On structures of rough topological spaces based on neighborhood systems

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Abstract. Keeping in view the generalized approximation space, the goal of this paper is to suggest and investigate four different styles for approximating rough sets. The proposed approximations are based on various general topologies. In fact, we first generalize the notion of the initial-neighborhood and thus we construct four different topologies generated from these neighborhoods. The relationships between the new neighborhoods (respectively, topologies) and the previous are studied. Comparisons of the degrees of different accuracy of the presented approximations are investigated. The essential characteristics of these operators are obtained.

Keywords: initial-neighborhoods, rough sets, topology.

1. Introduction

The number of research articles published has been rapidly increasing, particularly in Topology and its applications. Several proposals were made for using mathematical methodologies and relevant formulas to solve real-world problems in order to assist decision-makers in making the best decisions possible to deal with unpredictability in challenges (see [1, 4, 5, 6, 13, 14, 17, 18, 19, 20, 21, 22, 23, 24, 28, 30, 31, 36]). In 1982, Pawlak [25] proposed rough set theory as a new mathematical technique or set of simple tools for dealing with ambiguity in knowledge-based systems and data dissection. This theory has a wide range of applications, including process control, economics, medical diagnosis, and others (see [5, 6, 10, 13, 14, 18, 19, 20, 21, 22, 23, 28, 30, 31]). To extend the field of application for this theory, many papers were published (see [1]-[18], [27]-[28], [32]-[36]).

The novel notion "the $\mathcal{J}$-neighborhood space" (in short, $\mathcal{J} - NS$) was suggested by Abd El-Monsef et al. [1] as a general frame of neighborhood space. In fact, they hosted a structure for extending Pawlak's approach [25, 26] and some of the other generalizations. As a result, they devised various rough approximations to fulfill all properties of the rough sets without any constraints.

These methods paved the way for more topological applications of rough sets, as well as assisting in the formalization of many real-worlds applications.

The involvement of this article is to suggest a generalization for the idea of "initial-neighborhood" given by El-Sayed et al. [18]. It must be mentioned that the concept of "initial-neighborhood" was proposed by another notion (namely, "subset neighborhood") by Al-shami and Ciucci [9] in 2022 as an extension of the concept of initial-neighborhood. Hence, we produce four topologies and then we investigate the relationships among these topologies and the previous ones [1, 18]. Accordingly, we achieve four techniques to find the approximations of rough sets. Comparisons of the degrees of different accuracy of the presented approximations are investigated. Therefore, we ascertain that the recommended ways are extra precise than the others.

The present manuscript is prepared as follows: In section 2, we outline the main ideas about the $\mathcal{J}-NS$ cited in [1] and the basic properties of the initial-neighborhood [18]. Section 3 is devoted to introducing and studying new generalizations to the concept of "initial-neighborhood". We define three different types of initial neighborhoods and compare them with the previous one [18]. Moreover, using Theorem 1 in [1], we purpose a new method to generate four different topologies induced by the new neighborhoods. A comparison between these topologies and the previous one is investigated. Finally, in section 4, we use these new topologies to generate new generalizations to Pawlak rough sets and study their properties. We compare the suggested approaches with the previous one [1, 18] and verify that these techniques are more perfect than other approaches.

2. Preliminaries

The central ideas about $\mathcal{J}-NS$ cited in [1] and properties of the initial-neighborhood [18] are provided in the present part.

Definition 2.1 ([1]). Suppose that $\mathcal{U}$ be a non-empty finite set and $\mathcal{R}$ be a binary relation on it. Therefore, we define a $\mathcal{J}$-neighborhood of $x \in \mathcal{U}$, denoted by $\mathcal{N}_{\mathcal{J}}(x), \mathcal{J} \in \{r, \urcorner, \curlywedge, \curlyvee\}$ as follows:

(i) r-neighborhood: $\mathcal{N}_r(x) = \{\mathcal{Y} \in \mathcal{U} : x\mathcal{R}y\}$.

(ii) $\urcorner$-neighborhood: $\mathcal{N}_{\urcorner}(x) = \{\mathcal{Y} \in \mathcal{U} : y\mathcal{R}x\}$.

(iii) $\curlywedge$-neighborhood: $\mathcal{N}_{\curlywedge}(x) = \mathcal{N}_r(x) \cap \mathcal{N}_{\urcorner}(x)$.

(iv) $\curlyvee$-neighborhood: $\mathcal{N}_{\curlyvee}(x) = \mathcal{N}_r(x) \cup \mathcal{N}_{\urcorner}(x)$.

Definition 2.2 ([1]). Consider $\mathcal{R}$ be a binary relation on $\mathcal{U}$ and $\xi_{\mathcal{J}} : \mathcal{U} \to \mathcal{P}(\mathcal{U})$ represents a map that gives for every x in $\mathcal{U}$ its $\mathcal{J}$-neighborhood $\mathcal{N}_{\mathcal{J}}(x)$. Thus, triple $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ is said to be a $\mathcal{J}$-neighborhood space (in briefly, $\mathcal{J} - NS$).

Theorem 2.3 ([1]). *If* $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ *is a* $\mathcal{J} - NS$*, then for each* $\mathcal{J} \in \{r, \upharpoonleft, \curlywedge, \curlyvee\}$ *the collection*

$$\mathcal{T}_{\mathcal{J}} = \{\mathcal{M} \subseteq \mathcal{U} : \forall \mathrm{m} \in \mathcal{M}, \mathcal{N}_{\mathcal{J}}(\mathrm{m}) \subseteq \mathcal{M}\}$$

represents a topology on $\mathcal{U}$*.*

Definition 2.4 ([1]). Consider $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ be a $\mathcal{J} - NS$. The subset $\mathcal{M} \subseteq \mathcal{U}$ is said to be an "$\mathcal{J}$-open set" if $\mathcal{M} \in \mathcal{T}_{\mathcal{J}}$, its complement is an "$\mathcal{J}$-closed set". The family $\mathcal{F}_{\mathcal{J}}$ of all $\mathcal{J}$-closed sets of a $\mathcal{J} - NS$ is defined by $\mathcal{F}_{\mathcal{J}} = \{F \subseteq \mathcal{U} : F^c \in \mathcal{T}_{\mathcal{J}}\}$.

Definition 2.5 ([1]). Suppose that $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ be a $\mathcal{J} - NS$ and $\mathcal{M} \subseteq \mathcal{U}$. The "$\mathcal{J}$-lower" (respectively, "$\mathcal{J}$-upper") approximation of $\mathcal{M}$ is provided by

$$\underline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M}) = \cup\{G \in \mathcal{T}_{\mathcal{J}} : G \subseteq \mathcal{M}\} = int_{\mathcal{J}}(\mathcal{M})$$

(respectively, $\overline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M}) = \cap\{H \in \mathcal{F}_{\mathcal{J}} : \mathcal{M} \subseteq H\} = cl_{\mathcal{J}}(\mathcal{M})$), where $int_{\mathcal{J}}(\mathcal{M})$ (respectively, $cl_{\mathcal{J}}(\mathcal{M})$) is the $\mathcal{J}$-interior of $\mathcal{M}$ (respectively, $\mathcal{J}$-closure of $\mathcal{M}$).

Definition 2.6 ([1]). Let $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ be a $\mathcal{J} - NS$ and $\mathcal{M} \subseteq \mathcal{U}$. Then, for each $\mathcal{J} \in \{r, \upharpoonleft, \curlywedge, \curlyvee\}$, the subset $\mathcal{M}$ is called "$\mathcal{J}$-exact" set if $\underline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M}) = \overline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M}) = \mathcal{M}$. Else, it is "$\mathcal{J}$-rough".

Definition 2.7 ([1]). Consider $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ to be a $\mathcal{J} - NS$ and $\mathcal{M} \subseteq \mathcal{U}$. The "$\mathcal{J}$-boundary", "$\mathcal{J}$-positive" and "$\mathcal{J}$-negative" regions of $\mathcal{M}$ are defined respectively by $\mathcal{B}_{\mathcal{J}}(\mathcal{M}) = \overline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M}) - \underline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M})$, $POS_{\mathcal{J}}(\mathcal{M})= \underline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M})$ and $NEG_{\mathcal{J}}(\mathcal{M}) =U - \overline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M})$.

The "$\mathcal{J}$-accuracy" of $\mathcal{J}$-approximations of $\mathcal{M} \subseteq \mathcal{U}$ is given as follows: $\delta_{\mathcal{J}}(\mathcal{M}) = \frac{|\underline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M})|}{|\overline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M})|}$, where $|\overline{\mathcal{R}}_{\mathcal{J}}(\mathcal{M})| \neq 0$. Clearly, $0 \leq \delta_{\mathcal{J}}(\mathcal{M}) \leq 1$ and if $\delta_{\mathcal{J}}(\mathcal{M}) = 1$, then $\mathcal{M}$ is a $\mathcal{J} - exact$ set. Else,it is $\mathcal{J}$-rough.

3. Topologies generated from neighborhoods

The main ideas of this part is to generalize the concept of "initial-neighborhood [18]" and thus we produce four different topologies from these neighborhoods.

Definition 3.1. For a binary relation $\mathcal{R}$ on $\mathcal{U}$, we define the following neighborhoods of $x \in \mathcal{U}$:

(*i*) r-initial neighborhood [18]: $\mathcal{N}^i_r(x) = \{\mathcal{Y} \in \mathcal{U} : \mathcal{N}_r(x) \subseteq \mathcal{N}_r(\mathcal{Y})\}$;

(*ii*) $\upharpoonleft$-initial neighborhood: $\mathcal{N}^i_{\upharpoonleft}(x) = \{\mathcal{Y} \in \mathcal{U} : \mathcal{N}_{\upharpoonleft}(x) \subseteq \mathcal{N}_{\upharpoonleft}(\mathcal{Y})\}$;

(*iii*) $\curlywedge$-initial neighborhood: $\mathcal{N}^i_{\curlywedge}(x) = \mathcal{N}^i_r(x) \cap \mathcal{N}^i_{\upharpoonleft}(x)$;

(*iv*) $\curlyvee$-initial neighborhood: $\mathcal{N}^i_{\curlyvee}(x) = \mathcal{N}^i_r(x) \cup \mathcal{N}^i_{\upharpoonleft}(x)$.

The next lemmas give the main properties of the above neighborhoods.

Lemma 3.2. *If* $\mathcal{R}$ *is a binary relation on* $\mathcal{U}$*. Then, for each* $\mathcal{J} \in \{r, \upharpoonleft, \curlywedge, \curlyvee\}$*:*

(i) $x \in \mathcal{N}^i_{\mathcal{J}}(x)$.

(ii) $\mathcal{N}^i_{\mathcal{J}}(x) \neq \varphi$.

(iii) *If* $\mathcal{Y} \in \mathcal{N}^i_{\mathcal{J}}(x)$, *then* $\mathcal{N}^i_{\mathcal{J}}(y) \subseteq \mathcal{N}^i_{\mathcal{J}}(x)$, *for each* $\mathcal{J} \in \{r, \upharpoonleft, \curlywedge\}$.

Proof. Firstly, the proof of (i) and (ii) is obvious by Definition 3.1.
(iii) According to Definition 3.1, if $\mathcal{Y} \in \mathcal{N}^i_{\mathcal{J}}(x)$. Then

$$\mathcal{N}_r(x) \subseteq \mathcal{N}_r(y) \tag{1}$$

Now, let $\mathcal{Z} \in \mathcal{N}^i_{\mathcal{J}}(\mathcal{Y})$. Then $\mathcal{N}_r(\mathcal{Y}) \subseteq \mathcal{N}_r(\mathcal{Z})$. Consequently, by(1), $\mathcal{N}_r(x) \subseteq \mathcal{N}_r(\mathcal{Z})$ and this implies $\mathcal{Z} \in \mathcal{N}^i_{\mathcal{J}}(x)$. Consequently, $\mathcal{N}^i_{\mathcal{J}}(\mathcal{Y}) \subseteq \mathcal{N}^i_{\mathcal{J}}(x)$. □

Lemma 3.3. *If* $\mathcal{R}$ *is a binary relation on* $\mathcal{U}$. *Then,* $\forall x \in \mathcal{U}$:

(i) $\mathcal{N}^i_{\curlywedge}(x) \subseteq \mathcal{N}^i_r(x) \subseteq \mathcal{N}^i_{\curlyvee}(x)$.

(ii) $\mathcal{N}^i_{\curlywedge}(x) \subseteq \mathcal{N}^i_{\upharpoonleft}(x) \subseteq \mathcal{N}^i_{\curlyvee}(x)$.

Proof. Straightforward. □

The relationships between the initial-neighborhoods and $\mathcal{J}$-neighborhoods are given by the next lemma.

Lemma 3.4. *Suppose that* $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ *represents a* $\mathcal{J} - NS$. *If* $\mathcal{R}$ *is a reflexive and symmetric relation. Then,* $\forall x \in \mathcal{U}$, $\mathcal{N}^i_{\mathcal{J}}(x) \subseteq \mathcal{N}_{\mathcal{J}}(x)$.

Proof. Let $\mathcal{Y} \in \mathcal{N}^i_{\mathcal{J}}(x)$, then $\mathcal{N}_{\mathcal{J}}(x) \subseteq \mathcal{N}_{\mathcal{J}}(\mathcal{Y})$. But, $\mathcal{R}$ is a reflexive relation which implies $x \subseteq \mathcal{N}_{\mathcal{J}}(x)$ and thus $x \subseteq \mathcal{N}_{\mathcal{J}}(\mathcal{Y})$. Since $\mathcal{R}$ is a symmetric relation, then $\mathcal{Y} \subseteq \mathcal{N}_{\mathcal{J}}(x)$. Therefore, $\mathcal{N}^i_{\mathcal{J}}(x) \subseteq \mathcal{N}_{\mathcal{J}}(x)$,$\forall x \in \mathcal{U}$. □

The following result (depends on Theorem 2.3.) discusses an exciting technique to create different topologies using the above neighborhoods.

Theorem 3.5. *Let* $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ *be a* $\mathcal{J} - NS$. *Then, for each* $\mathcal{J} \in \{r, \upharpoonleft, \curlywedge, \curlyvee\}$, *the collection* $\mathcal{T}^i_{\mathcal{J}} = \{\mathcal{M} \subseteq \mathcal{U} : \forall \mathrm{m} \in \mathcal{M}, \mathcal{N}^i_{\mathcal{J}}(\mathcal{M}) \subseteq \mathcal{M}\}$ *is a topology on* $\mathcal{U}$.

Proof.

$(T1)$ Clearly, $\mathcal{U}$ and φ belong to $\mathcal{T}^i_{\mathcal{J}}$.

$(T2)$ Let$\{A_n : n \in N\}$ be a family of members in $\mathcal{T}^i_{\mathcal{J}}$ and $p \in U_n A_n$. Then there exists $n_0 \in N$ such that $P \in A_{n_0}$. Thus $\mathcal{N}^i_{\mathcal{J}}(p) \subseteq A_{n_0}$ this implies $\mathcal{N}^i_{\mathcal{J}}(p) \subseteq U_n A_n$. Therefore, $U_n A_n \in \mathcal{N}^i_{\mathcal{J}}(p)$.

$(T3)$ Let $A_1, A_2 \in \mathcal{N}^i_{\mathcal{J}}$ and $p \in A_1 \cap A_2$. Then $p \in A_1$ and $p \in A_2$ which implies $\mathcal{N}^i_{\mathcal{J}}(p) \subseteq A_1$ and $\mathcal{N}^i_{\mathcal{J}}(p) \subseteq A_2$. Thus $\mathcal{N}^i_{\mathcal{J}}(p) \subseteq A_1 \cap A_2$ and hence $A_1 \cap A_2 \in \mathcal{T}^i_{\mathcal{J}}$.

From (T1), (T2)and (T3) $\mathcal{T}^i_{\mathcal{J}}$ forms a topology on $\mathcal{U}$. □

The next proposition gives the relationships among different topologies $\mathcal{T}^i_{\mathcal{J}}$.

Proposition 3.6. *If* $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ *be a* $\mathcal{J} - NS$*. Then:*

(i) $\mathcal{T}^i_{\curlyvee} \subseteq \mathcal{T}^i_r \subseteq \mathcal{N}^i_{\curlywedge}$.

(ii) $\mathcal{T}^i_{\curlyvee} \subseteq \mathcal{T}^i_{\upharpoonleft} \subseteq \mathcal{T}^i_{\curlywedge}$.

Proof. By Lemma 3.3, the proof is obvious. □

Example 3.7 demonstrates that the opposite of Proposition 3.6 is not correct in general.

Example 3.7. Suppose that $\mathcal{R} = \{(a,a),(a,d),(b,a),(b,c),(c,c),(c,d),(d,a)\}$ be a relation on $\mathcal{U} = \{a,b,c,d\}$. Accordingly, we obtain $\mathcal{N}_r(a) = \{a,d\}$, $\mathcal{N}_r(b) = \{a,c\}$, $\mathcal{N}_r(c) = \{c,d\}$, and $\mathcal{N}_r(d) = \{a\}$.

$$\mathcal{N}_{\upharpoonleft}(a) = \{a,b,d\}, \mathcal{N}_{\upharpoonleft}(b) = \varphi, \mathcal{N}_{\upharpoonleft}(c) = \{b,c\}, \mathcal{N}_{\upharpoonleft}(d) = \{a,c\},$$
$$\mathcal{N}_{\curlywedge}(a) = \{a,d\}, \mathcal{N}_{\curlywedge}(b) = \varphi, \mathcal{N}_{\curlywedge}(c) = \{c\}, \mathcal{N}_{\curlywedge}(d) = \{a\},$$
$$\mathcal{N}_{\curlyvee}(a) = \{a,b,d\}, \mathcal{N}_{\curlyvee}(b) = \{a,c\}, \mathcal{N}_{\curlyvee}(c) = \{b,c,d\}, \mathcal{N}_{\curlyvee}(d) = \{a,c\}.$$

Therefore, we obtain $\mathcal{N}^i_r(a) = \{a\}$, $\mathcal{N}^i_r(b) = \{b\}$, $\mathcal{N}^i_r(c) = \{c\}$, and $\mathcal{N}^i_r(d) = \{a,b,d\}$

$$\mathcal{N}^i_{\upharpoonleft}(a) = \{a\}, \mathcal{N}^i_{\upharpoonleft}(b) = \mathcal{U}, \mathcal{N}^i_{\upharpoonleft}(c) = \{c\}, \mathcal{N}^i_{\upharpoonleft}(d) = \{d\},$$
$$\mathcal{N}^i_{\curlywedge}(a) = \{a\}, \mathcal{N}^i_{\curlywedge}(b) = \{b\}, \mathcal{N}^i_{\curlywedge}(c) = \{c\}, \mathcal{N}^i_{\curlywedge}(d) = \{d\},$$
$$\mathcal{N}^i_{\curlyvee}(a) = \{a\}, \mathcal{N}^i_{\curlyvee}(b) = \mathcal{U}, \mathcal{N}^i_{\curlyvee}(c) = \{c\}, \mathcal{N}^i_{\curlyvee}(d) = \{a,b,d\}.$$

Consequently, we generate the following topologies:

$$\mathcal{T}^i_r = \{\mathcal{U}, \varphi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{a,b,d\}\},$$
$$\mathcal{T}^i_{\upharpoonleft} = \{\mathcal{U}, \varphi, \{a\}, \{c\}, \{d\}, \{a,c\}, \{a,d\}, \{c,d\}, \{a,c,d\}\},$$
$$\mathcal{T}^i_{\curlywedge} = \mathcal{P}(\mathcal{U}), \mathcal{T}^i_{\curlyvee} = \{\mathcal{U}, \varphi, \{a\}, \{c\}, \{a,c\}, \{b,c\}\}.$$

The subsequent proposition gives the connections amongst the topologies $\mathcal{T}^i_{\mathcal{J}}$ and $\mathcal{T}_{\mathcal{J}}$.

Proposition 3.8. *If* $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ *is a* $\mathcal{J} - NS$ *such that* $\mathcal{R}$ *is a reflexive and symmetric relation. Then, for each* $\mathcal{J} \in \{r, \upharpoonleft, \curlywedge, \curlyvee\} : \mathcal{T}_{\mathcal{J}} \subseteq \mathcal{T}^i_{\mathcal{J}}$.

Proof. By Lemma 3.4, the proof is clear. □

Remark 3.9. For any a $\mathcal{J} - NS(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$, Example 3.7 shows the following:

(i) The topologies $\mathcal{T}^i_{\mathcal{J}}$ and $\mathcal{T}_{\mathcal{J}}$ are independent in general case.

(ii) The topologies $\mathcal{T}_r^i$ and $\mathcal{T}_{\upharpoonleft}^i$ are independent in general case.

(iii) The property (iii) in Lemma 3.2 is not true for case $j = \curlyvee$.

Example 3.10 proves that the opposite of Proposition 3.8 is not correct generally.

Example 3.10. Let $\mathcal{U} = \{a, b, c, d\}$ and $\mathcal{R} = \{(a, a), (a, b), (b, a), (b, b), (b, c), (c, b), (c, c), (d, d)\}$ be a reflexive and symmetric relation on $\mathcal{U}$. Thus, we compute the topologies $\mathcal{T}_{\mathcal{J}}^i$, and $\mathcal{T}_{\mathcal{J}}$ in the case of $\mathcal{J} = r$, and the others similarly $\mathcal{T}_r = \{\mathcal{U}, \varphi, \{d\}, \{a, b, c\}\}$, and $\mathcal{T}_r^i = \{\mathcal{U}, \varphi, \{b\}, \{d\}, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$.

Diagram 1 summarize the relationships among different topologies such that $\mathcal{R}$ represents a reflexive and symmetric relation.

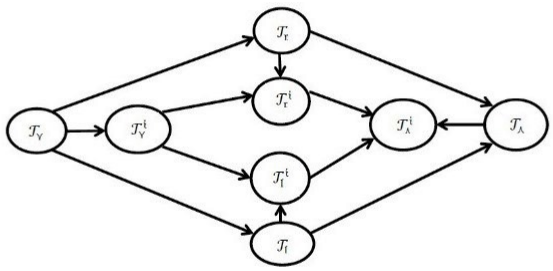

Diagram 1 The relationships among different topologies

4. Rough approximations based on topological structures

In this part, we present four new approximations called $\mathcal{J}$-initial lower and $\mathcal{J}$-initial upper approximations, which we use to define new regions and accuracy measures of a set using the interior and closure of the topologies $\mathcal{T}_{\mathcal{J}}^i$, for each $\mathcal{J} \in \{r, \upharpoonleft, \curlywedge, \curlyvee\}$. We show that these methods yield the best approximations and the highest accuracy measures. There are illustrative examples provided.

Definition 4.1. Suppose that $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ be a $\mathcal{J} - NS$ and $A \subseteq \mathcal{U}$. Therefore, A is called an $\mathcal{J}$-initial open set if $A \subseteq \mathcal{T}_{\mathcal{J}}^i$, and its complement is called an $\mathcal{J}$-initial closed set. The family $\mathcal{F}_{\mathcal{J}}^i$ of all $\mathcal{J}$-initial closed sets is defined by: $\mathcal{F}_{\mathcal{J}}^i = \{\mathcal{F} \subseteq \mathcal{U} : \mathcal{F}^c \in \mathcal{T}_{\mathcal{J}}^i\}$. Moreover, we define the following:

(i) The $\mathcal{J}$-*initial interior* of $A \subseteq \mathcal{U}$ is: $int_{\mathcal{J}}^i(A) = \cup\{G \in \mathcal{T}_{\mathcal{J}}^i : G \subseteq A\}$.

(ii) The $\mathcal{J}$-*initial closure* of $A \subseteq \mathcal{U}$ is: $cl_{\mathcal{J}}^i(A) = \cap\{H \in \mathcal{F}_{\mathcal{J}}^i : A \subseteq H\}$.

Definition 4.2. Let $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ be a $\mathcal{J} - NS$. Then, we define $\mathcal{J}$-*initial lower* and $\mathcal{J}$-*initial upper* approximations of A respectively as follows: $\underline{R}_{\mathcal{J}}^i(A) = int_{\mathcal{J}}^i(A)$, and $\overline{R}_{\mathcal{J}}^i(A) = cl_{\mathcal{J}}^i(A)$.

Definition 4.3. Suppose that $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ be a $\mathcal{J}-NS$. The $\mathcal{J}$-*initial boundary*, $\mathcal{J}$-*initial positive* and $\mathcal{J}$-*initial negative* regions of $A \subseteq \mathcal{U}$ are given, respectively, by $B^i_{\mathcal{J}}(A) = \overline{R}^i_{\mathcal{J}}(A) - \underline{R}^i_{\mathcal{J}}(A)$, $POS^i_{\mathcal{J}}(A) = \underline{R}^i_{\mathcal{J}}(A)$ and $NEG^i_{\mathcal{J}}(A) = \mathcal{U} - \overline{R}^i_{\mathcal{J}}(A)$.

Moreover, the $\mathcal{J}$-*initial accuracy* of the $\mathcal{J}$-*initial approximations* of $A \subseteq \mathcal{U}$ is defined by $\alpha^i_{\mathcal{J}}(A) = \frac{|\underline{R}^i_{\mathcal{J}}(A)|}{|\overline{R}^i_{\mathcal{J}}(A)|}$, where $|\overline{R}^i_{\mathcal{J}}(A)| \neq 0$. It is clear that, $0 \leq \alpha^i_{\mathcal{J}}(A) \leq 1$. In addition, A is called an $\mathcal{J}$-*initial* definable ($\mathcal{J}$-*initial exact*) set if $\alpha^i_{\mathcal{J}}(A) = 1$, and it is called $\mathcal{J}$-*initial rough* if $\alpha^i_{\mathcal{J}}(A) \neq 1$.

Example 4.4. By using Example 3.7, we get the following:

$$\begin{aligned}
&\mathcal{T}^i_r = \{\mathcal{U}, \varphi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}, \{a,b,d\}\},\\
&\mathcal{F}^i_r = \{\mathcal{U}, \varphi, \{c\}, \{d\}, \{a,d\}, \{b,d\}, \{c,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\},\\
&\mathcal{T}^i_{\upharpoonleft} = \{\mathcal{U}, \varphi, \{a\}, \{c\}, \{d\}, \{a,c\}, \{a,d\}, \{c,d\}, \{a,c,d\}\},\\
&\mathcal{F}^i_{\upharpoonleft} = \{\mathcal{U}, \varphi, \{b\}, \{a,b\}, \{b,c\}, \{b,d\}, \{a,b,c\}, \{a,b,d\}, \{b,c,d\}\},\\
&\mathcal{T}^i_{\curlywedge} = \mathcal{F}^i_{\curlywedge} = \mathcal{P}(\mathcal{U}), \mathcal{T}^i_{\curlyvee} = \{\mathcal{U}, \varphi, \{a\}, \{c\}, \{a,c\}\},\\
&\mathcal{F}^i_{\curlyvee} = \{\mathcal{U}, \varphi, \{b,d\}, \{a,b,d\}, \{b,c,d\}\}.
\end{aligned}$$

Thus, we can get Tables 1 and 2 that give the $\mathcal{J}$-*initial lower*, $\mathcal{J}$-*initial upper* approximations and the $\mathcal{J}$-*initial accuracy* of $\mathcal{J}$-initial approximations of all subsets of $\mathcal{U}$:

Table 1: Comparison among different types of $\mathcal{J}$-*initial approximations*

$\mathcal{P}(\mathcal{U})$	$\mathcal{T}^i_r$		$\mathcal{T}^i_{\upharpoonleft}$		$\mathcal{T}^i_{\curlywedge}$		$\mathcal{T}^i_{\curlyvee}$	
	$\underline{\mathcal{R}}^i_r(A)$	$\overline{\mathcal{R}}^i_r(A)$	$\underline{\mathcal{R}}^i_{\upharpoonleft}(A)$	$\overline{\mathcal{R}}^i_{\upharpoonleft}(A)$	$\underline{\mathcal{R}}^i_{\curlywedge}(A)$	$\overline{\mathcal{R}}^i_{\curlywedge}(A)$	$\underline{\mathcal{R}}^i_{\curlyvee}(A)$	$\overline{\mathcal{R}}^i_{\curlyvee}(A)$
$\{a\}$	$\{a\}$	$\{a,d\}$	$\{a\}$	$\{a,b\}$	$\{a\}$	$\{a\}$	$\{a\}$	$\{a,b,d\}$
$\{b\}$	$\{b\}$	$\{b,d\}$	φ	$\{b\}$	$\{b\}$	$\{b\}$	φ	$\{b,d\}$
$\{c\}$	$\{c\}$	$\{c\}$	$\{c\}$	$\{b,c\}$	$\{c\}$	$\{c\}$	$\{c\}$	$\{b,c,d\}$
$\{d\}$	φ	$\{d\}$	$\{d\}$	$\{b,d\}$	$\{d\}$	$\{d\}$	φ	$\{b,d\}$
$\{a,b\}$	$\{a,b\}$	$\{a,b,d\}$	$\{a\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	$\{a\}$	$\{a,b,d\}$
$\{a,c\}$	$\{a,c\}$	$\{a,c,d\}$	$\{a,c\}$	$\{a,b,c\}$	$\{a,c\}$	$\{a,c\}$	$\{a,c\}$	$\mathcal{U}$
$\{a,d\}$	$\{a\}$	$\{a,d\}$	$\{a,d\}$	$\{a,b,d\}$	$\{a,d\}$	$\{a,d\}$	$\{a\}$	$\{a,b,d\}$
$\{b,c\}$	$\{b,c\}$	$\{b,c,d\}$	$\{c\}$	$\{b,c\}$	$\{b,c\}$	$\{b,c\}$	$\{c\}$	$\{b,c,d\}$
$\{b,d\}$	$\{b\}$	$\{b,d\}$	$\{d\}$	$\{b,d\}$	$\{b,d\}$	$\{b,d\}$	φ	$\{b,d\}$
$\{c,d\}$	$\{c\}$	$\{c,d\}$	$\{c,d\}$	$\{b,c,d\}$	$\{c,d\}$	$\{c,d\}$	$\{c\}$	$\{b,c,d\}$
$\{a,b,c\}$	$\{a,b,c\}$	$\mathcal{U}$	$\{c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,b,c\}$	$\{a,c\}$	$\mathcal{U}$
$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a,b,d\}$	$\{a\}$	$\{a,b,d\}$
$\{a,c,d\}$	$\{a,c\}$	$\{a,c,d\}$	$\{a,c,d\}$	$\mathcal{U}$	$\{a,c,d\}$	$\{a,c,d\}$	$\{a,c\}$	$\mathcal{U}$
$\{b,c,d\}$	$\{b,c\}$	$\{b,c,d\}$	$\{c,d\}$	$\{b,c,d\}$	$\{b,c,d\}$	$\{b,c,d\}$	$\{c\}$	$\{b,c,d\}$
$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$	$\mathcal{U}$

Table 2: Comparison among different types of $\mathcal{J}$-*initial accuracy*

$\mathcal{P}(\mathcal{U})$	$\alpha^i_r(A)$	$\alpha^i_{\urcorner}(A)$	$\alpha^i_{\curlywedge}(A)$	$\alpha^i_{\curlyvee}(A)$
$\{a\}$	1/2	1/2	1	1/3
$\{b\}$	1/2	0	1	0
$\{c\}$	1	1/2	1	1/3
$\{d\}$	0	1/2	1	0
$\{a,b\}$	2/3	1/2	1	1/3
$\{a,c\}$	2/3	2/3	1	1/2
$\{a,d\}$	1/2	2/3	1	1/3
$\{b,c\}$	2/3	1/2	1	1/3
$\{b,d\}$	1/2	1/2	1	0
$\{c,d\}$	1/2	2/3	1	1/3
$\{a,b,c\}$	3/4	1/3	1	1/2
$\{a,b,d\}$	1	2/3	1	1/3
$\{a,c,d\}$	2/3	3/4	1	1/2
$\{b,c,d\}$	2/3	2/3	1	1/3
$\mathcal{U}$	1	1	1	1

Remark 4.5. According to Tables 1 and 2 of Example 4.4, we conclude that by using different types of $\mathcal{T}^i_{\mathcal{J}}$ in constructing the approximations of sets, the best of them is that given by $\mathcal{T}^i_{\curlywedge}$ since $\alpha^i_{\curlyvee}(A) \leq \alpha^i_r(A) \leq \alpha^i_{\curlywedge}(A)$ and $\alpha^i_{\curlyvee}(A) \leq \alpha^i_{\urcorner}(A) \leq \alpha^i_{\curlywedge}(A)$. In addition, these approaches are more accurate than the previous one in [18].

Some properties of the $\mathcal{J}$-*initial approximations* are provided in the next result. Moreover, it represents one of the distinctions between our approaches and other generalizations such as [1, 12-16, 21, 22, 25-28, and 33-36].

Proof. Suppose that $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ be a $\mathcal{J}-NS$ and $A,\ B \subseteq \mathcal{U}$. Thus:

(1) $\underline{R}^i_{\mathcal{J}}(A) \subseteq A \subseteq \overline{R}^i_{\mathcal{J}}(A)$.

(2) $\underline{R}^i_{\mathcal{J}}(\mathcal{U}) = \overline{R}^i_{\mathcal{J}}(\mathcal{U}) = \mathcal{U}$, and $\underline{R}^i_{\mathcal{J}}(\varphi) = \overline{R}^i_{\mathcal{J}}(\varphi) = \varphi$.

(3) $\overline{R}^i_{\mathcal{J}}(A \cup B) = \overline{R}^i_{\mathcal{J}}(A) \cup \overline{R}_{\mathcal{J}}(B)$.

(4) $\underline{R}^i_{\mathcal{J}}(A \cap B) = \underline{R}^i_{\mathcal{J}}(A) \cap \underline{R}^i_{\mathcal{J}}(B)$.

(5) If $A \subseteq B$, then $\underline{R}^i_{\mathcal{J}}(A) \subseteq \underline{R}^i_{\mathcal{J}}(B)$.

(6) If $A \subseteq B$, then $\overline{R}^i_{\mathcal{J}}(A) \subseteq \overline{R}^i_{\mathcal{J}}(B)$.

(7) $\underline{R}^i_{\mathcal{J}}(A \cup B) \supseteq \underline{R}^i_{\mathcal{J}}(A) \cup \underline{R}^i_{\mathcal{J}}(B)$.

(8) $\overline{R}^i_{\mathcal{J}}(A \cap B) \subseteq \overline{R}^i_{\mathcal{J}}(A) \cap \overline{R}^i_{\mathcal{J}}(B)$.

(9) $\underline{R}^i_{\mathcal{J}}(A) = [\overline{R}^i_{\mathcal{J}}(A^c)]^c$, A^c is the complement of A.

(10) $\overline{R}^i_{\mathcal{J}}(A) = [\underline{R}^i_{\mathcal{J}}(A^c)]^c$.

(11) $\underline{R}^i_{\mathcal{J}}(\underline{R}^i_{\mathcal{J}}(A) = \underline{R}^i_{\mathcal{J}}(A)$.
(12) $\overline{R}^i_{\mathcal{J}}(\overline{R}^i_{\mathcal{J}}(A)) = \overline{R}^i_{\mathcal{J}}(A)$. □

Proof. The proof is directly simple by applying the properties of interior $int^i_{\mathcal{J}}$ and closure $cl^i_{\mathcal{J}}$. □

The subsequent results illustrate the relationships among the suggested approximations ($\mathcal{J}$-*initial approximations*).

Proposition 4.6. *If* $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ *is a* $\mathcal{J} - NS$ *and* $\mathrm{A} \subseteq \mathcal{U}$*. Then:*
(1) $\underline{R}^i_{\curlyvee}(\mathrm{A}) \subseteq \underline{R}^i_{r}(\mathrm{A}) \subseteq \underline{R}^i_{\curlywedge}(\mathrm{A})$.
(2) $\underline{R}^i_{\curlyvee}(\mathrm{A}) \subseteq \underline{R}^i_{\restriction}(\mathrm{A}) \subseteq \underline{R}^i_{\curlywedge}(\mathrm{A})$.
(3) $\overline{R}^i_{\curlywedge}(\mathrm{A}) \subseteq \overline{R}^i_{r}(\mathrm{A}) \subseteq \overline{R}^i_{\curlyvee}(\mathrm{A})$.
(4) $\overline{R}^i_{\curlywedge}(\mathrm{A}) \subseteq \overline{R}^i_{\restriction}(\mathrm{A}) \subseteq \overline{R}^i_{\curlyvee}(\mathrm{A})$.

Proof. By using Proposition 3.6, the proof is obvious. □

Corollary 4.7. *If* $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ *is a* $\mathcal{J} - NS$ *and* $\mathrm{A} \subseteq \mathcal{U}$*. Then:*
(1) $\underline{B}^i_{\curlywedge}(\mathrm{A}) \subseteq \underline{B}^i_{r}(\mathrm{A}) \subseteq \underline{B}^i_{\curlyvee}(\mathrm{A})$.
(2) $\underline{B}^i_{\curlywedge}(\mathrm{A}) \subseteq \underline{B}^i_{\restriction}(\mathrm{A}) \subseteq \underline{B}^i_{\curlyvee}(\mathrm{A})$.
(3) $\alpha^i_{\curlyvee}(\mathrm{A}) \leq \alpha^i_{r}(\mathrm{A}) \leq \alpha^i_{\curlywedge}(\mathrm{A})$.
(4) $\alpha^i_{\curlyvee}(\mathrm{A}) \leq \alpha^i_{\restriction}(\mathrm{A}) \leq \alpha^i_{\curlywedge}(\mathrm{A})$.
(5) *The subset* A *is an* $\curlyvee$*-initial exact set* $\Rightarrow$ A *is* r*-initial exact* $\Rightarrow$ A *is* $\curlywedge$*-initial exact.*
(6) *The subset* A *is an* $\curlyvee$*-initial exact set* $\Rightarrow$ A *is* $\restriction$*-initial exact* $\Rightarrow$ A *is* $\curlywedge$*-initial exact.*

Remark 4.8. The converse of the above results is not true in general as illustrated in Example 4.4.

The following results introduce comparisons between the proposed approximations ($\mathcal{J}$-*initial approximations*) and the previous approximations ($\mathcal{J}$-*initial approximations* [1]).

Theorem 4.9. *If* $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ *is a* $\mathcal{J}$-NS *and* $\mathrm{A} \subseteq \mathcal{U}$ *such that* $\mathcal{R}$ *is a reflexive and symmetric relation on* $\mathcal{U}$*. Then, for each* $\mathcal{J} \in \{r, \restriction, \curlywedge, \curlyvee\}$*:*
(1) $\underline{\mathcal{R}}_{\mathcal{J}}(\mathrm{A}) \subseteq \underline{\mathcal{R}}^i_{\mathcal{J}}(\mathrm{A})$.
(2) $\overline{\mathcal{R}}^i_{\mathcal{J}}(\mathrm{A}) \subseteq \overline{\mathcal{R}}_{\mathcal{J}}(\mathrm{A})$.

Proof. We shall prove the first statement and the other similarly.

Let $x \in \underline{\mathcal{R}}_{\mathcal{J}}(A)$, then $\exists G \in \mathcal{T}_{\mathcal{J}}$ such that $x \in G \subseteq A$. But, from Proposition 3.8, $\mathcal{T}_{\mathcal{J}} \subseteq \mathcal{T}^i_{\mathcal{J}}$. Therefore, $G \in \mathcal{T}^i_{\mathcal{J}}$ such that $x \in G \subseteq A$ which implies $x \in \mathcal{R}^i_{\mathcal{J}}(A)$. □

Corollary 4.10. *Let* $(\mathcal{U}, \mathcal{R}, \xi_{\mathcal{J}})$ *be a* $\mathcal{J}$-NS . *Then:*

(1) $\mathcal{B}^i_{\mathcal{J}}(\mathrm{A}) \subseteq \mathcal{B}_{\mathcal{J}}(\mathrm{A})$.

(2) $\alpha_{\mathcal{J}}(\mathrm{A}) \leq \alpha^i_{\mathcal{J}}(\mathrm{A})$.

(3) *The subset* A *is an* $\mathcal{J}$-exact *set if it is* $\mathcal{J}$-initial exact.

Remark 4.11. The inverse of the above results is not true in general as illustrated by Example 4.12.

Example 4.12. Consider Example 3.10, we compare between the $\mathcal{J}$-*approximations* and $\mathcal{J}$-*initial approximations* in the case of $\mathcal{J} = r$ and the others similarly.

First, the topologies $\mathcal{T}^i_{\mathcal{J}}$ and $\mathcal{T}_{\mathcal{J}}$ in the case of $\mathcal{J} = r$ are:

$\mathcal{T}_r = \mathcal{F}_r = \{\mathcal{U}, \varphi, \{d\}, \{a, b, c\}\}$,

$\mathcal{T}^i_r = \{\mathcal{U}, \varphi, \{b\}, \{d\}, \{a, b\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$ and

$\mathcal{F}^i_r = \{\mathcal{U}, \varphi, \{a\}, \{c\}, \{d\}, \{a, c\}, \{a, d\}, \{c, d\}, \{a, b, c\}, \{a, c, d\}\}$.

Therefore, we get Table 3 which represents a comparison between the r-accuracy of $\mathcal{J}$- *approximations* and r-initial accuracy of r-initial approximations of all subsets of $\mathcal{U}$.

Table 3: Comparison between r-accuracies and r-initial accuracies

$\mathcal{P}(\mathcal{U})$	$\alpha_r(A)$	$\alpha^i_r(A)$
$\{a\}$	0	0
$\{b\}$	0	1/3
$\{c\}$	0	0
$\{d\}$	1	1
$\{a, b\}$	0	2/3
$\{a, c\}$	0	0
$\{a, d\}$	1/4	1/2
$\{b, c\}$	0	2/3
$\{b, d\}$	1/4	1/2
$\{c, d\}$	1/4	1/2
$\{a, b, c\}$	1	1
$\{a, b, d\}$	1/4	3/4
$\{a, c, d\}$	1/4	1/3
$\{b, c, d\}$	1/4	3/4
$\mathcal{U}$	1	1

Remark 4.13. According to Table 3 of Example 4.12, we notice that r-initial approximations are more accurate than r-approximations of sets since $\alpha_r(A) \leq \alpha^i_r(A)$. Therefore, we can say that the proposed approximations $\mathcal{J}$-*initial approximations* represent golden tools in removing the vagueness of sets. For example, in Table 3, the subset $A = \{b, c\}$ its r-approximations are $\underline{R}_r(A) = \varphi$ and $\overline{R}_r(A) = \{a, b, c\}$ which implies $B_r(A) = \{a, b, c\}$ and $\alpha_r(A) = 0$ and this means

that A is a r-rough set. Moreover, the r-positive region of A is $POS_r(A) = \varphi$ although A consist of two elements which is a contradiction to the knowledge of Example 4.12. On the other hand, we find r-initial approximations of A are $\underline{R}^i_r(A) = \{b, c\}$ and $\overline{R}^i_r(A) = \{a, b, c\}$ that is the r-initial positive region of A is $POS^i_r(A) = A$ and $\alpha^i_r(A) = 2/3$.

Conclusion

The present paper is devoted to introducing and studying new generalizations to the concept of "initial-neighborhood". We defined three different types and compare them with the previous one [18]. Moreover, using Theorem 1 in [1], we purposed a new method to generate four different topologies induced by the new neighborhoods. A comparison between these topologies and the previous one was investigated. Finally, we used these new topologies to generate new generalizations to Pawlak rough sets and study their properties. We compared the suggested approaches with the previous one [1, 18] and proved that these methods are more accurate than other methods. Theorem 3.5 gives an easy method to generate these topologies directly from relations without using sub-base or base. We believe that the using of this technique is easier in application fields and useful for applying many topological concepts in future studies.

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