $R = A \times A[\alpha]$, then $CTD(R)$ is disconnected.*

Proof. Let $B = \{(a,a), (-a,a), (a,-a) : a \in A^*\}$ and let $x \in B$. Suppose that $y \in R^*$, that is $y = (y_{11}, y_{21} + y_{22}\alpha)$, such that $x.y = 0$. Since A is an integral domain, one can deduce $y \in B$ (in general, $N^n(y) \subseteq B$ for any *positive* integer n)

Let $M = \{(a, b\alpha) : a \in A^*$ and $b \in A\} \cup \{(0, a + b\alpha) : a, b \in A$ not both zero$)\}$ and let $m \in M$. Suppose that $m.r = 0$ for some $r \in R^*$. Again, since A is an integral domain, we deduce that $r \in M$ (in general, $N^m(r) \subseteq M$ for any *positive* integer m). It is clear that B and M are disjoint sets.

We claim here that the sets B and M are disconnected in the graph $CTD(R)$. To see this, suppose the contrary. If $x \in M$ and $y \in B$ and there is a path between x and y in the graph $CTD(R)$, then by Lemma 2.1 there exist two non negative integers n, m such that $N^n(x) \cap N^m(y)$ is nonempty, which is a conradiction, since $N^n(x) \cap N^m(y) \subseteq B \cap M$. Thus, the graph $CTD(R)$ is disconnected. $\square$

The following theorem establishes the necessary conditions for the chain zero-divisor dot product graph $CZD(R)$ to be equal to the known zero-divisor graph $\Gamma(R)$.

Theorem 2.2. *Let A be a ring, $2 \leq n < \infty$, and $R = A \times A[\alpha_1] \times A[\alpha_1, \alpha_2] \times \cdots \times A[\alpha_1, \alpha_2, \ldots, \alpha_{n-1}]$. Then, $CZD(R) = \Gamma(R)$ if and only if $n = 2$ and A is an integral domain.*

Proof. Suppose that A is an integral domain and $R = A \times A[\alpha]$. Then, $Z(R) = \{(a, b\alpha) : a \in A^*$ and $b \in A\} \cup \{(0, a + b\alpha) : a, b \in A \}$. Let $x, y \in Z^*(R)$ such that $x.y = 0$. Hence, we have three cases to consider, which are $x = (x_{11}, x_{22}\alpha)$ and $y = (y_{11}, y_{22}\alpha)$, $x = (x_{11}, x_{22}\alpha)$ and $y = (0, y_{21} + y_{22}\alpha)$ or $x = (0, x_{21} + x_{22}\alpha)$ and $y = (0, y_{21} + y_{22}\alpha)$. In all three cases it is clear that $x.y = 0$ if and only if $xy = (0, 0)$. Hence, $CZD(R) = \Gamma(R)$.

Conversely, suppose that $CZD(R) = \Gamma(R)$. Assume that $n \geq 3$, then there exist $x = (0, \alpha_1, \alpha_1, 0, \ldots, 0)$, $y = (0, 1, -1, 0, \ldots, 0) \in Z^*(R)$, with $x.y = 0$, but $xy \neq (0, 0, 0, \ldots, 0)$. Thus, $x - y$ is an edge of $CZD(R)$ that is not an edge of $\Gamma(R)$, a contradiction. Thus, $n = 2$. Now, if A is not an integral domain, then there are $a, b \in A^*$ such that $ab = 0$. Hence, $x = (1, a)$, $y = (a, -1 + b\alpha) \in Z^*(R)$, and $x.y = 0$, but $xy \neq (0, 0)$. Again, $x - y$ is an edge of $CZD(R)$ that is not an edge of $\Gamma(R)$, a contradiction. Thus, A must be an integral domain. $\square$

Corollary 2.1. *Let A be an integral domain. If $R = A \times A[\alpha]$, then $CZD(R)$ is connected with $diam(CZD(R)) = 3$.*

Proof. Since A is an integral domain, the vertex set of $CZD(R)$ can be divided into three disjoint sets $X = \{(a, b\alpha) : a \in A^*$ and $b \in A\}$, $Y = \{(0, a + b\alpha) : a \in A^*$ and $b \in A\}$ and $Z = \{(0, b\alpha) : b \in A^*\}$. It is clear that X, Y are independent sets (that is any two vertices in X or Y are not adjacent). Also, Z forms a complete subgraph of $CZD(R)$. Now, by Theorem 2.2 and since X is an independent set, we deduce that $CZD(R)$ is connected with $2 \leq diam(CZD(R)) \leq 3$. Now, let $x = (1, \alpha)$ and $y = (0, 1+\alpha)$. Then, $x.y \neq 0$. Let $t = (t_{11}, t_{21}+t_{22}\alpha) \in Z^*(R)$ such that $x.t = t.y = 0$. Then, we conclude that $t = (0, 0)$ which is a contradiction. Thus, $d_{cz}(x, y) = 3$. Hence, $diam(CZD(R)) = 3$.

Or (Another Proof) By Theorem (2.2) and since R is nonreduced ring and the zero divisors of R does not form an ideal, then by [1], $diam(CZD(R)) = 3$. □

Theorem 2.3. *Let A be a ring that is not an integral domain, and let $R = A \times A[\alpha]$. Then:*

1. *$CTD(R)$ is connected with $diam(CTD(R)) = 3$.*
2. *$CZD(R)$ is connected with $diam(CZD(R)) = 3$.*

Proof. 1) Let $x = (x_{11}, x_{21} + x_{22}\alpha)$, $y = (y_{11}, y_{21} + y_{22}\alpha) \in R^*$, where $x \neq y$, and assume that $x.y \neq 0$. Since A is not an integral domain, there are $a, b \in A^*$ (not necessarily distinct) such that $ab = 0$. Let $w = (ax_{21}, -ax_{11} + ax_{22}\alpha)$ and $v = (by_{21}, -by_{11} + by_{22}\alpha)$. Note that $w, v \in Z(R)$. It is clear that $x.w = w.v = v.y = 0$. Since $x.y \neq 0$, $w \neq y$ and $v \neq x$. Now, there are two cases:

Case 1. Suppose that $w \neq (0, 0)$ and $v \neq (0, 0)$. If $x.v = 0$ or $y.w = 0$, then $x - v - y$ or $x - w - y$ is a path of length 2 in $CTD(R)$ from x to y. But, if $x.v \neq 0$ or $y.w \neq 0$, then x, w, v and y are distinct and $x - w - v - y$ is a path of length 3 in $CTD(R)$ from x to y.

Case 2. Suppose that $w = (0, 0)$ and $v = (0, 0)$. If $w = (0, 0)$, then replace w by $(a, a) \in Z^*(R)$, and hence $x.w = (x_{11}, x_{21} + x_{22}\alpha).(a, a) = (ax_{11} + ax_{21}) + ax_{22}\alpha = 0$. Again, if $v = (0, 0)$, then replace v by $(b, b) \in Z^*(R)$, and hence, $y.v = 0$. Thus, as we have done, we can redefine w and v so that $w, v \in Z^*(R)$ and $x.w = w.v = v.y = 0$. Hence, as in the earlier argument, we can conclude that there is a path of length at most 3 in $CTD(R)$ from x to y.

Thus, $CTD(R)$ is connected with $d_{CT}(x, y) \leq 3$, for every $x, y \in R^*$. Now, let $x = (1, 1)$ and $y = (1, 0)$. It is clear that, $x.y \neq 0$. Let $t = (t_{11}, t_{21} + t_{22}\alpha) \in R^*$ such that $x.t = t.y = 0$. Then, $t_{11} = t_{21} = t_{22} = 0$, so $t = (0, 0)$ a contradiction. Therefore, $d_{CT}(x, y) = 3$, and hence, $diam(CTD(R)) = 3$. □

Theorem 2.4. *Let A be a ring , $4 \leq n < \infty$, and let $R = A \times A[\alpha_1] \times A[\alpha_1, \alpha_2] \times \cdots \times A[\alpha_1, \alpha_2, \ldots, \alpha_{n-1}]$. Then, $CTD(R)$ is connected with $diam(CTD(R)) = 2$.*

Proof. Let $x = (x_{11}, x_{21} + x_{22}\alpha_1, x_{31} + x_{32}\alpha_1 + x_{33}\alpha_2, \ldots, x_{n1} + \sum_{i=1}^{n} x_{ni}\alpha_{i-1})$, $y = (y_{11}, y_{21} + y_{22}\alpha_1, y_{31} + y_{32}\alpha_1 + y_{33}\alpha_2, \ldots, y_{n1} + \sum_{i=1}^{n} y_{ni}\alpha_{i-1}) \in R^*$, and suppose that $x.y \neq 0$. Then, let $M = \{j : x_{ji} = y_{ji} = 0, 1 \leq j \leq n$ and $1 \leq i \leq j\}$. Now, we have two cases:

Case 1. Suppose that M is not empty set. Then, choose $k \in M$, and let $w = (w_{11}, w_{21} + w_{22}\alpha_1, w_{31} + w_{32}\alpha_1 + w_{33}\alpha_2, \ldots, \sum_{i=1}^{n} w_{ni}\alpha_{i-1}) \in R^*$, where

$$w_{ij} = \begin{cases} 1, & j = k \text{ and } i = 1, \\ 0, & j = k \text{ and } 1 < i \leq j, \\ 0, & j \neq k. \end{cases}$$

Then, $x - w - y$ is a path of length 2 in $CTD(R)$ from x to y.

Case 2. Suppose that M is empty set. Then, let $f(x) = \min\{j : x_{j1} \neq 0,\ 2 \leq j \leq n\}$ and $f(y) = \min\{j : y_{j1} \neq 0, 2 \leq j \leq n\}$. Since M is empty set, we deduce that $f(x) = 2$ or $f(y) = 2$, without loss of generality, assume that $f(x) = 2$. Let $v = (0, (x_{31}y_{41} - x_{41}y_{31})\alpha_1, (x_{41}y_{21} - x_{21}y_{41})\alpha_1, (x_{21}y_{31} - x_{31}y_{21})\alpha_1, 0, \ldots, 0)$. Now, we have two subcases:

Subcase 2.1. Suppose that $v \neq (0, 0, \ldots, 0)$. Then, $x.v = v.y = 0$. Since $x.y \neq 0$, $x \neq v$ and $y \neq v$. Hence, $x - v - y$ is a path of length 2 in $CTD(R)$ from x to y.

Subcase 2.2. Suppose that $v = (0, 0, \ldots, 0)$. Then, $x_{21}y_{31} - x_{31}y_{21} = 0$. Let $w = (0, -x_{31}\alpha_1, x_{21}\alpha_1, 0, \ldots, 0)$ Since $x_{21} \neq 0$, $w \in R^*$. Hence, $x.w = -x_{31}x_{21} + x_{21}x_{31} = 0$ and $w.y = -x_{31}y_{21} + x_{21}y_{31} = 0$. Since $x.w = w.y = 0$, and $x.y \neq 0$, $x \neq w$ and $y \neq w$. Thus, $x - w - y$ is a path of length 2 in $CTD(R)$ from x to y. Hence, $CTD(R)$ is connected with $diam(CTD(R)) = 2$. □

Theorem 2.5. *Let A be a ring, and let $R = A \times A[\alpha_1] \times A[\alpha_1, \alpha_2]$. Then, $CTD(R)$ is connected with $diam(CTD(R)) = 2$.*

Proof. Let $x = (x_{11}, x_{21}+x_{22}\alpha_1, x_{31}+x_{32}\alpha_1+x_{33}\alpha_2)$, $y = (y_{11}, y_{21}+y_{22}\alpha_1, y_{31}+y_{32}\alpha_1 + y_{33}\alpha_2) \in R^*$, and suppose that $x.y \neq 0$. Then, let $M = \{j : x_{j1} = y_{j1} = 0, 1 \leq j \leq 3\}$. Now, we have two cases:

Case 1. Suppose that M is not empty set. Then, choose $k \in M$, and let $z =$, where

$$z = \begin{cases} (1, 0, 0), & \text{if } k = 1 \\ (0, \alpha_1, 0), & \text{if } k = 2 \in R^*. \\ (0, 0, \alpha_1), & \text{if } k = 3 \end{cases}$$

Then, $x - z - y$ is a path of length 2 in $CTD(R)$ from x to y.

Case 2. Suppose that M is an empty set. Then, define $f(x) = \min\{j : x_{j1} \neq 0,$ $2 \leq j \leq 3\}$ and $f(y) = \min\{j : y_{j1} \neq 0,\ 2 \leq j \leq 3\}$. Since M is an empty set, we deduce that $f(x) = 2$ or $f(y) = 2$, without loss of generality, assume that $f(x) = 2$, that is $x_{21} \neq 0$. Now, we have three subcases:

Subcase 2.1. Suppose that $x_{31} \neq 0$, $y_{21} = 0$. If $y_{31}x_{21} \neq 0$, then select $v_1 = (0, x_{31}\alpha_1, -x_{21}\alpha_1)$, $v_2 = (0, \alpha_1, 0) \in R^*$. Thus, $x.v_1 = v_1.v_2 = v_2.y = 0$. Since $x.y \neq 0$, $x.v_2 \neq 0$, $y.v_1 \neq 0$, $x \neq v_1$ and $y \neq v_2$. Hence, $x - v_1 - v_2 - y$ is a path of length 3 in $CTD(R)$ from x to y. If $y_{31}x_{21} = 0$, then select $v = (0, x_{31}\alpha_1, -x_{21}\alpha_1) \in R^*$. So, $x.v = v.y = 0$. Since $x.y \neq 0$, $x \neq v$ and $y \neq v$. Hence, $x - v - y$ is a path of length 2 in $CTD(R)$ from x to y.

Subcase 2.2. Suppose that $x_{31} = 0$, $y_{21} = 0$. If $y_{31} \neq 0$, then select $v_1 = (0, 0, \alpha_1)$, $v_2 = (0, \alpha_1, 0) \in R^*$. Then, $x.v_1 = v_1.v_2 = v_2.y = 0$. Since $x.y \neq 0$, $x.v_2 \neq 0$, $y.v_1 \neq 0$, $x \neq v_1$ and $y \neq v_2$. Hence, $x - v_1 - v_2 - y$ is a path of length 3 in $CTD(R)$ from x to y. If $y_{31} = 0$, then select $v = (0, 0, \alpha_1) \in R^*$. So, $x.v = v.y = 0$. Since $x.y \neq 0$, $x \neq v$ and $y \neq v$. Hence, $x - v - y$ is a path of length 2 in $CTD(R)$ from x to y.

Subcase 2.3. Suppose that $x_{31} \neq 0$, $y_{21} \neq 0$. If $x_{21}y_{31} - x_{31}y_{21} = 0$, then select $v = (0, x_{31}\alpha_1, -x_{21}\alpha_1) \in R^*$. So, $x.v = v.y = 0$. Since $x.y \neq 0$, $x \neq v$ and $y \neq v$, we have $x - v - y$ a path of length 2 in $CTD(R)$ from x to y. If $x_{21}y_{31} - x_{31}y_{21} \neq 0$, then select $v_1 = (0, x_{31}\alpha_1, -x_{21}\alpha_1)$, $v_2 = (0, y_{31}\alpha_1, -y_{21}\alpha_1) \in R^*$. Since $x.y \neq 0$, $x.v_2 \neq 0$, $y.v_1 \neq 0$, $x \neq v_1$ and $y \neq v_2$, we have $x - v_1 - v_2 - y$ a path of length 3 in $CTD(R)$ from x to y.

Therefore, by the previous cases we deduce that $diam(CTD(R)) \leq 3$. Now, let $x = (1, \alpha_1, 1 + \alpha_1 + \alpha_2)$ and $y = (1, 1 + \alpha_1, \alpha_1 + \alpha_2)$. Suppose there exists $(v_{11}, v_{21} + v_{22}\alpha_1, v_{31} + v_{32}\alpha_1 + v_{33}\alpha_2) \in R^*$ such that $x - v - y$ is a path of length 2 in $CTD(R)$ from x to y. Since $x.v = v.y = 0$, we have the following equations

$$\begin{aligned} v_{11} + v_{31} &= 0 \\ v_{21} + v_{32} + v_{31} &= 0 \\ v_{33} + v_{31} &= 0 \end{aligned}$$

$$\begin{aligned} v_{11} + v_{21} &= 0 \\ v_{21} + v_{22} + v_{31} &= 0 \\ v_{31} &= 0 \end{aligned}$$

Solving these equations produces that $v = (0, 0, 0)$ which is a contradiction. Thus, $d_{CT}(x, y) = 3$, and hence, $diam(CTD(R)) = 3$. □

Theorem 2.6. *Let A be a ring, and let $R = A \times A[\alpha_1] \times A[\alpha_1, \alpha_2]$.If A is an integral domain, then $CZD(R)$ is connected with $diam(CZD(R)) = 3$.*

Proof. Every path in $\Gamma(R)$ is also a path in $CZD(R)$. Now, since $\Gamma(R)$ is connected with $diam(\Gamma(R)) \leq 3$ by [3], we conclude that $CZD(R)$ is connected with $diam(CZD(R)) \leq diam(\Gamma(R))$. Thus, $diam(CZD(R)) \leq 3$. Let $x = (1, -1, 0)$, $y = (1, 0, -1) \in Z(R)^*$. It is clear that $x.y = 1 \neq 0$. Hence, $1 < d_{CZ}(x, y) \leq 3$. Suppose that $d_{CZ}(x, y) = 2$. Then, there is $w = (w_{11}, w_{21} +$

$w_{22}\alpha_1, w_{31} + w_{32}\alpha_1 + w_{33}\alpha_2) \in Z(R)^*$ (Since A is an integral domain $w_{11}, w_{21} or$ w_{31} must be zero) such that $x.w = w.y = 0$. By direct calculations, we deduce that $w = (0, 0, 0)$ which is a contradiction. Hence, $d_{CZ}(x, y) = 3$. Therefore, $diam(CZD(R)) = 3$. □

Theorem 2.7. *Let A be a ring, and let $R = A \times A[\alpha_1] \times A[\alpha_1, \alpha_2] \times \cdots \times A[\alpha_1, \alpha_2, \ldots, \alpha_{n-1}]$.*

(1) If $|A| > 2$ and $2 \leq n < \infty$, then $gr(CTD(R)) = gr(CZD(R)) = 3$.

(2) If A is isomorphic to $\mathbb{Z}_2$, and $3 \leq n < \infty$, then $gr(CTD(R)) = gr(CZD(R)) = 3$.

(3) If A is isomorphic to $\mathbb{Z}_2$, and $n = 2$ then $gr(CZD(R)) = \infty$.

Proof. (1) Since $|A| > 2$, there is $a \in A \backslash \{0, 1\}$. Let $x = (1, 0, \ldots, 0)$, $y = (0, \alpha_1, \ldots, 0)$, and $z = (0, a\alpha_1, \ldots, 0)$. Then, $x - y - z - x$ is a cycle of length 3.

(2) Let $x = (1, 0, 0, \ldots, 0)$, $y = (0, 1, 0, \ldots, 0)$, and $z = (0, 0, 1, 0 \ldots, 0)$. Then, $x - y - z - x$ is a cycle of length 3.

(3) Clear. □

According to the previous results, one can conclude the following corollaries.

Corollary 2.2. *Let A be a ring, and let $R = A \times A[\alpha_1] \times A[\alpha_1, \alpha_2] \times \cdots \times A[\alpha_1, \alpha_2, \ldots, \alpha_{n-1}]$ (with $2 \leq n < \infty$). Then, the following are equivalent:*

(1) $gr(CTD(R)) = 3$.

(2) $gr(CZD(R)) = 3$.

(3) $|A| > 2$ or A is isomorphic to $\mathbb{Z}_2$, and $3 \leq n$.

Proof. Obvious, by Theorem 2.7. □

Corollary 2.3. *Let A be a ring, and let $R = A \times A[\alpha_1] \times A[\alpha_1, \alpha_2] \times \cdots \times A[\alpha_1, \alpha_2, \ldots, \alpha_{n-1}]$ (with $2 \leq n < \infty$). Then, the following are equivalent:*

(1) $gr(CZD(R)) = \infty$.

(2) A is isomorphic to $\mathbb{Z}_2$, and $n = 2$.

Proof. Obvious, by Theorem 2.7. □

Corollary 2.4. *Let A be a ring, and let $R = A \times A[\alpha_1] \times A[\alpha_1, \alpha_2] \times \cdots \times A[\alpha_1, \alpha_2, \ldots, \alpha_{n-1}]$ (with $2 \leq n < \infty$). Then, the following are equivalent:*

(1) $CZD(R) = \Gamma(R)$.

(2) $CTD(R)$ is disconnected.

(3) A is an integral domain and $n = 2$.

3. Conclusion

Let A be a commutative ring with nonzero identity 1. for the natural number n, we use the ring $R = A\times A[\alpha_1]\times A[\alpha_1,\alpha_2]\cdots\times A[\alpha_1,\alpha_2,\ldots\alpha_n]$ to construct what we call the chain total dot product graph (the chain zero-divisor dot product graph), denoted by $CTD(R)$ ($CZD(R)$). These two graphs are considered to be a generalization of the total and the zero-divisor dot product graphs in [2]. In this article, we studied some basic graph properties for the graphs $CTD(R)$ and $CZD(R)$ such as connectedness, diameter and the girth. Many graph properties, such as the graph's core, center, and median, as well as planarity, can be explored in the future for the graphs $CTD(R)$ and $CZD(R)$.

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