Some separation axioms via nano S_β-open sets in nano topological spaces

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Abstract. In this present study, we shed light on some separation axioms via nano S_β-open sets including nano S_β-regular, S_β-normal, $S_\beta - S_0$ and $S_\beta - S_1$ axioms in nano topological spaces where nano S_β-open set is defined and related to nano semi-open and nano β-closed sets. Here, we implement each axiom on the family of all nano S_β-open sets according to upper and lower approximations in which there exist exactly six families of nano S_β-open sets. This research work brings out some interesting results such as it is shown that in which condition a nano topological space is always nano S_β-normal space where upper and lower approximations are leading conditions. In addition, the relationship among those axioms is also considered.

Keywords: nano S_β-open sets, nano S_β-regular, nano S_β-normal, nano $S_\beta - S_0$, nano $S_\beta - S_1$.

1. Introduction

The concept of nano topological space is introduced by Thivagar and Richard [2] with respect to a subset X of U as the universe. Then, some types of nano open sets are defined and introduced such as nano semi-open sets, nano α-open sets and nano pre-open sets in [2] and nano *rare* sets by Thivagar et al., [7]. After that, nano β-open sets are introduced by Revathy and Ilango [3]. By using nano semi-open sets with nano β-open sets, nano S_β-open sets are introduced by Pirbal and Ahmed [4]. Moreover, regarding the structure of nano S_β-open sets, nano S_C-open sets defined by Pirbal and Ahmed [10]. The authors of [4] studied connectedness by using nano S_β-open sets in [11]. In addition, some separation

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axioms via nano β-open sets studied by Ghosh [8] and almost nano regular space by David et al., [9]. So, in this study, since separation axioms are the main tool to distinguish two points, two sets or a point with a set topologically, it was such an inspiration for the authors to introduce some separation axioms such as nano S_β-regular, S_β-normal, $S_\beta - S_0$ and $S_\beta - S_1$ axioms in nano topological spaces. Then, each axiom is applied on the family of all nano S_β-open sets in terms of upper and lower approximations.

2. Preliminaries

Definition 2.1 ([1])**.** Let U be a non-empty finite set of objects called the universe and R be an equivalence relation on U called as the indiscernibility relation. Elements belonging to the same equivalence class are said to be indiscernible with one another. The pair $(U,\ R)$ is said to be the approximation space. Let $X \subseteq U$:

1. The lower approximation of X with respect to R is the set of all objects which can be for certain classified as X with respect to R and it is denoted by $L_R(X)$. That is, $L_R(X) = \bigcup_{x\in U} \{R(x); R(x) \subseteq X\}$, where $R(x)$ denotes the equivalence class determined by x.
2. The upper approximation of X with respect to R is the set of all objects which can be possibly classified as X with respect to R and it is denoted by $U_R(X)$. That is, $U_R(X) = \bigcup_{x\in U} \{R(x); R(x) \cap X \neq \phi\}$.
3. The boundary region of X with respect to R is the set of all objects which can be classified neither as X nor as not-X with respect to R and it is denoted by $B_R(X)$. That is, $B_R(X) = U_R(X) - L_R(X)$.

Definition 2.2 ([2])**.** Let U be the universe and R be an equivalence relation on U and $\tau_R(X) = \{\phi,\ U,\ L_R(X),\ U_R(X),\ B_R(X)\}$ where $X \subseteq U$. Then, $\tau_R(X)$ satisfies the followings axioms:

1. U and $\phi \in \tau_R(X)$;
2. the union of elements of any subcollection of $\tau_R(X)$ is in $\tau_R(X)$;
3. the intersection of elements of any finite subcollection of $\tau_R(X)$ is in $\tau_R(X)$.

That is, $\tau_R(X)$ forms a topology on U and called the nano topology on U with respect to X. We call $(U,\ \tau_R(X))$ as the nano topological space. The elements of $\tau_R(X)$ are called as nano open sets and $[\tau_R(X)]^c$ is called as the nano dual topology of $\tau_R(X)$.

Definition 2.3. Let $(U,\ \tau_R(X))$ be a nano topological space and $A \subseteq U$. The set A is said to be:

1. Nano semi-open [2], if $A \subseteq ncl\,(nint\,(A))$.
2. Nano β-open (nano semi pre-open) [3], if $A \subseteq ncl\,(nint\,(ncl\,(A)))$.
3. Nano S_β-open [4], if A is nano semi-open and $A = \cup\{F_\alpha;\ F_\alpha$ nano β-closed sets$\}$.

The set of all nano semi-open, nano β-open and nano S_β-open sets denoted by $nSO\,(U,\ X)$, $n\beta O\,(U,\ X)$ and $nS_\beta O\,(U,\ X)$.

Theorem 2.4 ([5]). *Let A be any subset of a nano topological space $(U, \tau_R(X))$, then:*

1. $nS_\beta int(A) = \cup\{G : G$ *is* nS_β*-open and* $G \subseteq A\}$*;*
2. $nS_\beta cl(A) = \cap\{F : F$ *is* nS_β*-closed and* $A \subseteq F\}$*;*

Theorem 2.5 ([4]). *If $U_R\,(X) = U$ and $L_R\,(X) = \phi$ in a nano topological space $(U,\ \tau_R\,(X))$, then $nS_\beta O\,(U, X) = \{U,\ \phi\}$.*

Theorem 2.6 ([4]). *If $U_R\,(X) = U$ and $L_R\,(X) \neq \phi$ in a nano topological space $(U,\ \tau_R\,(X))$, then $\tau_R\,(X) = \tau_R^{S_\beta}\,(X)$.*

Theorem 2.7 ([4]). *Let $(U,\ \tau_R\,(X))$ be a nano topological space. If $U_R\,(X) = L_R\,(X) = \{x\}$, $x \in U$, then $nS_\beta O\,(U, X) = \{\phi,\ U\}$.*

Theorem 2.8 ([4]). *Let $(U,\ \tau_R\,(X))$ be a nano topological space. If $U_R\,(X) = L_R(X) \neq U$ and $U_R\,(X)$ contains more than one element of U, then the set of all nS_β-open sets in U are ϕ and those sets A for which $U_R(X) \subseteq A$.*

Theorem 2.9 ([4]). *Let $(U,\ \tau_R\,(X))$ be a nano topological space. If $U_R\,(X) \neq U$, $L_R\,(X) = \phi$ and $U_R\,(X)$ contains more than one element of U, then the set of all nS_β-open sets in U are ϕ and those sets A for which $U_R(X) \subseteq A$.*

Theorem 2.10 ([4]). *Let $(U,\ \tau_R\,(X))$ be a nano topological space. If $U_R\,(X) \neq L_R(X)$ where $U_R(X) \neq U$ and $L_R\,(X) \neq \phi$, then ϕ, $L_R\,(X)$, $B_R\,(X)$, $L_R\,(X) \cup B$, $B_R\,(X) \cup B$ and any set containing $U_R\,(X)$ where $B \subseteq [U_R\,(X)]^c$ are the only nS_β-open sets in U.*

Theorem 2.11 ([5]). *Let $(U,\ \tau_R\,(X))$ be a nano topological space. If $U_R\,(X) = L_R(X) \neq U$ and $U_R\,(X)$ contains more than one element of U, then for any non-empty subset A of U:*

$$nS_\beta cl\,(A) = \begin{cases} A, & if\ A \subset [U_R\,(X)]^c \\ U, & otherwise \end{cases}.$$

Theorem 2.12. *Let $(U,\ \tau_R\,(X))$ be a nano topological space. The only nS_β-clopen subset of U are ϕ and U if:*

1. *$U_R(X) = U$ and $L_R(X) = \phi$;*

2. *$U_R(X) = L_R(X) = \{x\}$, $x \in U$;*

3. *$U_R(X) = L_R(X) \neq U$ and $U_R(X)$ contains more than one element of U;*

4. *$U_R(X) \neq U$, $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of U.*

Proof. Obvious. □

Theorem 2.13. *If $U_R(X) = U$ and $L_R(X) \neq \phi$ in a nano topological space $(U,\ \tau_R(X))$, then*

$$\left[\tau\begin{matrix}S_\beta\\R\end{matrix}(X)\right] = \left[\tau\begin{matrix}S_\beta\\R\end{matrix}(X)\right]^c$$

.

Proof. Obvious. □

Theorem 2.14. *Let $(U,\ \tau_R(X))$ be a nano topological space. If $U_R(X) \neq L_R(X)$, where $U_R(X) \neq U$ and $L_R(X) \neq \phi$, then $L_R(X)$, $B_R(X)$, $L_R(X) \cup B$ and $B_R(X) \cup B$ are non-empty proper nS_β-clopen in U where $B \subseteq [U_R(X)]^c$.*

Proof. By Theorem 2.10, $L_R(X)$, $B_R(X)$, $L_R(X) \cup B$ *and* $B_R(X) \cup B$ are non-empty proper nS_β-open set in U where $B \subseteq [U_R(X)]^c$. We have to show that they are also nS_β-closed in U.

Now, $nS_\beta cl(L_R(X)) = nS_\beta cl([B_R(X) \cup B]^c)$ where $B = [U_R(X)]^c$, but $[B_R(X) \cup B]^c$ is nS_β-closed, so $nS_\beta cl([B_R(X) \cup B]^c) = [B_R(X) \cup B]^c = L_R(X)$. Also, $nS_\beta cl(B_R(X)) = nS_\beta cl([L_R(X) \cup B]^c)$, where $B \subseteq [U_R(X)]^c$, but $[L_R(X) \cup B]^c$ is nS_β-closed, so $nS_\beta cl([L_R(X) \cup B]^c) = [L_R(X) \cup B]^c = B_R(X)$.

Also, $nS_\beta cl(L_R(X) \cup B) = nS_\beta cl([B_R(X)]^C) = [B_R(X)]^C = L_R(X) \cup B$ and $nS_\beta cl(B_R(X) \cup B) = nS_\beta cl([L_R(X)]^C) = [L_R(X)]^C = B_R(X) \cup B$, where $B \subset [U_R(X)]^c$. Hence, $L_R(X)$, $B_R(X)$, $L_R(X) \cup B$ and $B_R(X) \cup B$ are nS_β-clopen where $B \subseteq [U_R(X)]^c$. □

3. Nano S_β-regular spaces

Definition 3.1. A nano topological space $(U,\ \tau_R(X))$ is said to be nS_β-regular if for each $x \in U$ and each nano closed set A such that $x \notin$A, there exist two nS_β-open sets G and H such that $x \in G, A \subseteq H$ and $G \cap H = \phi$.

Remark 3.2. Nano indiscrete topological space is nS_β-regular space.

Theorem 3.3. *Let $(U,\ \tau_R(X))$ be a nano topological space. Then, U is a nS_β-regular space if and only if for each $x \in U$ and each nano open set G containing x, there exist a nS_β-open set V containing x such that $x \in V \subseteq nS_\beta cl(V) \subseteq G$.*

Proof. Let G be a nano open set and $x\in G$. Then, $U-$G is nano closed set such that $x\notin U-G$. By nS_β-regularity of U, there are nS_β-open sets M and W such that $x\in M$, $U-G\subseteq$W and $M\cap$W $=\phi$. Therefore, $x\in M\subseteq U-$W$\subseteq G$, Hence, $x\in M\subseteq nS_\beta cl\,(M)\subseteq nS_\beta cl\,(U-\text{W})=U-\text{W}\subseteq G$. Thus, $nS_\beta cl(M)\subseteq U-\text{W}\subseteq G$. Conversely, let F be nano closed set in U and let $x\notin F$. Then, $U-F$ is a nano open set and $x\in U-F$. By assumption, there exist a nS_β-open set H such that $x\in H$ and $nS_\beta cl(H)\subseteq U-F$. Define $K=U-nS_\beta cl(H)$. Then, $K\in nS_\beta O(U,X)$ and $H\subseteq nS_\beta cl\,(H)$, then $H\cap K=H\cap(U-nS_\beta cl(H))=\phi$ $(U-nS_\beta cl(H)\subseteq U-H)$. Thus, for $x\notin F$,$\exists$ disjoint nS_β-open sets H and K such that $x\in H$ and $F\subseteq K$. Hence, U is a nS_β-regular space. □

Theorem 3.4. *Let $(U,\ \tau_R\,(X))$ be a nano topological space, then U is nS_β-regular if:*

1. *$U_R\,(X)=U$ and $L_R\,(X)=\phi$;*
2. *$U_R\,(X)=U$ and $L_R(X)\neq\phi$;*
3. *$U_R\,(X)\neq L_R(X)$, where $U_R(X)\neq U$ and $L_R\,(X)\neq\phi$.*

Proof.

1. By Theorem 2.5, $\tau_R\,(X)\ =\ nS_\beta O\,(U,X)=\{\phi,\,U\}$. Hence, U is nS_β-regular.
2. By Theorem 2.6, $\tau_R\,(X)\ =nS_\beta O\,(U,X)=\{\phi,\,U,\,L_R\,(X)\,,\,B_R\,(X)\}$. Let $x\in U$, since $L_R\,(X)\cap B_R\,(X)=\phi$ and $L_R\,(X)\cup B_R\,(X)=U$, then either $x\in L_R\,(X)$ or $x\in B_R\,(X)$. Also, $L_R\,(X)=[B_R\,(X)]^c$. Let say $x\in L_R\,(X)$, then $x\in\ L_R\,(X)\subseteq nS_\beta cl(L_R\,(X))=[B_R\,(X)]^c\subseteq L_R\,(X)$. If $x\in B_R(X)$, then $x\in\ B_R\,(X)\subseteq nS_\beta cl(B_R\,(X))=[L_R\,(X)]^c\subseteq B_R\,(X)$. Hence, U is nS_β-regular.
3. Let $x\in U_R(X)$, then either $x\in L_R\,(X)$ or $x\in B_R\,(X)$. Since by Theorem 2.14 $L_R\,(X)$ and $B_R(X)$ are nS_β-clopen in U, then let $x\in U_R(X)$:

   If $x\in L_R\,(X)$, then

   $$x\in L_R\,(X)\subseteq nS_\beta cl\,(L_R\,(X))\subseteq\begin{cases}U_R(X)\\L_R(X)\end{cases}.$$

   If $x\in B_R(X)$, then

   $$x\in B_R\,(X)\subseteq nS_\beta cl\,(B_R\,(X))\subseteq\begin{cases}U_R(X)\\L_R(X)\end{cases}.$$

   If $x\notin U_R(X)$, then the only nano open set containing x is U.

   Therefore, U is nS_β-regular space. □

Remark 3.5. Let $(U,\ \tau_R(X))$ be a nano topological space, then U is not nS_β-regular if:

1. $U_R(X) = L_R(X) \neq U$ and $U_R(X)$ contains more than one element of U. Since $\tau_R(X) = \{\phi,\ U,\ U_R(X)\}$ and by Theorem 2.8, ϕ and those subsets A for which $U_R(X) \subseteq A$ are nS_β-open sets in U. Let $x \in U_R(X)$, then there is no nS_β-open set V such that $x \in V \subseteq nS_\beta cl(V) \subseteq U_R(X)$, since by Theorem 2.11, $nS_\beta cl(V) = U$. Hence, U is not nS_β-regular space.

2. $U_R(X) \neq U$, $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of U. Since $\tau_R(X) = \{\phi,\ U,\ U_R(X)\}$ and by Theorem 2.9, ϕ and those subsets A for which $U_R(X) \subseteq A$ are nS_β-open sets in U. Let $x \in U_R(X)$, then there is no nS_β-open set V such that $x \in V \subseteq nS_\beta cl(V) \subseteq U_R(X)$, since by Theorem 2.11, $nS_\beta cl(V) = U$. Hence, U is not nS_β-regular space.

3. $U_R(X) = L_R(X) \neq U$ and $U_R(X) = \{x\}$, $x \in U$. Since $\tau_R(X) = \{\phi,\ U, \{x\}\}$ and by Theorem 2.7, $nS_\beta O(U, X) = \{\phi,\ U\}$. Then, there is no nS_β-open set V such that $x \in V \subseteq nS_\beta cl(V) \subseteq \{x\}$. Hence, U is not nS_β-regular space.

4. Nano S_β-normal spaces

Definition 4.1. A nano topological space U is said to be nS_β-normal if for any disjoint nano closed sets A, B of U, there exist nS_β-open sets G and H such that $A \subseteq G, B \subseteq H$ and $G \cap H = \phi$.

Theorem 4.2. *A topological space U is nS_β-normal if and only if for each nano closed set F in U and nano open set G containing F, there is an nS_β-open set H such that $F \subseteq H \subseteq nS_\beta cl(H) \subseteq G$.*

Proof. Suppose that G is nano open set containing F, then $U - G$ and F are disjoint nano closed sets in U. Since U is nS_β-normal, so there exist nS_β-open sets H and V such that $F \subseteq H, U - G \subseteq V$ and $H \cap V = \phi$. Hence, $F \subseteq H \subseteq nS_\beta cl(H) \subseteq nS_\beta cl(U - V) = U - V \subseteq G$, or $F \subseteq H \subseteq nS_\beta cl(H) \subseteq G$.

Conversely, assume that for any nano-closed F and nano open set G containing F, there exists a nS_β-open set H such that $F \subseteq H \subseteq nS_\beta cl(H) \subseteq G$. Let F and K be disjoint nS_β-closed sets in U. So $F \cap K = \phi$ then $F \subseteq U - K$.As F is a nS_β-closed set and $U - K$ is a nS_β-open set, by assumption $\exists$ nS_β-open sets H in U such that, $F \subseteq H \subseteq nS_\beta cl(H) \subseteq U - K$. We get $K \subseteq U - nS_\beta cl(H)$. Define $G = U - nS_\beta cl(H)$. Thus $\exists$ $G, H \in nS_\beta O(U, X)$ such that $F \subseteq H, K \subseteq G$ and $H \cap G = \phi$. Hence, U is a nS_β-normal space. □

Theorem 4.3. *Let $(U,\ \tau_R(X))$ be a nano topological space, then U is nS_β-normal if:*

1. *$U_R(X) = U$ and $L_R(X) = \phi$;*

2. *$U_R(X)=U$ and $L_R(X)\neq\phi$;*

3. *$U_R(X)=L_R(X)=\{x\}$, $x\in U$;*

4. *$U_R(X)=L_R(X)\neq U$ and $U_R(X)$ contains more than one element of U;*

5. *$U_R(X)\neq U$, $L_R(X)=\phi$ and $U_R(X)$ contains more than one element of U;*

6. *$U_R(X)\neq L_R(X)$, where $U_R(X)\neq U$ and $L_R(X)\neq\phi$.*

Proof.

1. Since $\tau_R(X)=\{\phi, U\}$. By Theorem 2.5, $\tau_R(X)=nS_\beta O(U,X)=\{\phi, U\}$. Hence, U is nS_β-normal space.

2. Since $\tau_R(X)=\{\phi, U, L_R(X), B_R(X)\}$. By Theorem 2.6, $\tau_R(X)=nS_\beta O(U,X)=\{\phi, U, L_R(X), B_R(X)\}$. Since $L_R(X)\cap B_R(X)=\phi$, $L_R(X)\cup B_R(X)=U$ and $L_R(X)=[B_R(X)]^c$. Hence, U is nS_β-normal space.

3. Since $\tau_R(X)=\{\phi, U,\{x\}\}$ and by Theorem 2.7, $nS_\beta O(U,X)=\{\phi, U\}$. Then, it is clear that U is nS_β-normal space.

4. Since $\tau_R(X)=\{\phi, U, U_R(X)\}$, by Theorem 2.8, ϕ, U and those sets A for which $U_R(X)\subseteq A$ are nS_β-open sets in U. In this case, 'ϕ with $[U_R(X)]^c$' and 'ϕ with U' are the disjoint nano closed sets in U. For 'ϕ with $[U_R(X)]^c$, $\phi\subseteq\phi$ and $[U_R(X)]^c\subseteq U$ and for 'ϕ and U' the result is clear. Hence, U is nS_β-normal.

5. Similar to part (i).

6. Since the only disjoint nano closed sets are ϕ and U. Therefore, U is nS_β-normal space. □

5. Nano S_β-S$_0$ and S_β-S$_1$ spaces

Definition 5.1. A nano topological space $(U, \tau_R(X))$ is called nS_β-S_0 if for every non-empty nS_β- open set A, $A\subseteq nS_\beta cl(\{x\})$, $\forall x\in A$.

Definition 5.2. A topological space $(U, \tau_R(X))$ is called nS_β-S_1 if for any distinct points $x,y\in U$ with $nS_\beta cl(\{x\})\neq nS_\beta cl(\{y\})$, there exist non-empty disjoint nS_β-open sets G and H such that, $G\subseteq nS_\beta cl(\{x\})$ and $H\subseteq nS_\beta cl(\{y\})$.

Theorem 5.3. *Let $(U, \tau_R(X))$ be a nano topological space. Then, U is nS_β-S_0 space if:*

1. *$U_R(X)=U$ and $L_R(X)=\phi$;*

2. *$U_R(X) = L_R(X) \neq U$ and $U_R(X) = \{x\}$, $x \in U$;*
3. *$U_R(X) = U$ and $L_R(X) \neq \phi$;*
4. *$U_R(X) \neq U$ and $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of U;*
5. *$U_R(X) = L_R(X) \neq U$ and $U_R(X)$ contains more than one element of U.*

Proof.

1. Since $nS_\beta O(U, X) = \{\phi, U\}$. Then, only non-empty nS_β-open subset is U and $U = nS_\beta cl(\{x\})$, $\forall x \in U$. Hence, U is nS_β-S_0 space.
2. Since $nS_\beta O(U, X) = \{\phi, U\}$. Then, only non-empty nS_β-open subset is U and $U \subseteq nS_\beta cl(\{x\})$, $\forall x \in U$. Hence, U is nS_β-S_0 space.
3. Since $nS_\beta O(U, X) = \{\phi, U, L_R(X), B_R(X)\}$, then
$$L_R(X) \subseteq nS_\beta cl(\{x\}) = L_R(X), \forall x \in L_R(X)$$
and similarly for $B_R(X)$. Hence, U is nS_β-S_0.
4. Since ϕ, U and those sets A for which $U_R(X) \subseteq A$ are nS_β-open sets in U, but $nS_\beta cl(\{x\}) = U$, $\forall x \in U_R(X)$, then $U_R(X) \subseteq U$. Hence, U is nS_β-S_0 space.
5. Since ϕ, U and those sets A for which $U_R(X) \subseteq A$ are nS_β-open sets in U, but $nS_\beta cl(\{x\}) = U$, $\forall x \in U_R(X)$, then $U_R(X) \subseteq U$. Hence, U is nS_β-S_0 space. □

Remark 5.4. Let $(U, \tau_R(X))$ be a nano topological space. Then, U is not nS_β-S_0 space if $U_R(X) \neq L_R(X)$ where $U_R(X) \neq U$ and $L_R(X) \neq \phi$. Since $U_R(X) \in nS_\beta O(U, X)$ but $U_R(X) \nsubseteq nS_\beta cl(\{x\})$ for any $x \in U_R(X)$. Hence, U is not nS_β-S_0 space.

Theorem 5.5. *Let $(U, \tau_R(X))$ be a nano topological space. Then, U is nS_β-S_1 space if:*

1. *$U_R(X) = U$ and $L_R(X) = \phi$;*
2. *$U_R(X) = L_R(X) = \{x\}$, $x \in U$;*
3. *$U_R(X) = U$ and $L_R(X) \neq \phi$.*

Proof.

1. Obvious.
2. Obvious.

3. Since $nS_\beta O(U,X)=\{\phi, U, L_R(X), B_R(X)\}$ and $L_R(X)\cap B_R(X) = \phi$ also $L_R(X) \subseteq nS_\beta cl(\{x\}) = L_R(X), \forall x \in L_R(X)$ and $B_R(X) \subseteq nS_\beta cl(\{x\}) = B_R(X), \forall x \in B_R(X)$, then for any $x \in L_R(X)$ and $y \in B_R(X)$, $nS_\beta cl(\{x\}) \neq nS_\beta cl(\{y\})$ and $L_R(X) \subseteq nS_\beta cl(\{x\})$ and $B_R(X) \subseteq nS_\beta cl(\{y\})$. Hence, U is nS_β-S_1 space. □

Remark 5.6. Let $(U, \tau_R(X))$ be a nano topological space. Then, U is not nS_β-S_1 space if:

1. $U_R(X) \neq U$ and $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of U.

   Since for any distinct points $x, y \in U$ with $nS_\beta cl(\{x\}) \neq nS_\beta cl(\{y\})$, there is no non-empty disjoint nS_β-open sets G and H such that, $G \subseteq nS_\beta cl(\{x\})$ and $H \subseteq nS_\beta cl(\{y\})$, since every nS_β-open set containing $U_R(X)$. Hence, U is not nS_β-S_1 space.

2. Similar to part (i).

3. $U_R(X) \neq L_R(X)$ where $U_R(X) \neq U$ and $L_R(X) \neq \phi$. Since any non-empty proper subset A of U with less than one element of U is nS_β-open set and its complement is singleton nS_β-closed, then $nS_\beta cl(\{x\}) = \{x\}$, for any $x \in [U_R(X)]^c$. Also, $nS_\beta cl(\{y\}) = L_R(X)$, for any $y \in L_R(X)$, then $nS_\beta cl(\{x\}) \neq nS_\beta cl(\{y\})$, but there is no non-empty nS_β-open set such that $G \subseteq nS_\beta cl(\{x\})$. Hence, U is not nS_β-S_1 space.

Theorem 5.7. *Every nS_β-S_1 space is nS_β-S_0.*

Proof. The proof follows form Theorem 5.3 and Theorem 5.5. □

The converse of above theorem need not to be true, as it shown by the following example.

Example 5.8. Let $U = \{a, b, c\}$ with $U/R = \{\{a, b\}, \{c\}\}$ and $X = \{a, b\}$. Then, $\tau_R(X) = nS_\beta O(U,X) = \{\phi, U, \{a, b\}\}$. Then, $nS_\beta cl(\{a\}) = U$ and $nS_\beta cl(\{c\}) = \{c\}$ which there is no non-empty disjoint nS_β-open sets G and H containing $nS_\beta cl(\{a\})$ and $nS_\beta cl(\{c\})$ respectively. Hence, U is not nS_β-S_1 space.

6. Conclusion

In this paper, we have introduced the concepts of nano S_β-regular, S_β-normal, $S_\beta - S_0$ and $S_\beta - S_1$ axioms in nano topological spaces. According to the family of all nano S_β-open sets, the axioms are studied and the relationship among the axioms presented in the table below. For instance, we can see that every nS_β-regular space is nS_β-normal but the converse is proved that is not true in three cases.

Family of nS_β-open sets in term of upper and lower approximations if:	nS_β-regular	nS_β-normal	$nS_\beta - S_0$	$nS_\beta - S_1$
$U_R(X) = U$ and $L_R(X) = \phi$	1	1	1	1
$U_R(X) = U$ and $L_R(X) \neq \phi$	1	1	1	1
$U_R(X) = L_R(X) = \{x\}$, $x \in U$	0	1	1	1
$U_R(X) = L_R(X) \neq U$ and $U_R(X)$ contains more than one element of U.	0	1	1	0
$U_R(X) \neq U$, $L_R(X) = \phi$ and $U_R(X)$ contains more than one element of U.	0	1	1	0
$U_R(X) \neq U$, $L_R(X) \neq \phi$ and $U_R(X) \neq L_R(X)$	1	1	0	0

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