On graded weakly classical 2-absorbing submodules of graded modules over graded commutative rings

Shatha Alghueiri
Department of Mathematics and Statistics
Jordan University of Science and Technology
P.O.Box 3030, Irbid 22110
Jordan
soalghueiri@just.edu.jo

Khaldoun Al-Zoubi*
Department of Mathematics and Statistics
Jordan University of Science and Technology
P.O.Box 3030, Irbid 22110
Jordan
kfzoubi@just.edu.jo

Abstract. In this paper, we introduce the concept of graded weakly classical 2-absorbing submodule as a generalization of a graded classical 2-absorbing submodule. We give a number of results concerning this class of graded submodules and their homogeneous components.

Keywords: graded weakly classical 2-absorbing submodule, graded classical 2-absorbing submodule, graded 2-absorbing submodule.

1. Introduction and preliminaries

Throughout this paper all rings with identity and all modules are unitary.

Refai and Al-Zoubi in [23] introduced the concept of graded primary ideal. The concept of graded 2-absorbing ideal was introduced and studied by Al-Zoubi, Abu-Dawwas and Ceken in [5]. The concept of graded prime submodule was introduced and studied by many authors, see for example [2, 3, 12, 13, 15, 22]. The concept of graded classical prime submodules as a generalization of graded prime submodules was introduced in [17] and studied in [11]. The concept of graded weakly classical prime submodules, generalizations of graded classical prime submodules, was introduced by Abu-Dawwas and Al-Zoubi in [1]. The concept of graded 2-absorbing submodule, generalizations of graded prime submodule, was introduced by Al-Zoubi and Abu-Dawwas in [4] and studied in [8, 9]. Then, many generalizations of graded 2-absorbing submodules were studied such as graded 2-absorbing primary (see [16]), graded weakly 2-absorbing primary (see [7]) and graded 2-absorbing I_e-prime submodules (see [14]).

*. Corresponding author

Recently, Al-Zoubi and Al-Azaizeh, in [6] introduced the concept of graded classical 2-absorbing submodules over a graded commutative ring as a new generalization of graded 2-absorbing submodules.

Here, we introduce the concept of graded weakly classical 2-absorbing submodule as a new generalization of graded classical 2-absorbing submodule on the one hand and a generalization of a graded weakly classical prime submodule on other hand.

First, we recall some basic properties of graded rings and modules which will be used in the sequel. We refer to [18, 19, 20, 21] for these basic properties and more information on graded rings and modules. Let G be a group with identity element e. A ring R is called a graded ring (or G-graded ring) if there exist additive subgroups R_h of R indexed by the elements $h \in G$ such that $R = \oplus_{g\in G} R_g$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. The non-zero elements of R_g are said to be homogeneous of degree g and all the homogeneous elements are denoted by $h(R)$, i.e. $h(R) = \cup_{h\in G} R_h$. If $r \in R$, then r can be written uniquely as $\sum_{g\in G} r_g$, where r_g is called a homogeneous component of r in R_g. Moreover, R_e is a subring of R and $1 \in R_e$ (see [21]). Let $R = \oplus_{g\in G} R_g$ be a G-graded ring. An ideal I of R is said to be a graded ideal if $I = \sum_{h\in G}(I \cap R_h) := \sum_{h\in G} I_h$ (see [21]). Let $R = \oplus_{g\in G} R_g$ be a G-graded ring. A left R-module M is said to be a graded R-module (or G-graded R-module) if there exists a family of additive subgroups $\{M_g\}_{g\in G}$ of M such that $M = \oplus_{g\in G} M_g$ and $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Similarly, if an element of M belongs to $\cup_{g\in G} M_h = h(M)$, then it is called a homogeneous. Note that M_g is an R_e-module for every $g \in G$. Let $R = \oplus_{g\in G} R_g$ be a G-graded ring. A submodule N of M is said to be a graded submodule of M if $N = \oplus_{g\in G}(N \cap M_g) := \oplus_{g\in G} N_g$. In this case, N_g is called the g-component of N. Moreover, M/N becomes a G-graded R-module with g-component $(M/N)_g := (M_g + N)/N$ for $g \in G$.

2. Graded weakly classical 2-absorbing submodules

Definition 2.1. Let R be a G-graded ring, M a graded R-module, N a proper graded submodule of M and $g \in G$.

(i) We say that N_g is a weakly classical g-2-absorbing submodule of the R_e-module M_g if $N_g \neq M_g$; and whenever $r_e, s_e, t_e \in R_e$ and $m_g \in M_g$ with $0 \neq r_e s_e t_e m_g \in N_g$, then either $r_e s_e m_g \in N_g$ or $r_e t_e m_g \in N_g$ or $s_e t_e m_g \in N_g$.

(ii) We say that N is a graded weakly classical 2-absorbing submodule of M if $r_h, s_\alpha, t_\beta \in h(R)$ and $m_\lambda \in h(M)$ with $0 \neq r_h s_\alpha t_\beta m_\lambda \in N$, then either $r_h s_\alpha m_\lambda \in N$ or $r_h t_\beta m_\lambda \in N$ or $s_\alpha t_\beta m_\lambda \in N$.

Clearly, every graded classical 2-absorbing submodule is a graded weakly classical 2-absorbing. However, since $\{0\}$ is always a graded weakly classical 2-absorbing submodule (by defintion), a graded weakly classical 2-absorbing submodule need not be a graded classical 2-absorbing submodule.

Theorem 2.2. *Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. If N is a graded weakly classical 2-absorbing submodule of M, then for each $g \in G$ with $N_g \neq M_g$, N_g is a weakly classical g-2-absorbing submodule of the R_e-module M_g.*

Proof. Suppose that N is a graded weakly classical 2-absorbing submodule of M and $g \in G$ with $N_g \neq M_g$. Now, assume that $0 \neq r_e s_e t_e m_g \in N_g$ where $r_e, s_e, t_e \in R_e$ and $m_g \in M_g$. Then, $0 \neq r_e s_e t_e m_g \in N$. Since N is a graded weakly classical 2-absorbing submodule of M, either $r_e s_e m_g \in N$ or $r_e t_e m_g \in N$ or $s_e t_e m_g \in N$. But $N_g = N \cap M_g$, so we get that either $r_e s_e m_g \in N_g$ or $r_e t_e m_g \in N_g$ or $s_e t_e m_g \in N_g$. Hence, N_g is a weakly classical g-2-absorbing submodule of the R_e-module M_g. □

Let R be a G-graded ring, M a graded R-module and N a graded submodule of M. A proper submodule N_g of the R_e-module M_g is said to be a classical g-2-absorbing submodule if whenever $r_e, s_e, t_e \in R_e$ and $m_g \in M_g$ with $r_e s_e t_e m_g \in N_g$, then either $r_e s_e m_g \in N_g$ or $r_e t_e m_g \in N_g$ or $s_e t_e m_g \in N_g$ (see [6]).

Theorem 2.3. *Let R be a G-graded ring, M a graded R-module, N a graded submodule of M and $g \in G$. If N_g is a weakly classical g-2-absorbing submodule of the R_e-module M_g, then either N_g is a classical g-2-absorbing submodule of the R_e-module M_g or $(N_g :_{R_e} M_g)^3 N_g = 0$.*

Proof. Suppose that $(N_g :_{R_e} M_g)^3 N_g \neq 0$. Let $r_e, s_e, t_e \in R_e$ and $m_g \in M_g$ such that $r_e s_e t_e m_g \in N_g$. If $r_e s_e t_e m_g \neq 0$, then we get the result as N_g is a weakly classical g-2-absorbing of M_g. So, we assume $r_e s_e t_e m_g = 0$. Now, if $r_e s_e t_e N_g \neq 0$, then there exists $n_{1_g} \in N_g$ such that $r_e s_e t_e n_{1_g} \neq 0$, so $0 \neq r_e s_e t_e(m_g + n_{1_g}) \in N_g$ which yields either $r_e s_e(m_g + n_{1_g}) \in N_g$ or $r_e t_e(m_g + n_{1_g}) \in N_g$ or $s_e t_e(m_g + n_{1_g}) \in N_g$ and then either $r_e s_e m_g \in N_g$ or $r_e t_e m_g \in N_g$ or $s_e t_e m_g \in N_g$. So, we can assume that $r_e s_e t_e N_g = 0$. Now, if $r_e s_e(N_g :_{R_e} M_g) m_g \neq 0$, then there exists $t_{1_e} \in (N_g :_{R_e} M_g)$ such that $r_e s_e t_{1_e} m_g \neq 0$. Thus, $0 \neq r_e s_e(t_e + t_{1_e}) m_g \in N_g$ and then either $r_e s_e m_g \in N_g$ or $r_e(t_e + t_{1_e}) m_g \in N_g$ or $s_e(t_e + t_{1_e}) m_g \in N_g$ which follows either $r_e s_e m_g \in N_g$ or $r_e t_e m_g \in N_g$ or $s_e t_e m_g \in N_g$. We can assume that $r_e s_e(N_g :_{R_e} M_g) m_g = 0$, $r_e t_e(N_g :_{R_e} M_g) m_g = 0$ and $s_e t_e(N_g :_{R_e} M_g) m_g = 0$. Now, if $r_e(N_g :_{R_e} M_g)^2 m_g \neq 0$, then there exist $s_{2_e}, t_{2_e} \in (N_g :_{R_e} M_g)$ such that $r_e s_{2_e} t_{2_e} m_g \neq 0$. Thus, by our assumptions we get $0 \neq r_e(s_e + s_{2_e})(t_e + t_{2_e}) m_g \in N_g$ which gives either $r_e(s_e + s_{2_e}) m_g \in N_g$ or $r_e(t_e + t_{2_e}) m_g \in N_g$ or $(s_e + s_{2_e})(t_e + t_{2_e}) m_g \in N_g$, and then either $r_e s_e m_g \in N_g$ or $r_e t_e m_g \in N_g$ or $s_e t_e m_g \in N_g$. So, we assume that $r_e(N_g :_{R_e} M_g)^2 m_g = 0$, $s_e(N_g :_{R_e} M_g)^2 m_g = 0$ and $t_e(N_g :_{R_e} M_g)^2 m_g = 0$. Now, if $r_e s_e(N_g :_{R_e} M_g) N_g \neq 0$, then there exist $t_{3_e} \in (N_g :_{R_e} M_g)$ and $n_{2_g} \in N_g$ such that $r_e s_e t_{3_e} n_{2_g} \neq 0$. Hence, by our assumptions we get $0 \neq r_e s_e(t_e + t_{3_e})(m_g + n_{2_g}) \in N_g$ and then either $r_e s_e(m_g + n_{2_g}) \in N_g$ or $r_e(t_e + t_{3_e})(m_g + n_{2_g}) \in N_g$ or $s_e(t_e + t_{3_e})(m_g + n_{2_g}) \in N_g$ which yields either $r_e s_e m_g \in N_g$ or $r_e t_e m_g \in N_g$ or $s_e t_e m_g \in N_g$. We assume that $r_e s_e(N_g :_{R_e} M_g) N_g = 0$, $r_e t_e(N_g :_{R_e} M_g) N_g = 0$ and $s_e t_e(N_g :_{R_e} M_g) N_g = 0$.

Now, if $r_e(N_g :_{R_e} M_g)^2 N_g \neq 0$, then there exist $s_{4_e}, t_{4_e} \in (N_g :_{R_e} M_g)$ and $n_{3_g} \in N_g$ such that $r_e s_{4_e} t_{4_e} n_{3_g} \neq 0$. Thus, by assumptions, $0 \neq r_e(s_e + s_{4_e})(t_e + t_{4_e})(m_g + n_{3_g}) \in N_g$, then either $r_e(s_e + s_{4_e})(m_g + n_{3_g}) \in N_g$ or $r_e(t_e + t_{4_e})(m_g + n_{3_g}) \in N_g$ or $(s_e + s_{4_e})(t_e + t_{4_e})(m_g + n_{3_g}) \in N_g$, and then either $r_e s_e m_g \in N_g$ or $r_e t_e m_g \in N_g$ or $s_e t_e m_g \in N_g$. So, we can assume that $r_e(N_g :_{R_e} M_g)^2 N_g = 0$, $s_e(N_g :_{R_e} M_g)^2 N_g = 0$ and $t_e(N_g :_{R_e} M_g)^2 N_g = 0$. Since $(N_g :_{R_e} M_g)^3 N_g \neq 0$, there exist $r_{5_e}, s_{5_e}, t_{5_e} \in (N_g :_{R_e} M_g)$ and $n_{4_g} \in N_g$ such that $r_{5_e} s_{5_e} t_{5_e} n_{4_g} \neq 0$. Hence, by our assumptions we get $0 \neq (r_e + r_{5_e})(s_e + s_{5_e})(t_e + t_{5_e})(m_g + n_{4_g}) \in N_g$ which follows that either $(r_e + r_{5_e})(s_e + s_{5_e})(m_g + n_{4_g}) \in N_g$ or $(r_e + r_{5_e})(t_e + t_{5_e})(m_g + n_{4_g}) \in N_g$ or $(s_e + s_{5_e})(t_e + t_{5_e})(m_g + n_{4_g}) \in N_g$, and then either $r_e s_e m_g \in N_g$ or $r_e t_e m_g \in N_g$ or $s_e t_e m_g \in N_g$. Therefore, N_g is a classical g-2-absorbing submodule of the R_e-module M_g. □

Let R be a G-graded ring and M a graded R-module. A proper graded submodule N of M is said to be a graded weakly classical prime submodule if whenever $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ such that $0 \neq r_g s_h m_\lambda \in N$, then either $r_g m_\lambda \in N$ or $s_h m_\lambda \in N$ (see [1]).

It is easy to see that every graded weakly classical prime submodule is a graded weakly classical 2-absorbing. The following example shows that the converse is not true in general.

Example 2.4. Let $G = \mathbb{Z}_2$, then $R = \mathbb{Z}$ is a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}$ be a graded R-module with $M_0 = \mathbb{Z}$ and $M_1 = \{0\}$. Now, consider the graded submodule $N = 4\mathbb{Z}$ of M. Then, N is not a graded weakly classical prime submodule of M since $0 \neq 2 \cdot 2 \cdot 3 \in N$ but $2 \cdot 3 \notin N$. However, easy computations show that N is a graded weakly classical 2-absorbing submodule of M.

Theorem 2.5. *Let R be a G-graded ring, M a graded R-module and N and K be two graded submodules of M with $N \subsetneq K$. If N is a graded weakly classical 2-absorbing submodule of M, then N is a graded weakly classical 2-absorbing submodule of K.*

Proof. Let $r_g, s_h, t_\alpha \in h(R)$ and $m_\lambda \in K \cap h(M)$ such that $0 \neq r_g s_h t_\alpha m_\lambda \in N$, then either $r_g s_h m_\lambda \in N$ or $r_g t_\alpha m_\lambda \in N$ or $s_h t_\alpha m_\lambda \in N$ as N is a graded weakly classical 2-absorbing submodule of M. So, we get the result. □

The following example shows that a graded submodule of a graded weakly classical 2-absorbing submodule need not be a graded weakly classical 2-absorbing.

Example 2.6. Let $G = \mathbb{Z}_2$, then $R = \mathbb{Z}$ is a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}$ be a graded R-module with $M_0 = \mathbb{Z}$ and $M_1 = \{0\}$. Now, consider the graded submodules $N = 4\mathbb{Z}$ and $K = 16\mathbb{Z} \subseteq N$ of M. It is easy to see that N is a graded weakly classical 2-absorbing submodule of M but K is not a graded weakly classical 2-absorbing since $0 \neq 2 \cdot 2 \cdot 2 \cdot 2 \in K$ and $2 \cdot 2 \cdot 2 \notin K$.

Theorem 2.7. *Let R be a G-graded ring, M a graded R-module and N and K be two proper graded R-submodules of M such that $K \subseteq N$. Then, the following statements hold:*

(i) *If N is a graded weakly classical 2-absorbing submodule of M, then N/K is a graded weakly classical 2-absorbing submodule of M/K.*

(ii) *If K is a graded weakly classical 2-absorbing submodule of M and N/K is a graded weakly classical 2-absorbing submodule of M/K, then N is a graded weakly classical 2-absorbing submodule of M.*

Proof. (i) Let $r_g, s_h, t_\alpha \in h(R)$ and $m_\lambda + K \in h(M/K)$ such that $0_{M/K} \neq r_g s_h t_\alpha m_\lambda + K \in N/K$. Hence, $0_M \neq r_g s_h t_\alpha m_\lambda \in N$ which implies that either $r_g s_h m_\lambda \in N$ or $r_g t_\alpha m_\lambda \in N$ or $s_h t_\alpha m_\lambda \in N$ and then either $r_g s_h m_\lambda + K \in N/K$ or $r_g t_\alpha m_\lambda + K \in N/K$ or $s_h t_\alpha m_\lambda + K \in N/K$. Therefore, N/K is a graded weakly classical 2-absorbing submodule of M/K.

(ii) Let $r_g, s_h, t_\alpha \in h(R)$ and $m_\lambda \in h(M)$ such that $0_M \neq r_g s_h t_\alpha m_\lambda \in N$. Now, if $0_M \neq r_g s_h t_\alpha m_\lambda \in K$, then either $r_g s_h m_\lambda \in K \subseteq N$ or $r_g t_\alpha m_\lambda \in K \subseteq N$ or $s_h t_\alpha m_\lambda \in K \subseteq N$. Otherwise, we get $0_{M/K} \neq r_g s_h t_\alpha m_\lambda + K \in N/K$ and then either $r_g s_h m_\lambda + K \in N/K$ or $r_g t_\alpha m_\lambda + K \in N/K$ or $s_h t_\alpha m_\lambda + K \in N/K$. Thus, either $r_g s_h m_\lambda \in N$ or $r_g t_\alpha m_\lambda \in N$ or $s_h t_\alpha m_\lambda \in N$. Therefore, N is a graded weakly classical 2-absorbing submodule of M. □

The following example shows that the intersection of two graded weakly classical 2-absorbing submodules need not be a graded weakly classical 2-absorbing submodule.

Example 2.8. Let $G = \mathbb{Z}_2$. Then, $R = \mathbb{Z}$ is a G-graded ring with $R_0 = \mathbb{Z}$ and $R_1 = \{0\}$. Let $M = \mathbb{Z}$ be a graded R-module with $M_0 = \mathbb{Z}$ and $M_1 = \{0\}$. Now, consider the graded submodules $N = 4\mathbb{Z}$ and $K = 9\mathbb{Z}$ of M. It is easy to see that N and K are graded weakly classical 2-absorbing submodules of M. But $N \cap K = 36\mathbb{Z}$ is not a graded weakly classical 2-absorbing submodule of M, since $0 \neq 2 \cdot 2 \cdot 3 \cdot 3 \in 36\mathbb{Z}$ and neither $2 \cdot 2 \cdot 3 \in 36\mathbb{Z}$ nor $2 \cdot 3 \cdot 3 \in 36\mathbb{Z}$.

Theorem 2.9. *Let R be a G-graded ring, M a graded R-module and N and K be two graded submodules of M. If N and K are graded weakly classical prime submodules of M, then $N \cap K$ is a graded weakly classical 2-absorbing submodule of M.*

Proof. Let $r_g, s_h, t_\alpha \in h(R)$ and $m_\lambda \in h(M)$ such that $0 \neq r_g s_h t_\alpha m_\lambda \in N \cap K$. Hence, $0 \neq r_g s_h t_\alpha m_\lambda \in N$ and $0 \neq r_g s_h t_\alpha m_\lambda \in K$. This yields that either $r_g m_\lambda \in N$ or $s_h m_\lambda \in N$ or $t_\alpha m_\lambda \in N$ and either $r_g m_\lambda \in K$ or $s_h m_\lambda \in K$ or $t_\alpha m_\lambda \in K$ as N and K are graded classical prime submodules of M. Assume, without loss of generality, $r_g m_\lambda \in N$ and $s_h m_\lambda \in K$. Thus, $r_g s_h m_\lambda \in N \cap K$. Therefore, $N \cap K$ is a graded weakly classical 2-absorbing submodule of M. □

Let M and M' be two graded R-modules. A homomorphism of graded R-modules $f : M \to M'$ is a homomorphism of R-modules which satisfies $f(M_g) \subseteq M'_g$ for every $g \in G$ (see [21]).

Theorem 2.10. *Let R be a G-graded ring, M and M' be two graded R-modules and $f : M \to M'$ be a graded homomorphism.*

(i) *If f is a graded epimorphism and N is a graded weakly classical 2-absorbing submodule of M with $ker(f) \subseteq N$, then $f(N)$ is a graded weakly classical 2-absorbing submodule of M'.*

(ii) *If f is a graded isomorphism and N' is a graded weakly classical 2-absorbing submodule of M', then $f^{-1}(N')$ is a graded weakly classical 2-absorbing submodule of M.*

Proof. (i) Clearly, $f(N)$ is a proper graded submodule of M'. Now, let $r_g, s_h, t_\alpha \in h(R)$ and $m'_\lambda \in h(M')$ such that $0 \neq r_g s_h t_\alpha m'_\lambda \in f(N)$. Since f is a graded epimorphism, there exists $m_\lambda \in h(M)$ such that $f(m_\lambda) = m'_\lambda$. Hence, $0 \neq r_g s_h t_\alpha m'_\lambda = f(r_g s_h t_\alpha m_\lambda) \in f(N)$ and then there exists $n \in N \cap h(M)$ such that $f(r_g s_h t_\alpha m_\lambda) = f(n)$ which yields that $r_g s_h t_\alpha m_\lambda - n \in ker(f) \subseteq N$, so $0 \neq r_g s_h t_\alpha m_\lambda \in N$. Thus, as N is a graded weakly classical 2-absorbing submodule of M we get either $r_g s_h m_\lambda \in N$ or $r_g t_\alpha m_\lambda \in N$ or $s_h t_\alpha m_\lambda \in N$. So, either $r_g s_h m'_\lambda \in f(N)$ or $r_g t_\alpha m'_\lambda \in f(N)$ or $s_h t_\alpha m'_\lambda \in f(N)$. Therefore, $f(N)$ is a graded weakly classical 2-absorbing submodule of M'.

(ii) It is easy to see that $f^{-1}(N')$ is a proper graded submodule of M. Now, let $r_g, s_h, t_\alpha \in h(R)$ and $m_\lambda \in h(M)$ such that $0 \neq r_g s_h t_\alpha m_\lambda \in f^{-1}(N')$. Thus, $0 \neq r_g s_h t_\alpha f(m_\lambda) \in N'$ and then either $r_g s_h f(m_\lambda) \in N'$ or $r_g t_\alpha f(m_\lambda) \in N'$ or $s_h t_\alpha f(m_\lambda) \in N'$ as N' is a graded weakly classical 2-absorbing submodule of M'. Hence, either $r_g s_h m_\lambda \in f^{-1}(N')$ or $r_g t_\alpha m_\lambda \in f^{-1}(N')$ or $s_h t_\alpha m_\lambda \in f^{-1}(N')$. Therefore, $f^{-1}(N')$ is a graded weakly classical 2-absorbing submodule of M. □

Recall from [4] that a proper graded submodule N of a graded R-module M is said to be a graded weakly 2-absorbing submodule of M if whenever $r_g, s_h \in h(R)$ and $m_\lambda \in h(M)$ with $0 \neq r_g s_h m_\lambda \in N$, then either $r_g s_h \in (N :_R M)$ or $r_g m_\lambda \in N$ or $s_h m_\lambda \in N$.

Theorem 2.11. *Let R be a G-graded ring, M a graded gr-cyclic R-module and N a proper graded submodule of M. If N is a graded weakly classical 2-absorbing submodule of M, then N is a graded weakly 2-absorbing submodule of M.*

Proof. Since M is a gr-cyclic, there exists $m_{\lambda_1} \in h(M)$ such that $M = Rm_{\lambda_1}$. Now, let $r_g, s_h \in h(R)$ and $m_{\lambda_2} \in h(M)$ with $0 \neq r_g s_h m_{\lambda_2} \in N$. Hence, there exists $t_\alpha \in h(R)$ such that $0 \neq r_g s_h m_{\lambda_2} = r_g s_h t_\alpha m_{\lambda_1} \in N$. This yields that either $r_g m_{\lambda_2} = r_g t_\alpha m_{\lambda_1} \in N$ or $s_h m_{\lambda_2} = s_h t_\alpha m_{\lambda_1} \in N$ or $r_g s_h \in (N :_R m_{\lambda_1}) = (N :_R M)$ as N is a graded weakly classical 2-absorbing submodule of M. Therefore, N is a graded weakly 2-absorbing submodule of M. □

Recall from [5] that a proper graded ideal I of R is said to be a graded weakly 2-absorbing ideal of R if whenever $r_g, s_h, t_\alpha \in h(R)$ with $0 \neq r_g s_h t_\alpha \in I$, then $r_g s_h \in I$ or $r_g t_\alpha \in I$ or $s_h t_\alpha \in I$.

Theorem 2.12. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M.*

(i) *If N is a graded weakly classical 2-absorbing submodule of M and $m_\lambda \in h(M)\backslash N$ with $Ann_R(m_\lambda) = \{0\}$, then $(N :_R m_\lambda)$ is a graded weakly 2-absorbing ideal of R.*

(ii) *If $(N :_R m_\lambda)$ is a graded weakly 2-absorbing ideal of R for each $m_\lambda \in h(M)\backslash N$, then N is a graded weakly classical 2-absorbing submodule of M.*

Proof. (i) Let $m_\lambda \in h(M)\backslash N$, so $(N :_R m_\lambda)$ is a proper graded ideal of R. Now, let $r_g, s_h, t_\alpha \in h(R)$ with $0 \neq r_g s_h t_\alpha \in (N :_R m_\lambda)$. Since $Ann_R(m_\lambda) = \{0\}$, $0 \neq r_g s_h t_\alpha m_\lambda \in N$. Hence, we get either $r_g s_h m_\lambda \in N$ or $r_g t_\alpha m_\lambda \in N$ or $s_h t_\alpha m_\lambda \in N$ as N is a graded weakly classical 2-absorbing submodule of M. This yields that either $r_g s_h \in (N :_R m_\lambda)$ or $r_g t_\alpha \in (N :_R m_\lambda)$ or $s_h t_\alpha \in (N :_R m_\lambda)$. Therefore, $(N :_R m_\lambda)$ is a graded weakly 2-absorbing ideal of R.

(ii) Let $r_g, s_h, t_\alpha \in h(R)$ and $m_\lambda \in h(M)$ such that $0 \neq r_g s_h t_\alpha m_\lambda \in N$. If $m_\lambda \in N$, the we get the result. So, we assume $m_\lambda \notin N$, then $(N :_R m_\lambda)$ is a graded weakly 2-absorbing ideal of R. Hence, $0 \neq r_g s_h t_\alpha \in (N :_R m_\lambda)$ which yields that $r_g s_h \in (N :_R m_\lambda)$ or $r_g t_\alpha \in (N :_R m_\lambda)$ or $s_h t_\alpha \in (N :_R m_\lambda)$ and then either $r_g s_h m_\lambda \in N$ or $r_g t_\alpha m_\lambda \in N$ or $s_h t_\alpha m_\lambda \in N$. Therefore, N is a graded weakly classical 2-absorbing submodule of M. □

A graded zero-divisor on a graded R-module M is an element $r \in h(R)$ for which there exists $m \in h(M)$ such that $m \neq 0$ but $rm = 0$. The set of all graded zero-divisors on M is denoted by $G\text{-}Zdv_R(M)$.

The following result studies the behavior of graded weakly classical 2-absorbing submodules under localization.

Theorem 2.13. *Let R be a G-graded ring, M a graded R-module, $S \subseteq h(R)$ a multiplication closed subset of R and N a graded submodule of M. Then, the following statements hold.*

(i) *If N is a graded weakly classical 2-absorbing submodule of M and $(N :_R M) \cap S = \emptyset$, then $S^{-1}N$ is a graded weakly classical 2-absorbing submodule of $S^{-1}M$.*

(ii) *If $S^{-1}N$ is a graded weakly classical 2-absorbing submodule of $S^{-1}M$ such that $S \cap G\text{-}Zdv_R(N) = \emptyset$ and $S \cap G\text{-}Zdv_R(M/N) = \emptyset$, then N is a graded weakly classical 2-absorbing submodule of M.*

Proof. (i) Suppose that N is a graded weakly classical 2-absorbing submodule of M. Since $(N :_R M) \cap S = \emptyset$, $S^{-1}N$ is a proper graded submodule of

$S^{-1}M$. Now, let $\frac{r_g}{s_1}, \frac{s_h}{s_2}, \frac{t_\alpha}{s_3} \in h(S^{-1}R)$ and $\frac{m_\lambda}{s_4} \in h(S^{-1}M)$ such that $0_{S^{-1}M} \neq \frac{r_g}{s_1}\frac{s_h}{s_2}\frac{t_\alpha}{s_3}\frac{m_\lambda}{s_4} = \frac{r_g s_h t_\alpha m_\lambda}{s_1 s_2 s_3 s_4} \in S^{-1}N$. Hence, there exists $s_5 \in S$ such that $s_5 r_g s_h t_\alpha m_\lambda \in N$. If $s_5 r_g s_h t_\alpha m_\lambda = 0_M$, then $\frac{r_g}{s_1}\frac{s_h}{s_2}\frac{t_\alpha}{s_3}\frac{m_\lambda}{s_4} = \frac{s_5 r_g s_h t_\alpha m_\lambda}{s_5 s_1 s_2 s_3 s_4} = 0_{S^{-1}M}$, a contradiction. So, $0_M \neq s_5 r_g s_h t_\alpha m_\lambda \in N$. This yields that either $s_5 r_g s_h m_\lambda \in N$ or $s_5 r_g t_\alpha m_\lambda \in N$ or $s_5 s_h t_\alpha m_\lambda \in N$. Thus, either $\frac{r_g}{s_1}\frac{s_h}{s_2}\frac{m_\lambda}{s_4} = \frac{s_5 r_g s_h m_\lambda}{s_5 s_1 s_2 s_4} \in S^{-1}N$ or $\frac{r_g}{s_1}\frac{t_\alpha}{s_3}\frac{m_\lambda}{s_4} = \frac{s_5 r_g t_\alpha m_\lambda}{s_5 s_1 s_3 s_4} \in S^{-1}N$ or $\frac{s_h}{s_2}\frac{t_\alpha}{s_3}\frac{m_\lambda}{s_4} = \frac{s_5 s_h t_\alpha m_\lambda}{s_5 s_2 s_3 s_4} \in S^{-1}N$. Therefore, $S^{-1}N$ is a graded weakly classical 2-absorbing submodule of $S^{-1}M$.

(ii) Let $r_g, s_h, t_\alpha \in h(R)$ and $m_\lambda \in h(M)$ such that $0_M \neq r_g s_h t_\alpha m_\lambda \in N$. Hence, $\frac{r_g}{1}\frac{s_h}{1}\frac{t_\alpha}{1}\frac{m_\lambda}{1} \in S^{-1}N$. If $\frac{r_g}{1}\frac{s_h}{1}\frac{t_\alpha}{1}\frac{m_\lambda}{1} = 0_{S^{-1}M}$, then there exists $s \in S$ with $s r_g s_h t_\alpha m_\lambda = 0_M$, but $S \cap G\text{-}Zdv_R(N) = \emptyset$, a contradiction. So, $0_{S^{-1}M} \neq \frac{r_g}{1}\frac{s_h}{1}\frac{t_\alpha}{1}\frac{m_\lambda}{1} \in S^{-1}N$. Thus, either $\frac{r_g}{1}\frac{s_h}{1}\frac{m_\lambda}{1} \in S^{-1}N$ or $\frac{r_g}{1}\frac{t_\alpha}{1}\frac{m_\lambda}{1} \in S^{-1}N$ or $\frac{s_h}{1}\frac{t_\alpha}{1}\frac{m_\lambda}{1} \in S^{-1}N$ as $S^{-1}N$ is a graded weakly classical 2-absorbing submodule of $S^{-1}M$. If $\frac{r_g}{1}\frac{s_h}{1}\frac{m_\lambda}{1} \in S^{-1}N$, then there exists $s \in S$ with $s r_g s_h m_\lambda \in N$ and this follows that $r_g s_h m_\lambda \in N$ since $S \cap G\text{-}Zdv_R(M/N) = \emptyset$. Similarly, if either $\frac{r_g}{1}\frac{t_\alpha}{1}\frac{m_\lambda}{1} \in S^{-1}N$ or $\frac{s_h}{1}\frac{t_\alpha}{1}\frac{m_\lambda}{1} \in S^{-1}N$, then $r_g t_\alpha m_\lambda \in N$ or $s_h t_\alpha m_\lambda \in N$. Therefore, N is a graded weakly classical 2-absorbing submodule of M. □

Theorem 2.14. *Let R be a G-graded ring, M_1 and M_2 be two graded R-modules and N_1 and N_2 be two proper graded submodules of M_1 and M_2, respectively. Let $M = M_1 \times M_2$. Then, the following statements hold.*

(i) *N_1 is a graded weakly classical 2-absorbing submodule of M_1 and for each $r_g, s_h, t_\alpha \in h(R)$ and $m_{1_\lambda} \in h(M_1)$ with $r_g s_h t_\alpha m_{1_\lambda} = 0$, $r_g s_h m_{1_\lambda} \notin N_1$, $r_g t_\alpha m_{1_\lambda} \notin N_1$ and $s_h t_\alpha m_{1_\lambda} \notin N_1$, implies $r_g s_h t_\alpha \in Ann_R(M_{2_\lambda})$ if and only if $N_1 \times M_2$ is a graded weakly classical 2-absorbing submodule of M.*

(ii) *N_2 is a graded weakly classical 2-absorbing submodule of M_2 and for each $r_g, s_h, t_\alpha \in h(R)$ and $m_{2_\lambda} \in h(M_2)$ with $r_g s_h t_\alpha m_{2_\lambda} = 0, r_g s_h m_{2_\lambda} \notin N_2$, $r_g t_\alpha m_{2_\lambda} \notin N_2$ and $s_h t_\alpha m_{2_\lambda} \notin N_2$, implies $r_g s_h t_\alpha \in Ann_R(M_{1_\lambda})$ if and only if $M_1 \times N_2$ is a graded weakly classical 2-absorbing submodule of M.*

Proof. (i) Suppose that $N_1 \times M_2$ is a graded weakly classical 2-absorbing submodule of M. Let $r_g, s_h, t_\alpha \in h(R)$ and $m_{1_\lambda} \in h(M_1)$ such that $0 \neq r_g s_h t_\alpha m_{1_\lambda} \in N_1$. Hence, $(0,0) \neq r_g s_h t_\alpha (m_{1_\lambda}, 0) \in N_1 \times M_2$ and then either $r_g s_h (m_{1_\lambda}, 0) \in N_1 \times M_2$ or $r_g t_\alpha (m_{1_\lambda}, 0) \in N_1 \times M_2$ or $s_h t_\alpha (m_{1_\lambda}, 0) \in N_1 \times M_2$, and so either $r_g s_h m_{1_\lambda} \in N_1$ or $r_g t_\alpha m_{1_\lambda} \in N_1$ or $s_h t_\alpha m_{1_\lambda} \in N_1$. Thus, N_1 is a graded weakly classical 2-absorbing submodule of M_1. Now, let $r_g, s_h, t_\alpha \in h(R)$ and $m_{1_\lambda} \in h(M_1)$ such that $r_g s_h t_\alpha m_{1_\lambda} = 0$ and neither $r_g s_h m_{1_\lambda} \in N_1$ nor $r_g t_\alpha m_{1_\lambda} \in N_1$ nor $s_h t_\alpha m_{1_\lambda} \in N_1$. And assume $r_g s_h t_\alpha \notin Ann_R(M_{2_\lambda})$, then there exists $m_{2_\lambda} \in M_{2_\lambda}$ such that $r_g s_h t_\alpha m_{2_\lambda} \neq 0$. Thus, $(0,0) \neq r_g s_h t_\alpha (m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$, which yields either $r_g s_h (m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$ or $r_g t_\alpha (m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$ or $s_h t_\alpha (m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$ and then either $r_g s_h m_{1_\lambda} \in N_1$ or $r_g t_\alpha m_{1_\lambda} \in N_1$ or $s_h t_\alpha m_{1_\lambda} \in N_1$, a contradiction. Therefore, $r_g s_h t_\alpha \in Ann_R(M_{2_\lambda})$. Conversely, let $r_g, s_h, t_\alpha \in h(R)$ and $(m_{1_\lambda}, m_{2_\lambda}) \in h(M)$ such that $(0,0) \neq r_g s_h t_\alpha (m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$. If $0 \neq r_g s_h t_\alpha m_{1_\lambda} \in N_1$, then either $r_g s_h m_{1_\lambda} \in N_1$ or $r_g t_\alpha m_{1_\lambda} \in N_1$

or $s_h t_\alpha m_{1_\lambda} \in N_1$ as N_1 is a graded weakly classical 2-absorbing submodule of M_1, so either $r_g s_h(m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$ or $r_g t_\alpha(m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$ or $s_h t_\alpha(m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$. Now, if $r_g s_h t_\alpha m_{1_\lambda} = 0$, then $r_g s_h t_\alpha m_{2_\lambda} \neq 0$ and so $r_g s_h t_\alpha \notin Ann_R(M_{2_\lambda})$. Thus, either $r_g s_h m_{1_\lambda} \in N_1$ or $r_g t_\alpha m_{1_\lambda} \in N_1$ or $s_h t_\alpha m_{1_\lambda} \in N_1$ and then either $r_g s_h(m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$ or $r_g t_\alpha(m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$ or $s_h t_\alpha(m_{1_\lambda}, m_{2_\lambda}) \in N_1 \times M_2$. Therefore, $N_1 \times M_2$ is a graded weakly classical 2-absorbing submodule of M.

(ii) The proof is similar to that in part (i). □

Theorem 2.15. *Let R be a G-graded ring, M a graded R-module and N a proper graded submodule of M. If N is a graded weakly classical 2-absorbing submodule of M, then for each $r_g, s_h, t_\alpha \in h(R)$ and $m_\lambda \in h(M)$, then $(N :_R r_g s_h t_\alpha m_\lambda) = (0 :_R r_g s_h t_\alpha m_\lambda) \cup (N :_R r_g s_h m_\lambda) \cup (N :_R r_g t_\alpha m_\lambda) \cup (N :_R s_h t_\alpha m_\lambda)$.*

Proof. Let $r_g, s_h, t_\alpha \in h(R)$ and $m_\lambda \in h(M)$. It is easy to see that $(0 :_R r_g s_h t_\alpha m_\lambda) \cup (N :_R r_g s_h m_\lambda) \cup (N :_R r_g t_\alpha m_\lambda) \cup (N :_R s_h t_\alpha m_\lambda) \subseteq (N :_R r_g s_h t_\alpha m_\lambda)$. Now, let $l_\beta \in (N :_R r_g s_h t_\alpha m_\lambda) \cap h(R)$, then $l_\beta r_g s_h t_\alpha m_\lambda \in N$. If $l_\beta r_g s_h t_\alpha m_\lambda = 0$, then $l_\beta \in (0 :_R r_g s_h t_\alpha m_\lambda)$. If $0 \neq l_\beta r_g s_h t_\alpha m_\lambda \in N$, then either $l_\beta r_g s_h m_\lambda \in N$ or $l_\beta r_g t_\alpha m_\lambda \in N$ or $l_\beta s_h t_\alpha m_\lambda \in N$. Thus, either $l_\beta \in (N :_R r_g s_h m_\lambda)$ or $l_\beta \in (N :_R r_g t_\alpha m_\lambda)$ or $l_\beta \in (N :_R s_h t_\alpha m_\lambda)$. Hence, we get the result. □

Theorem 2.16. *Let R_i be a G-graded ring and M_i a graded R_i-module, for $i = 1, 2$. Let $R = R_1 \times R_2$, $M = M_1 \times M_2$ and $g \in G$ with $M_{2_g} \neq 0$. Suppose that $N = N_1 \times M_2$ is a proper graded submodule of M. Then, the following statements are equivalent:*

(i) *N_{1_g} is a classical g-2-absorbing submodule of an R_{1_e}-module M_{1_g}.*

(ii) *N_g is a classical g-2-absorbing submodule of an R_e-module M_g.*

(iii) *N_g is a weakly classical g-2-absorbing submodule of an R_e-module M_g.*

Proof. $(i) \Rightarrow (ii)$ Let $(r_{1_e}, r_{2_e}), (s_{1_e}, s_{2_e}), (t_{1_e}, t_{2_e}) \in R_e$ and $(m_{1_g}, m_{2_g}) \in M_g$ such that $(r_{1_e}, r_{2_e})(s_{1_e}, s_{2_e})(t_{1_e}, t_{2_e})(m_{1_g}, m_{2_g}) \in N_g$. Then, $r_{1_e} s_{1_e} t_{1_e} m_{1_g} \in N_{1_g}$, so we get either $r_{1_e} s_{1_e} m_{1_g} \in N_{1_g}$ or $r_{1_e} t_{1_e} m_{1_g} \in N_{1_g}$or $s_{1_e} t_{1_e} m_{1_g} \in N_{1_g}$ as N_{1_g} is a classical g-2-absorbing submodule of M_{1_g}.

Hence, either $(r_{1_e}, r_{2_e})(s_{1_e}, s_{2_e})(m_{1_g}, m_{2_g}) \in N_g$ or $(r_{1_e}, r_{2_e})(t_{1_e}, t_{2_e})(m_{1_g}, m_{2_g}) \in N_g$ or $(s_{1_e}, s_{2_e})(t_{1_e}, t_{2_e})(m_{1_g}, m_{2_g}) \in N_g$. Therefore, N_g is a classical g-2-absorbing submodule of M_g.

$(ii) \Rightarrow (iii)$ It is easy to see that every classical g-2-absorbing submodule is a weakly classical g-2-absorbing.

$(iii) \Rightarrow (i)$ Let $r_{1_e}, s_{1_e}, t_{1_e} \in R_{1_e}$ and $m_{1_g} \in M_{1_g}$ such that $r_{1_e} s_{1_e} t_{1_e} m_{1_g} \in N_{1_g}$. Hence, for any $0 \neq m_{2_g} \in M_{2_g}$, we get $0 \neq (r_{1_e}, 1_{2_e})(s_{1_e}, 1_{2_e})(t_{1_e}, 1_{2_e})(m_{1_g}, m_{2_g}) \in N_g$. So, either $(r_{1_e}, 1_{2_e})(s_{1_e}, 1_{2_e})(m_{1_g}, m_{2_g}) \in N_g$ or $(r_{1_e}, 1_{2_e})(t_{1_e}, 1_{2_e})(m_{1_g}, m_{2_g}) \in N_g$ or $(s_{1_e}, 1_{2_e})(t_{1_e}, 1_{2_e})(m_{1_g}, m_{2_g}) \in N_g$. Then, either $r_{1_e} s_{1_e} m_{1_g} \in N_{1_g}$ or $r_{1_e} t_{1_e} m_{1_g} \in N_{1_g}$ or $s_{1_e} t_{1_e} m_{1_g} \in N_{1_g}$. Therefore, N_{1_g} is a classical g-2-absorbing submodule of an R_{1_e}-module M_{1_g}. □

Theorem 2.17. *Let R_i be a G-graded ring, M_i a graded R_i-module and N_i a proper graded submodule of M_i, for $i = 1, 2$. Let $R = R_1 \times R_2$, $M = M_1 \times M_2$, $N = N_1 \times N_2$ and $g \in G$. If N_g is a weakly classical g-2-absorbing submodule of an R_e-module M_g and $N_{2_g} \neq M_{2_g}$, then N_{1_g} is a weakly classical g-prime submodule of an R_{1_e}-module M_{1_g}.*

Proof. Let $r_{1_e}, s_{1_e} \in R_{1_e}$ and $m_{1_g} \in M_{1_g}$ such that $0 \neq r_{1_e} s_{1_e} m_{1_g} \in N_{1_g}$. Since $N_{2_g} \neq M_{2_g}$, there exists $m_{2_g} \in M_{2_g} \backslash N_{2_g}$. Hence,

$$(0_{1_g}, 0_{2_g}) \neq (r_{1_e}, 1_{2_e})(s_{1_e}, 1_{2_e})(1_{1_e}, 0_{2_e})(m_{1_g}, m_{2_g}) = (r_{1_e} s_{1_e} m_{1_g}, 0_{2_g}) \in N_g.$$

This implies that either $(r_{1_e}, 1_{2_e})(1_{1_e}, 0_{2_e})(m_{1_g}, m_{2_g}) \in N_g$ or

$$(s_{1_e}, 1_{2_e})(1_{1_e}, 0_{2_e})(m_{1_g}, m_{2_g}) \in N_g$$

as N_g is a weakly classical g-2-absorbing submodule of M_g and $m_{2_g} \notin N_{2_g}$. Thus, either $r_{1_e} m_{1_g} \in N_{1_g}$ or $s_{1_e} m_{1_g} \in N_{1_g}$. Therefore, N_{1_g} is a weakly classical g-prime submodule of an R_{1_e}-module M_{1_g}. □

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