Petrov-discontinuous Galerkin finite element method for solving diffusion-convection problems

Mohammed Waleed AbdulRidha
Department of Mathematics
College of Education for Pure Sciences
University of Basrah
Basrah
Iraq
mohammed.abdul_ridha@uobasrah.edu.iq

Hashim A. Kashkool
Department of Mathematics
College of Education for Pure Sciences
University of Basrah
Basrah
Iraq
hashim.kashkool@uobasrah.edu.iq

Ali Hasan Ali*
Department of Mathematics
College of Education for Pure Sciences
University of Basrah
Basrah
Iraq
and
College of Engineering Technology
National University of Science and Technology
Dhi Qar 64001
Iraq
and
Institute of Mathematics University of Debrecen
Pf. 400, H-4002 Debrecen
Hungary
ali.hasan@science.unideb.hu

Abstract. In this paper, we present a new modification of the discontinuous Galerkin Finite element method (DGFEM). The proposed modification is considered when the symmetric interior penalty Galerkin scheme involves only space variables by using the Petrov discontinuous Galerkin Finite element method (PDGFEM), while the time in the linear diffusion-convection problem remains continuous. We prove the properties of the bi-linear form (V-elliptic, continuity and stability), and we show that the error estimate is of second order with respect to the space. We also present some numerical

∗. Corresponding author

experiments to validate the proposed method, and we simulate these peppermints to illustrate the theoretical results.

Keywords: linear diffusion-convection, Petrov-discontinuous, Galerkin finite element method, error estimate.

1. Introduction

We consider the problem mentioned in [1, 2, 3] of the diffusion-convection, $U \in Q_T \longrightarrow \mathbb{R}$, suth that $Q_T = \Omega \times (0, T)$:

$$U_t - \lambda \Delta U + \boldsymbol{b} \cdot \nabla U = f \quad \text{in} \quad Q_T, \tag{1.1}$$

$$U = U^D \quad \text{on} \quad \partial\Omega^D \times (0, T), \tag{1.2}$$

$$\lambda \frac{\partial U}{\partial n} = U^N \quad \text{on} \quad \partial\Omega^N \times (0, T), \tag{1.3}$$

$$U(x, 0) = U^0(x), \quad \forall x \in \Omega, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^2$ denotes a polygonal domain and $T > 0$.

Assume that $\partial\Omega = \partial\Omega^D \cup \partial\Omega^N$

$$\begin{array}{lllll} \boldsymbol{b} \cdot n & \leq & 0 & on & \partial\Omega^D, \\ \boldsymbol{b} \cdot n & \geq & 0 & on & \partial\Omega^N, \quad ; \quad \forall t \in [0, T]. \end{array} \tag{1.5}$$

Here n is the unit outer normal to the boundary $\partial\Omega$ of Ω, the inflow boundary is $\partial\Omega^D$, and the outflow boundary is $\partial\Omega^N$.

Assumptions:
a) $U_t \in L^2(Q_T)$, $U, U^0 \in L^2(\Omega)$,
b)$f \in C([0, T]; L^2(\Omega))$,
c) U^D is the trace of some $U \in C([0, T]; H^1(\Omega)) \cap L^\infty(Q_T)$ on $\partial\Omega^D \times (0, T)$,
d) $U^N \in C([0, T]; L^2(\partial\Omega^N))$,
e) $|K| =$ is the area of $K \in T_h$,
f) $\sigma = \frac{\sigma^0}{|E|^{\beta_0}}$, $\beta_0 \geq (d-1)^{-1}$, $\sigma^0 > 0$.

This problem consists mainly of two components: the terms of diffusion with the coefficient of diffusion and the terms of convection with the field of convection velocity. When using the Galerkin Finite Element Method (GFEM) to solve one-sided turbulent convection problems, the approximate solutions show pseudo-oscillation, i.e. $\forall h > 0$, $\frac{\lambda}{|b|h} << 1$, this condition can occur as any combination of weak diffusion (small), strong convection (large), alternatively, as a result of a large domain, the last case accurate frequently in geophysical applications. Several approaches have been intensively researched to eliminate such a downside adding stabilization terms to the problem's formulation is a common concept. This is done predominantly by stabilized processes such as upwinding methods [4, 5], Petrov-Galerkin approach [6, 7], nonlinear diffusivity method [8, 9, 10], Weak Galerkin method [11, 12] and oscillation theory [13, 14, 15].

Researchers devised a new approach to address these problems in the 1970s called the Discontinues Galerkin finite element method (DGFEM). Without having any consistency criteria, the DGFEM approach approximates the approximate limits of the ideal grid solution on finite elements. The DGFEM utilizes the same function space as the finite volume method (FVM) and continuous finite element method (FEM), but also with relaxed continuity at inter-element borders, and may be thought of as a hybrid of the two. The convection component dominates over diffusion when $\lambda < h$, where h is mesh size, and the usual Galerkin finite element technique generates an oscillating solution that is not near to the exact solution ([16]). The PDGFEM is an improvement and provident of DGFEM.In DGFEM, the shape function and trail function are in the same field, but in PDGFEM, the test function space differs from the trial function space. In this paper, we shall show and analyze the PDGFEM in the case of the SIPG for the linear diffusion-convection problem.$V-$elliptic,continuity, stability, and convergence were demonstrated in thesemi-discrete PDGFEM. We found the L^2-error and H^1-error of PDGFEM and DGFEM for solving a linear diffusion-convection problem to discuss the approximation between the L^2-error and the order of error. The following is how this paper is structured. In the section 2, we have shown the discretization. The variation formulation of PDGFEM and the semi-discrete PDGFEM are presented in the section 3. In the section 4, we proved the properties of the bilinear form and stability. The error estimate is presented in the section 5. In the section 6, we showed numerical results to confirm the theoretical results. Finally, the conclusions are shown in the section 6.

2. The discretization

Let T_h $(h > 0)$ represent a limited number of closed triangles with mutually disjoint interiors divided by $\overline{\Omega}$ (the domain closure Ω).A triangulation of Ω is what we'll call T_h . The conforming qualities of T_h that are employed in the FEM are denoted by T_h. That suggests that we recall what are known as "hanging nodes". Neighbors are two elements $K^i, K^j \in \mathrm{T}_h$ that share a non-empty open portion of their sides. If we provide $\partial K^1 \cap \partial K^2$ to has $(d-1)$ a positive dimensional measure, suppose that $E \in K$ is the edge of K if it is a maximum connected open subset either of $K^1 \cap K^2$, where K^1 is a neighbor of K^2or a subset of $\partial K \cap \partial\Omega$.The term $\partial\mathrm{T}_h$ refers to the system of all sides of all elements $K \in \mathrm{T}_h$. In addition, all inner and border edges are specified in [17] by

$$\begin{aligned}
\partial\mathrm{T}_h^I &= \{E \subset \Omega, E \in \partial\mathrm{T}_h\},\\
\partial\mathrm{T}_h^B &= \{E \subset \partial\Omega, E \in \partial\mathrm{T}_h\},\\
\Gamma^D &= \{E \subset \partial\Omega^D, E \in \partial\mathrm{T}_h^B\},\\
\Gamma^N &= \{E \subset \partial\Omega^N, E \in \partial\mathrm{T}_h^B\}.
\end{aligned}$$

Obviously $\partial \mathrm{T}_h = \partial \mathrm{T}_h^I \cup \partial \mathrm{T}_h^B$ for $\varphi \in H^1(\Omega, \mathrm{T}_h)$, $\partial \mathrm{T}_h^B = \Gamma^D \cup \Gamma^N$ for each $E \in \partial \mathrm{T}_h$.

Each edge $E \in K$ has elements on both sides, and they are called outside and inside elements, respectively, with arbitrary constants. The assessment of a function v in the inside of E is defined as $\forall x \in E; v^-(x) = v(x)|_{inside}$ where $v^-(x) = \lim_{\epsilon \longrightarrow 0}(x - \epsilon); \epsilon > 0$, and the external or the outside elements are defined as $\forall x \in E; v^+(x) = v(x)|_{outside}$ where $v^+(x) = \lim_{\epsilon \longrightarrow 0}(x + \epsilon); \epsilon > 0$.

On the side E, the function v is discontinuous. The discontinuity size must be quantified. Let us define $[v](x) = -(v^-(x) - v^+(x))$ as the function v jumping on the side E for each $x \in E$. On the discontinuity side E, a function v is undefined, and the average v is used to close this gap in the definition. For each $x \in E$, let it be $v(x) = \frac{(v^+(x)+v^-(x))}{2}$, defined as the average of function v on side E.

2.1 Broken Sobolev spaces

Discontinuous approximations are used in the DGFEM. This is why, for each r∈ℕ, the so-called broken Sobolev space is defined over triangulation T_h:

$$H^r(\Omega, \mathrm{T}_h) = \{\forall K \in \mathrm{T}_h; v \in L^2(\Omega); v|_K \in H^r(K)\}.$$

The norm of $v \in H^r(\Omega, \mathrm{T}_h)$ is defined

$$||U||_{H^r(\Omega,\mathrm{T}_h)} = (\sum_{K \in \mathrm{T}_h} ||U||^2_{H^r(\Omega)})^{1/2},$$

and semi-norm $|U|_{H^r(\Omega,\mathrm{T}_h)} = (\sum_{K \in \mathrm{T}_h} |U|^2_{H^r(\Omega)})^{1/2}$. Assume that $l \geq 0$ is a positive integer. Piecewise polynomial functions with discontinuous coefficients have a space represented by

$$S_h = \{\forall K \in \mathrm{T}_h; v \in L^2(\Omega); v|_K \in P_l(K)\},$$

where P_l (K) represents the space occupied by all degree $\leq$ polynomials on K. The number$l$ represents the degree of polynomial approximation ([18]). Obviously, $\mathrm{S_h} \subset \mathrm{H^r}(\Omega,\mathrm{T_h})$.

Let ϑ be trial space and $\emptyset$ be a test space

$$\begin{aligned} \vartheta &= H^r(\Omega, \mathrm{T}_h), \\ \emptyset &= \{w : w = v + \delta \boldsymbol{b} \cdot \nabla v; v \in \vartheta\}, \end{aligned}$$

and dim ϑ = dim $\emptyset$.

We defined PDGFE space

$$\begin{aligned} \vartheta_h &= S_h, \\ \emptyset_h &= \{w : w = v + \delta \boldsymbol{b} \cdot \nabla v; v \in \vartheta_h\}, \end{aligned}$$

where δ denotes a constant stability parameter in Q_T. It can be selected as [19],

$$\delta \equiv \begin{cases} \eta h, & \text{if} \quad \lambda < h \\ 0, & \text{if} \quad \lambda \geq h \end{cases}; 0 < \eta < \frac{1}{4}.$$

3. The variation formulation of PDGFEM

By multiplying equation (1.1) by the test function w, we can get $U \in \vartheta$ in the SIPG form of the PDGFEM approximation:

$$
\begin{aligned}
&(U_t, w) + \sum_{K \in \mathrm{T}_h} \lambda(\nabla U, \nabla w)_K - \sum_{E \in \partial \mathrm{T}_h} \int (\{\lambda \nabla U \cdot n\}[w] - \varepsilon [U]\{\lambda \nabla w \cdot n\}) ds \\
&\quad + \sum_{E \in \partial \mathrm{T}_h} \int (\{|\boldsymbol{b} \cdot n| U\}[w]) ds + \sigma \sum_{E \in \partial \mathrm{T}_h} \int [U][w] ds - \sum_{K \in \mathrm{T}_h} (\boldsymbol{b} \cdot \nabla U, w)_K \\
&\quad = (f, w) + \sum_{E \in \Gamma^N} \int U^N w ds + \sum_{E \in \Gamma^D} \int \lambda \nabla w \cdot n U^D ds - \sum_{E \in \Gamma^D} \int |\boldsymbol{b} \cdot n| U^D w ds \\
&\quad - \sigma \sum_{E \in \Gamma^D} \int U^D w ds; \qquad \forall w \in \emptyset.
\end{aligned}
$$

Since $\varepsilon = -1$(SIPG) ([1]) and $w = v + \delta \boldsymbol{b} \cdot \nabla v$ then

$$
\begin{aligned}
&(U_t, v + \delta \boldsymbol{b} \cdot \nabla v) + \sum_{K \in \mathrm{T}_h} \lambda(\nabla U, \nabla(v + \delta \boldsymbol{b} \cdot \nabla v))_K \\
&\quad - \sum_{E \in \partial \mathrm{T}_h} \int (\{\lambda \nabla U \cdot n\}[v + \delta \boldsymbol{b} \cdot \nabla v] \\
&\quad + \sum_{E \in \partial \mathrm{T}_h} \int (\{|\boldsymbol{b} \cdot n| U\}[v + \delta \boldsymbol{b} \cdot \nabla v]) ds + [U]\{\lambda \nabla(v + \delta \boldsymbol{b} \cdot \nabla v) \cdot n\}) ds \\
&\quad - \sum_{K \in \mathrm{T}_h} (\boldsymbol{b} \cdot \nabla U, v + \delta \boldsymbol{b} \cdot \nabla v)_K + \sigma \sum_{E \in \partial \mathrm{T}_h} \int [U][v + \delta \boldsymbol{b} \cdot \nabla v] ds \\
&\quad = (f, v + \delta \boldsymbol{b} \cdot \nabla v) + \sum_{E \in \Gamma^N} \int U^N (v + \delta \boldsymbol{b} \cdot \nabla v) ds \\
&\quad + \sum_{E \in \Gamma^D} \int \lambda \nabla(v + \delta \boldsymbol{b} \cdot \nabla v) \cdot n U^D ds - \sum_{E \in \Gamma^D} \int |\boldsymbol{b} \cdot n| U^D (v + \delta \boldsymbol{b} \cdot \nabla v) ds \\
&\quad - \sigma \sum_{E \in \Gamma^D} \int U^D (v + \delta \boldsymbol{b} \cdot \nabla v) ds, \qquad \forall v \in \vartheta.
\end{aligned} \tag{3.1}
$$

The variation formulation of PDGFEM is find $U \in \vartheta \ni$

$$
\begin{aligned}
&(U_t, v) + (U_t, \delta \boldsymbol{b} \cdot \nabla v) + a_{PD}(U, v) = (f, \delta \boldsymbol{b} \cdot \nabla v) + (f, v) \\
&\qquad - \sum_{E \in \Gamma^D} \int |\boldsymbol{b} \cdot n| U^D (v + \delta \boldsymbol{b} \cdot \nabla v) ds \\
&\qquad + \sum_{E \in \Gamma^N} \int U^N (v + \delta \boldsymbol{b} \cdot \nabla v) ds - \sigma \sum_{E \in \Gamma^D} \int U^D (v + \delta \boldsymbol{b} \cdot \nabla v) ds \\
&\qquad + \sum_{E \in \Gamma^D} \int \lambda \nabla(v + \delta \boldsymbol{b} \cdot \nabla v) \cdot n U^D, \quad \forall v \in \vartheta,
\end{aligned} \tag{3.2}
$$

where

$$
\begin{aligned}
a_{PD}(U,v) = &\sum_{K\in \mathrm{T}_h} \lambda(\nabla U,\nabla v)_K - \sum_{E\in\partial \mathrm{T}_h}\int([U]\{\lambda\nabla v\cdot n\}+\{\lambda\nabla U\cdot n\}[v])ds \\
&-\sum_{K\in \mathrm{T}_h}(\boldsymbol{b}\cdot\nabla U, v+\delta\boldsymbol{b}\cdot\nabla v)_K+\sum_{E\in\partial \mathrm{T}_h}\int(\{|\boldsymbol{b}\cdot n|U\}[v])ds \\
&+\sigma\sum_{E\in\partial \mathrm{T}_h}\int[U][v]ds. \qquad (3.3)
\end{aligned}
$$

3.1 The semi-discrete PDGFEM

The semi-discrete solution: find $U_h\in\vartheta_h, \forall v\in\vartheta_h$, such that:

$$
\begin{aligned}
&(U_{h,t},v)+a_{PD}\left(U_h,v\right)+(U_{h,t},\delta\boldsymbol{b}\cdot\nabla v)=(f,v)+(f,\delta\boldsymbol{b}\cdot\nabla v) \\
&\quad+\sum_{E\in\Gamma^N}\int U^N(v+\delta\boldsymbol{b}\cdot\nabla v)ds+\sum_{E\in\Gamma^D}\int\lambda\nabla(v+\delta\boldsymbol{b}\cdot\nabla v)\cdot nU^D \\
&\quad-\sum_{E\in\Gamma^D}\int|\boldsymbol{b}\cdot n|U^D(v+\delta\boldsymbol{b}\cdot\nabla v)ds-\sigma\sum_{E\in\Gamma^D}\int U^D(v+\delta\boldsymbol{b}\cdot\nabla v)ds, \qquad (3.4)
\end{aligned}
$$

where

$$
\begin{aligned}
a_{PD}(U_h,v) = &\sum_{K\in \mathrm{T}_h}\lambda(\nabla U_h,\nabla v)_K-\sum_{E\in\partial \mathrm{T}_h}\int(\{\lambda\nabla U_h\cdot n\}[v]+[U_h]\{\lambda\nabla v\cdot n\})ds \\
&+\sum_{E\in\partial \mathrm{T}_h}\int(\{|b\cdot n|U_h\}[v])ds-\sum_{K\in \mathrm{T}_h}(\boldsymbol{b}\cdot\nabla U_h, v+\delta\boldsymbol{b}\cdot\nabla v)_K \\
&+\sigma\sum_{E\in\partial \mathrm{T}_h}\int[U_h][v]ds. \qquad (3.5)
\end{aligned}
$$

4. The properties of $a_{PD}(U,v)$aPD and stability

In this section, we prove some impotent lemmas for the bilinear form ($V-$elliptic, continuous) and stability.

Lemma 4.1 ($V-$elliptic)**.** *Assume the penalty σ is large enough, and there is a positive constant α independent of h, $\beta_0\geq(d-1)^{-1}$ such that*

$$a_{PD}(U,U)\geq\ \alpha\ \|U\|^2_{H^1(\mathrm{T}_h)}, \qquad (4.1)$$

where

$$\|U\|_{H^1(\mathrm{T}_h)}=\left(\sum_{K\in \mathrm{T}_h}\left\|\lambda^{\frac{1}{2}}\nabla U\right\|^2_{L^2(K)}+\left(\sum_{E\in\partial \mathrm{T}_h}\int\sigma^{-1}(\{\lambda\nabla U\cdot n\})^2ds\right)^{\frac{1}{2}}\right)^2$$

$$+\sum_{E\in\partial \mathrm{T}_h}\sigma\|[U]\|^2_{L^2(E)}+\|U\|^2_{H^1(\ \mathrm{T}_h)}+\sum_{K\in \mathrm{T}_h}\|\boldsymbol{b}\cdot\nabla U\|^2_{L^2(\ \mathrm{K})}$$
$$+\left(\left(\left(\sum_{E\in\partial \mathrm{T}_h}\int\sigma^1[U]^2ds\right)^{\frac{1}{2}}\right)^2\right)^{\frac{1}{2}}.$$

Proof. In the equation (3.3), put $v=U$

$$\begin{aligned} a_{PD}(U,U) &= \sum_{K\in \mathrm{T}_h}\lambda(\nabla U,\nabla U)_K-\sum_{E\in\partial \mathrm{T}_h}\int(\{\lambda\nabla U\cdot n\}[U] \\ &+ \sum_{E\in\partial \mathrm{T}_h}\int(\{|\boldsymbol{b}\cdot n|U\}[U])ds+[U]\{\lambda\nabla U\cdot n\})ds \\ &+ \sigma\sum_{E\in\partial \mathrm{T}_h}\int[U][U]ds-\sum_{K\in \mathrm{T}_h}(\boldsymbol{b}\cdot\nabla U,U+\delta\boldsymbol{b}\cdot\nabla U)_K+. \end{aligned} \tag{4.2}$$

From [1]

$$a_{PD}(U,U)=\sum_{K\in \mathrm{T}_h}\left\|\lambda^{\frac{1}{2}}\nabla U\right\|^2_{L^2(K)}+\frac{\beta}{2}\left(\left(\sum_{E\in\partial \mathrm{T}_h}\int\sigma^{-1}(\{\lambda\nabla U\cdot n\})^2ds\right)^{\frac{1}{2}}\right)^2$$
$$+\frac{2}{\beta}\left(\left(\sum_{E\in\partial T_h}\int\sigma^1[U]^2ds\right)^{\frac{1}{2}}\right)^2+\varrho\|U\|^2_{H^1(T_h)}+\sigma^2G_t\|U\|^2_{H^1(\ \mathrm{T}_h)}$$
$$+\frac{\beta}{2}\sum_{E\in\partial T_h}\sigma\|[U]\|^2_{L^2(E)}+\frac{\omega^2}{2\beta}\|U\|^2_{H^1(\mathrm{T}_h)}+\sum_{K\in \mathrm{T}_h}\delta\|\boldsymbol{b}\cdot\nabla U\|^2_{L^2(\mathrm{K})},$$

$$a_{PD}(U,U)\geq g\left(\sum_{K\in \mathrm{T}_h}\left\|\lambda^{\frac{1}{2}}\nabla U\right\|^2_{L^2(K)}+\left(\left(\sum_{E\in\partial \mathrm{T}_h}\int\sigma^{-1}(\{\lambda\nabla U\cdot n\})^2ds\right)^{\frac{1}{2}}\right)^2\right.$$
$$+\sum_{E\in\partial T_h}\sigma\|[U]\|^2_{L^2(E)}+\|U\|^2_{H^1(\mathrm{T}_h)}+\sum_{K\in\mathbb{T}_h}\|\boldsymbol{b}\cdot\nabla U\|^2_{L^2(\ \mathrm{K})}$$
$$\left.+\left(\left(\sum_{E\in\partial T_h}\int\sigma^1[U]^2ds\right)^{\frac{1}{2}}\right)^2\right)+\varrho\|U\|^2_{H^1(\mathrm{T}_h)}+\sigma^2G_t\|U\|^2_{H^1(\mathrm{T}_h)},$$

where $g=\min(\frac{\beta}{2},\frac{\omega^2}{2\beta},\ 1,\frac{2}{\beta},\delta)$,

$$a_{PD}(U,U)\geq g\|U\|^2_{H^{1`}(\mathrm{T}_h)}+q\|U\|^2_{H^{1`}(\mathrm{T}_h)}$$

then

$$a_{PD}(U,U) \geq \alpha \, \|U\|^2_{H^1(\mathrm{T}_h)},$$

where $q \leq \left(\varrho + \sigma^2 G_t\right)$, and $\alpha \leq (g+q)$. □

Lemma 4.2 (countinuity)**.** *If U is the solution of equation (3.2), and $v \in \vartheta$ is the test function, then $a_{PD}(U,v)$ is continuous if κ is nonnegative,such that:*

$$\|a_{PD}(U,v)\| \leq \kappa \|U\|_{H^1(\mathrm{T}_h)} \, |v\|_{H^1(\mathrm{T}_h)} \,, \qquad \forall \; U, v \in \; \vartheta.$$

Proof. From the equation(3.3) we have

$$\begin{aligned} |a_{PD}(U,v)| =| \, \sigma \sum_{E\in\partial\mathrm{T}_h} \int [U][v]ds - \sum_{E\in\partial\mathrm{T}_h} \int (\{\lambda\nabla U\cdot n\}[v] + [U]\{\lambda\nabla v\cdot n\})ds \\ - \sum_{K\in\mathrm{T}_h} (\boldsymbol{b}\cdot\nabla U, v+\delta\boldsymbol{b}\cdot\nabla v)_K + \sum_{E\in\partial\mathrm{T}_h} \int (\{|\boldsymbol{b}\cdot n|U\}[v])ds \\ + \sum_{K\in\mathrm{T}_h} \lambda(\nabla U, \nabla v)_K \; |, \end{aligned}$$

$$\begin{aligned} |a_{PD}(U,v)| \leq \sigma \sum_{E\in\partial\mathrm{T}_h} \int |[U][v]|ds - \sum_{E\in\partial\mathrm{T}_h} \int |[U]\{\lambda\nabla v\cdot n\} + \{\lambda\nabla U\cdot n\}[v]|ds \\ + \sum_{E\in\partial\mathrm{T}_h} \int |(\{|\boldsymbol{b}\cdot n|U\}[v])|ds - \sum_{K\in\mathrm{T}_h} |(\boldsymbol{b}\cdot\nabla U, v+\delta\boldsymbol{b}\cdot\nabla v)_K| \\ + \sum_{K\in\mathrm{T}_h} |\lambda(\nabla U,\nabla v)_K| = \sum_{i=1}^{6} B_i. \end{aligned} \tag{4.3}$$

From [1], we get

$$\begin{aligned} |a(U,v)| \leq \varsigma ||U||_{H^1(\mathrm{T}_h)}||v||_{H^1(\mathrm{T}_h)} + 2|\lambda|\sigma G_t^2||U||_{H^1(\mathrm{T}_h)}||v||_{H^1(\mathrm{T}_h)} \\ + \sigma G_t^2||U||_{H^1(\mathrm{T}_h)}||v||_{H^1(\mathrm{T}_h)} + \sigma^2 G_t^2||U||_{L^2(\mathrm{T}_h)}||v||_{L^2(\mathrm{T}_h)}. \end{aligned} \tag{4.4}$$

To estimate B_5

$$\begin{aligned} B_5 = \sum_{K\in\mathrm{T}_h} (\boldsymbol{b}\cdot\nabla U, \delta\boldsymbol{b}\cdot\nabla v)_K \leq \sum_{K\in\mathrm{T}_h} |\delta|_{L_\infty}|\boldsymbol{b}\cdot\nabla U|_{L^2(K)}|\boldsymbol{b}\cdot\nabla v|_{L^2(K)} \\ \leq \sum_{K\in\mathrm{T}_h} |\delta|_{L_\infty} \left|\boldsymbol{b}^2\right|_{L^\infty} |\nabla U|_{L^2(K)}|\nabla v|_{L^2(K)} = \Lambda \sum_{K\in\mathrm{T}_h} \|U\|_{H^1(K)}\|v\|_{H^1(K)} \\ = \Lambda\|U\|_{H^1(\mathrm{T}_h)}\|v\|_{H^1(\mathrm{T}_h)}, \end{aligned} \tag{4.5}$$

where $\Lambda = |\delta|_{L_\infty}\left|\boldsymbol{b}^2\right|_{L^\infty}$.

Substituting (4.4) and (4.5) in(4.3) we get,

$$\begin{aligned}\|a_{PD}(U,v)\| &\leq \Lambda\|U\|_{H^1(\mathrm{T}_h)}\|v\|_{H^1(\mathrm{T}_h)} + |a(U,v)| = \varsigma||U||_{H^1(\mathrm{T}_h)}||v||_{H^1(\mathrm{T}_h)}\\ &+ 2|\lambda|\sigma G_t^2||U||_{H^1(\mathrm{T}_h)}||v||_{H^1(\mathrm{T}_h)} + \sigma G_t^2||U||_{H^1(\mathrm{T}_h)}||v||_{H^1(\mathrm{T}_h)}\\ &+ \sigma^2 G_t^2||U||_{L^2(\mathrm{T}_h)}||v||_{L^2(\mathrm{T}_h)} + \Lambda\|U\|_{H^1(\mathrm{T}_h)}\|v\|_{H^1(\mathrm{T}_h)}\\ &= (\varsigma + 2|\lambda|\sigma G_t^2 + \sigma G_t^2 + \sigma^2 G_t^2 + \Lambda)\|U\|_{H^1(\mathrm{T}_h)}\|v\|_{H^1(\mathrm{T}_h)}\\ &\leq \kappa\|U\|_{H^1(\mathrm{T}_h)}\|v\|_{H^1(\mathrm{T}_h)}, \end{aligned} \tag{4.6}$$

where $\kappa \geq (\varsigma + 2\,|\lambda|\,\sigma {G_t}^2 + \sigma {G_t}^2 + \sigma^2 {G_t}^2 + \Lambda)$. □

Lemma 4.3 (stability)**.** *There are a set of variables* $\xi,\ \mathrm{A},\ \varpi > 0$ *that are independent of* h *and are as follows:*

$$\|U_h(t)\|^2_{L^2(\Omega)} + \Upsilon\,\|U_{h,t}\|^2_{L^2(0,T;L^2(\Omega))} + \xi\,\|U_h\|^2_{L^2(0,T;H^1(T_h))} \leq \varpi\left(\|f\|^2_{L^2(0,T;L^2(\mathrm{T}_h))}\right.$$
$$\left. + \left\|U_h(0)\|^2_{L^2(\Omega)}\right.\right) + \varpi \sum_{E\in\partial\mathrm{T}_h}\left(\|U_N\|^2_{L^2(0,T;L^2(\Gamma^N))} + \|U_D\|^2_{L^2(0,T;L^2(\Gamma^D))}\right).$$

Proof. Let $v = U_h$ in equation (3.4), we obtain

$$\begin{aligned}&(U_{h,t}, U_h) + (U_{h,t}, \delta\boldsymbol{b}\cdot\nabla U_h) + a_{PD}\left(U_h, U_h\right) = (f, U_h) + (f, \delta\boldsymbol{b}\cdot\nabla U_h)\\ &+ \sum_{E\in\Gamma^N}\int U^N\left(U_h + \delta\boldsymbol{b}\cdot\nabla U_h\right)ds + \sum_{E\in\Gamma^D}\int \lambda\nabla\left(U_h + \delta\boldsymbol{b}\cdot\nabla U_h\right)\cdot nU^D\\ &- \sum_{E\in\Gamma^D}\int |\boldsymbol{b}\cdot n|U^D(U_h+\delta\boldsymbol{b}\cdot\nabla U_h)ds - \sigma\sum_{E\in\Gamma^D}\int U^D(U_h+\delta\boldsymbol{b}\cdot\nabla U_h)ds.\end{aligned} \tag{4.7}$$

From Lemma (4.1), we have

$$(U_{h,t}, U_h) + a_{PD}\left(U_h, U_h\right) \geq \frac{1}{2}\frac{d}{dt}\|U_h\|^2_{L^2(\Omega)} + \alpha\,\|U_h\|^2_{H^1(\mathrm{T}_h)}. \tag{4.8}$$

By Young's-inequality and Cauchy [18], we get

$$\begin{aligned}(U_{h,t}, \delta\boldsymbol{b}\cdot\nabla U_h) &\leq \|U_{h,t}\|_{L^2(\Omega)}\,\|\delta\boldsymbol{b}\cdot\nabla U_h\|_{L^2(\mathrm{T}_h)}\\ &\leq \frac{\beta}{2}\,\|U_{h,t}\|^2_{L^2(\Omega)} + \frac{1}{2\beta}\delta\,\|\boldsymbol{b}\cdot\nabla U_h\|^2_{L^2(\mathrm{T}_h)}\\ &\leq \Upsilon\left(\|U_h\|^2_{H^1(\mathrm{T}_h)} + \|U_{h,t}\|^2_{L^2(\Omega)}\right).\end{aligned} \tag{4.9}$$

By the Young's-inequality and using Cauchy inequality of equation (4.7), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|U_h\|^2_{L^2(\Omega)} + \Upsilon\left(\|U_{h,t}\|^2_{L^2(\Omega)} + \|U_h\|^2_{H^1(\mathrm{T}_h)}\right) + \alpha\|U_h\|^2_{H^1(\mathrm{T}_h)}\\
&\leq C\left(\|f\|^2_{L^2(\mathrm{T}_h)} + \|U_h\|_{L^2(\mathrm{T}_h)}\right) + \Upsilon\left(\|f\|^2_{L^2(\mathrm{T}_h)} + \|U_h\|^2_{H^1(\mathrm{T}_h)}\right)\\
&+ 2C\sum_{E\in\Gamma^D}\left(\|U_h\|^2_{H^1(K)} + \left\|U^D\right\|^2_{L^2(\Gamma^D)}\right) + 2C\sum_{E\in\Gamma^N}\left(\|U_h\|^2_{H^1(K)} + \left\|U^N\right\|^2_{L^2(\Gamma^N)}\right)\\
&+ 2C\sum_{E\in\Gamma^D}\left(\|U_h\|^2_{H^1(K)} + \left\|U^D\right\|^2_{L^2(\Gamma^D)}\right) + 2C\sum_{E\in\Gamma^D}\left(\|U_h\|^2_{H^1(K)} + \left\|U^D\right\|^2_{L^2(\Gamma^D)}\right),
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|U_h\|^2_{L^2(\Omega)}+\Upsilon\|U_{h,t}\|^2_{L^2(\Omega)}+(\alpha-9C)\|U_h\|^2_{H^1(\mathrm{T}_h)} \leq (C+\Upsilon)\|f\|^2_{L^2(\mathrm{T}_h)}\\
&+ 2C\sum_{E\in\partial T_h}(3\|U^D\|^2_{L^2(\Gamma^D)} + \|U^N\|^2_{L^2(\Gamma^N)}).
\end{aligned} \tag{4.10}$$

By integrating the equation (4.10) both sides from 0 to t, we get,

$$\begin{aligned}
&\|U_h(t)\|^2_{L^2(\Omega)|} - \|U_h(0)\|^2_{L^2(\Omega)} + \Upsilon\int_0^t \|U_{h,t}\|^2_{L^2(\Omega)} + \xi\int_0^t \|U_h\|^2_{H^1(\mathrm{T}_h)}\\
&\leq \mathrm{A}\int_0^t \|f\|^2_{L^2(\mathrm{T}_h)} + 2C\sum_{E\in\partial\mathrm{T}_h}\left(\int_0^t \left\|U^N\right\|^2_{L^2(\Gamma^N)} + 3\int_0^t \left\|U^D\right\|^2_{L^2(\Gamma^D)}\right),
\end{aligned}$$

where $\xi \leq (\alpha - 9C)$ and $\mathrm{A} = (C + \Upsilon)$, we obtain

$$\begin{aligned}
&\|U_h(t)\|^2_{L^2(\Omega)} + \Upsilon\|U_{h,t}\left\|^2_{L^2(0,T;L^2(\Omega))} + \xi\right\| U_h \left\|^2_{L^2(0,T;H^1(\mathrm{T}_h))}\right.\\
&\leq A\| f\|^2_{L^2(0,T;L^2(\mathrm{T}_h))} + \|U_h(0)\|^2_{L^2(\Omega)}\\
&+ 2C\sum_{E\in\partial\mathrm{T}_h}\left(\left\|U^N\right\|^2_{L^2(0,T;L^2(\Gamma^N))} + 3\left\|U^D\right\|^2_{L^2(0,T;L^2(\Gamma^D))}\right),
\end{aligned} \tag{4.11}$$

$$\begin{aligned}
&\|U_h(t)\|^2_{L^2(\Omega)} + \Upsilon\|U_{h,t}\|^2_{L^2(0,T;L^2(\Omega))} + \xi\|U_h\|^2_{L^2(0,T;H^1(\mathrm{T}_h))}\\
&\leq \varpi\left(\|f\|^2_{L^2(0,T;L^2(\mathrm{T}_h))} + \|U_h(0)\|^2_{L^2(\Omega)}\right)\\
&+ \varpi\sum_{E\in\partial\mathrm{T}_h}\left(\left\|U^N\right\|^2_{L^2(0,T;L^2(\Gamma^N))} + \left\|U^D\right\|^2_{L^2(0,T;L^2(\Gamma^D))}\right),
\end{aligned} \tag{4.12}$$

where $\varpi \geq 6C$. □

5. The error estimate

This section shows the semi-discrete PDGFEM error estimates in the SIPG case. The L^2-error will be used to estimate the $U - U_h$ error.

Theorem 5.1. *Let U represent the solution of equation (3.2), $U_h \in \vartheta_h$ represent the approximate solution of equation (3.4) and $U \in L^2(H^1(\Omega))$, $U_t \in L^2\left(0,\ T;\ H^1(\Omega)\right)$ and σ is large enough, then C is a constant such that:*

$$\|U-U_h\|_{L^2(\Omega)} \leq \mathrm{c}h^2\|U\|_{L^2(H^1)} + \sqrt{\frac{\beta}{2}}\mathrm{c}h^2\left(\|U_t\|_{L^2(0,T;H^1)} + \|U\|_{L^2(0,T;H^1)}\right).$$

Proof. Let ΠU be the interpolate of U, and $e = U - U_h = (U - \Pi U) + (\Pi U - U_h) = \Theta - \Xi$,So

$$\|U-U_h\|_{L^2(\Omega)} \leq \quad \|\Theta\|_{L^2(\Omega)} + \|\Xi\|_{L^2(\Omega)}. \tag{5.1}$$

From [3]

$$\|\Theta\|_{L^2(\Omega)} \leq ch^2\|U\|_{L^2(H^1)}. \tag{5.2}$$

Now,

$$\begin{aligned}(U_t,w)+a_{PD}(U,w) &= (f,w)+\sum_{E\in\Gamma^N}\int U^N w ds+\sum_{E\in\Gamma^D}\int \lambda\nabla w\cdot nU^D ds\\ &-\sum_{E\in\Gamma^D}\int|\boldsymbol{b}\cdot n|U^D w ds-\sigma\sum_{E\in\Gamma^D}\int U^D w ds,\ \ \forall w\in\emptyset,\end{aligned} \tag{5.3}$$

$$\begin{aligned}(U_{h,t},w)+a_{PD}(U_h,w) &= (f,w)+\sum_{E\in\Gamma^N}\int U^N w ds+\sum_{E\in\Gamma^D}\int \lambda\nabla w\cdot nU^D ds\\ &-\sum_{E\in\Gamma^D}\int|\boldsymbol{b}\cdot n|U^D w ds-\sigma\sum_{E\in\Gamma^D}\int U^D w ds,\ \ \forall w\in\emptyset_h.\end{aligned} \tag{5.4}$$

Subtracting (5.3) from (5.4), we obtain,

$$\begin{aligned}&((U-U_h)_t\ ,w)+a_{PD}(U-U_h,w)=((\Theta-\Xi)_t\ ,w)\\ &+a_{PD}(\Theta-\Xi,\ w)=0,\ \forall w\in\emptyset_h.\end{aligned} \tag{5.5}$$

Then

$$(\Theta_t,w)+a_{PD}(\Theta,w)=(\Xi_t,w)+a_{PD}(\Xi,w). \tag{5.6}$$

Let $w=\Xi$, we have,

$$(\Theta_t,\ \Xi)+a_{PD}(\Theta,\ \Xi)=(\Xi_t,\ \Xi)+a_{PD}(\Xi,\ \Xi)\ . \tag{5.7}$$

From Lemma 4.1, we have,

$$(\Xi_t,\ \Xi)+a_{PD}(\Xi,\ \Xi)\geq\frac{1}{2}\frac{d}{dt}\|\ \Xi\|^2_{L^2(\Omega)}+\alpha\ \|\Xi\|^2_{L^2(\mathrm{T}_h)}. \tag{5.8}$$

By Young inequality and Schwartz [18], we have,

$$(\Theta_t,\ \Xi) \leq \frac{\beta}{2} c^2 h^4 U_t{}^2_{L^2(H^1)} + \frac{1}{2\beta}\|\Xi\|^2_{L^2(\mathrm{T}_h)}. \tag{5.9}$$

From Lemma 4.2, we obtain,

$$\begin{aligned} a_{PD}(\Theta,\Xi) &\leq \kappa\|\Theta\|_{L^2(\mathrm{T}_h)}\|\Xi\|_{L^2(\mathrm{T}_h)} \\ &\leq \frac{\beta}{2}\|\Theta\|_{L^2(\mathrm{T}_h)} + \frac{\kappa^2}{2\beta}\|\Xi\|_{L^2(\ \mathrm{T}_h)} \\ &\leq \frac{\beta}{2} c^2 h^4 \|U\|^2_{L^2(H^1)} + \frac{\kappa^2}{2\beta}\|\Xi\|_{L^2(\mathrm{T}_h)}. \end{aligned} \tag{5.10}$$

Substituting (5.8),(5.9) and (5.10) in (5.7), we have,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\Xi\|^2_{L^2(\Omega)} + \alpha\|\Xi\|^2_{L^2(\mathrm{T}_h)} \leq \frac{\beta}{2} c^2 h^4 \|U_t\|^2_{L^2(H^1)} + \frac{1}{2\beta}\|\Xi\|^2_{L^2(\mathrm{T}_h)} \\ + \frac{\beta}{2} c^2 h^4 \|U\|^2_{L^2(H^1)} + \frac{\kappa^2}{2\beta}\|\Xi\|_{L^2(\mathrm{T}_h)}. \end{aligned} \tag{5.11}$$

Then

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \parallel \Xi\|^2_{L^2(\Omega)} + \left(\alpha - \frac{1}{2\beta} - \frac{\kappa^2}{2\beta}\right)\|\Xi\|^2_{L^2(\mathrm{T}_h)} \\ &\leq \frac{\beta}{2} c^2 h^4 \left(||U_t||^2_{L^2(H^1)} + \parallel U\|^2_{L^2(H^1)}\right). \end{aligned} \tag{5.12}$$

Then

$$\frac{1}{2}\frac{d}{dt} \parallel \Xi\|^2_{L^2(\Omega)} + C_1\|\Xi\|^2_{L^2(\mathrm{T}_h)} \leq \frac{\beta}{2} c^2 h^4 \left(||U_t||^2_{L^2(H^1)} + \parallel U\|^2_{L^2(H^1)}\right), \tag{5.13}$$

where $C_1 \leq (\alpha - \frac{1}{2\beta} - \frac{\kappa^2}{2\beta})$.

We can get the following result by integrating two sides of the equation (5.13) from 0 to t:

$$\parallel \Xi\,(t)\,\|^2_{L^2(\Omega)} - \parallel \Xi^0 \|^2_{L^2(\Omega)} \leq \frac{\beta}{2} c^2 h^4 \int_0^t \left(\parallel U_t \|^2_{L^2(H^1)} + \parallel U\|^2_{L^2(H^1)}\right). \tag{5.14}$$

Since $\Xi^0 = 0$, then

$$\|\Xi\,(t)\,\|^2{}_{L^2(\Omega)} \leq \frac{\beta}{2} c^2 h^4 \left(\|U_t\|^2{}_{L^2(0,T;H^1)} + \|U\|^2{}_{L^2(0,T;H^1)}\right). \tag{5.15}$$

Then

$$\parallel \Xi\|_{L^2(\Omega)} \leq \sqrt{\frac{\beta}{2}}\, c h^2 \left(\|U_t\|_{L^2(0,T;H^1)} + \|U\|_{L^2(0,T;H^1)}\right). \tag{5.16}$$

Substituting equations (5.2) and (5.16) in (5.1), we have,

$$\|U-U_h\|_{L^2(\Omega)}\le ch^2\|U\|_{L^2(H^1)}+\sqrt{\frac{\beta}{2}}ch^2\left(\|U\|_{L^2(0,T;H^1)}+\|U_t\|_{L^2(0,T;H^1)}\right). \tag{5.17}$$

Then

$$\|U-U_h\|_{L^2(\Omega)}\le Ch^2\left(||U||_{L^2(H^1)}+(||U_t||_{L^2(0,\ T;\ H^1(\Omega))}+||U||_{L^2(0,T;H^1(K)}))\right).$$

where $C\ge c+c\sqrt{\frac{\beta}{2}}$. □

6. Numerical results

In this section, we find the error $U-U_h$ of L^2-error and H^1-error of the semi-discrete PDGFEM and DGFEM in the SIPG case by using Matlab software. The problem of diffusion-convection is as follows:

$$U_t-\lambda\Delta U+\boldsymbol{b}\cdot\nabla U=f,\qquad \text{in }\Omega\times\text{J}. \tag{6.1}$$

A homogeneous Dirichlet border condition and a homogeneous beginning condition were used.The analytical solution to this problem is:

$$U(x,y,t)=e^{-t}\sin(\pi x)\,\sin(\pi y)\ .$$

Suppose that $\Omega=[0,1]\times[0,1]$, $\boldsymbol{b}=[0,1]$, as the time interval $J=(0,1)$, $\sigma=2782$, and f it is calculated by inserting the real solution into the left side of the equation (6.1). The square domain is evenly partitioned into N × N sub-squares by $\Omega=(0,1)\times(0,1)$. For triangular meshes, set h = 1/N($N=4,8,16,32,64$) as the mesh size. The numerical error outcomes and degree of convergence for DGFEM when $\delta=0$ in Table 1 and convergence rate in Figure 1, the results of the numerical error and degree of convergence for PDGFEM when $\delta=$ h/6 in Table 2 and convergence rate in Figure 2. In DGFEM, we can note that the numerical solution is not compatible with the precise solution (see Figure 3), but in PDGFEM, we note that the numerical solution is compatible with the precise solution (see Figure 4).

Table 1: Numerical results for $\lambda=0.001$ in DGFEM.

h	H^1-error	H^1-order	L^2-error	L^2-order
1/4	0.4075	0	0.1473	0
1/8	0.3227	0.3367	0.0557	1.4042
1/16	0.2373	0.4434	0.0191	1.5416
1/32	0.1734	0.4528	0.0066	1.5261
1/64	0.1283	0.4349	0.0024	1.4978

Table 2: Numerical results for $\lambda = 0.001$ in PDGFEM.

h	H^1-error	H^1-order	L^2-error	L^2-order
1/4	0.1992	0	0.0736	0
1/8	0.1021	0.9644	0.0196	1.9074
1/16	0.0533	0.9382	0.0048	2.0310
1/32	0.0293	0.8631	0.0012	2.0272
1/64	0.0158	0.8906	0.0003	2.0017

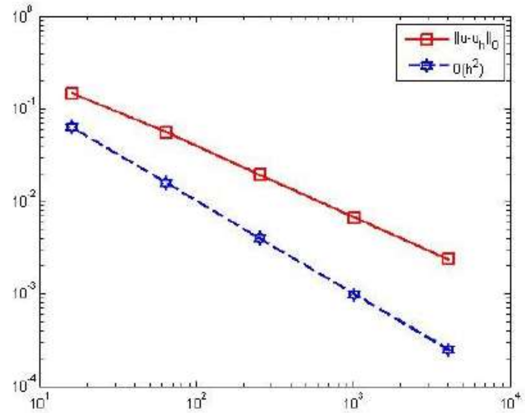


Figure 1: Convergence rate in DGFEM for $\lambda = 0.001$ in L^2norm.

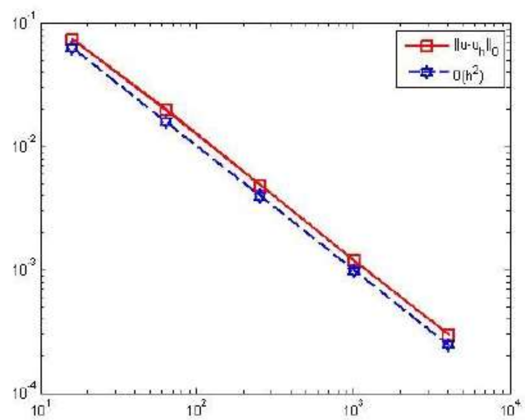
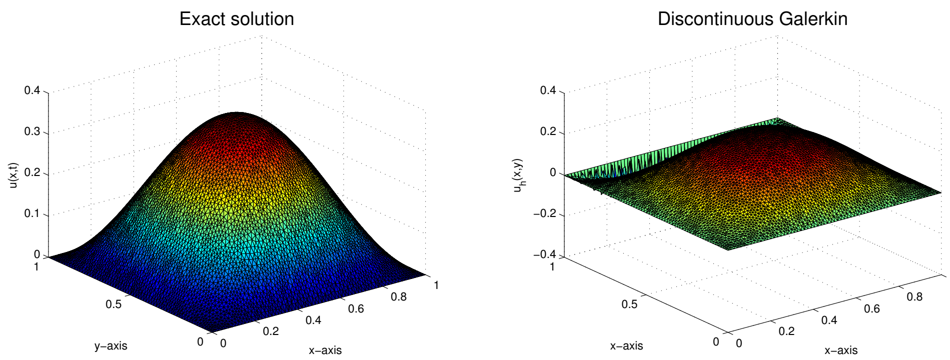
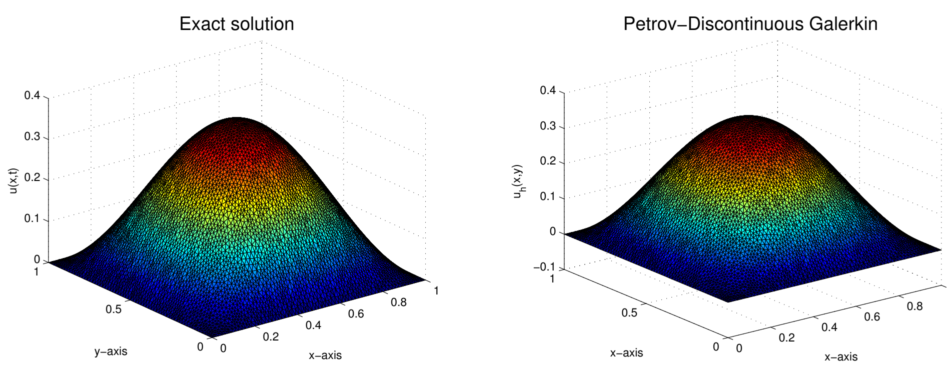


Figure 2: Convergence rate in PDGFEM for $\lambda = 0.001$ in L^2-norm.



(a) (b)

Figure 3: (a) The exact solution with $\lambda = 0.001$ and $h = 1/64$. (b) The numerical solution of DGFEM with $\lambda = 0.001$ and $h = 1/64$.



(a) (b)

Figure 4: (a) The exact solution with $\lambda = 0.001$ and $h = 1/64$. (b) The numerical solution of PDGFEM with $\lambda = 0.001$ and $h = 1/64$.

Conclusion

Throughout this current work, we have proved the continuity and $V-$elliptic properties of $a_{PD}(U, v)$ and the stability in PDGFEM. In addition, we demonstrated a theoretical analysis that shows how the PDGFEM is convergent of order $O(h^2)$. Moreover, depending on the comparison of Table 1 and Figure 1 for the DGFEM with Table 2 and Figure 2 for the PDGFEM, we stated that the numerical results of the PDGFEM showed improvement and regularity when compared to the numerical results of the DGFEM. Finally, when we smoothed the network with $n = 64$, we found that the numerical results in DGFEM are oscillated as shown in the Figure 3, but the numerical results in PDGFEM were appropriately approximated as well as free from oscillation as in the Figure 4.

References

[1] M. W. AbdulRidha, H. A. Kashkool, *The error analysis for the discontinuous galerkin finite element method of the convection- diffusion problem,* Journal of Basrah Researches (Sciences), 45 (2019).

[2] M. W. AbdulRidha, H. A. Kashkool, *Space-time petrov- discontinuous galerkin finite element method for solving linear convection-diffusion problems,* Journal of Physics: Conference Series, IOP Publishing, 2322 (2022), 012007.

[3] H. A. Kashkool, Y. H. Salim, G. A. Muften, *Error estimate of the discontinuous galerkin finite element method for convection-diffusion problems,* Basrah Journal of Science, 34(A(2)) (2016), 55-72.

[4] A. K. Hashim, *Upwind type finite element method for nonlinear convection-diffusion problem and applicationto numerical reservoir simulation,* Nankai Universty, China, 2002.

[5] A. H. Ali, A. S. Jaber, M. T. Yaseen, M. Rasheed, O. Bazighifan, T. A. Nofal, *A comparison of finite difference and finite volume methods with numerical simulations: burgers equation model,* Complexity, 2022.

[6] D. Broersen, R. Stevenson. *A robust Petrov-Galerkin discretisation of convection-diffusion equations,* Computers & Mathematics with Applications, 68 (2014), 1605-1618.

[7] J. Chan, J. A. Evans, W. Qiu, *A dual petrov-galerkin finite element method for the convection-diffusion equation,* Computers & Mathematics with Applications, 68 (2014), 1513-1529.

[8] G. A. Meften, A. H. Ali, K. S. Al-Ghafri, J. Awrejcewicz, O. Bazighifan, *Nonlinear stability and linear instability of double-diffusive convection in a rotating with ltne effects and symmetric properties: Brinkmannforchheimer model,* Symmetry, 14 (2022), 565.

[9] A. H. Ali, G. A. Meften, O. Bazighifan, M. Iqbal, S. Elaskar, J. Awrejcewicz, *A study of continuous dependence and symmetric properties of double diffusive convection: Forchheimer model,* Symmetry, 14 (2022), 682.

[10] B. Fiorina, S. K. Lele, *An artificial nonlinear diffusivity method for supersonic reacting flows with shocks,* Journal of Computational Physics, 222 (2007), 246-264.

[11] M. S. Cheichan, H. A. Kashkool, F. Gao, *A weak galerkin finite element method for solving nonlinear convection-diffusion problems in two dimensions,* Applied Mathematics and Computation, 354 (2019), 149-163.

[12] F. Gao, S. Zhang, P. Zhu, *Modified weak galerkin method with weakly imposed boundary condition for convection-dominated diffusion equations,* Applied Numerical Mathematics, 157 (2020), 490-504.

[13] O. Bazighifan, A. H. Ali, F. Mofarreh, Y. N. Raffoul, *Extended approach to the asymptotic behavior and symmetric solutions of advanced differential equations,* Symmetry, 14 (2022), 686.

[14] B. Almarri, A. H. Ali, K. S. Al-Ghafri, A. Almutairi, O. Bazighifan, J. Awrejcewicz, *Symmetric and non-oscillatory characteristics of the neutral differential equations solutions related to p-laplacian operators,* Symmetry, 14 (2022), 566.

[15] B. Almarri, A. H. Ali, A. M. Lopes, O. Bazighifan, *Nonlinear differential equations with distributed delay: some new oscillatory solutions,* Mathematics, 10 (2022), 995.

[16] C. Johnson, *Numerical solution of partial differential equations by the finite element method,* Courier Corporation, 2012.

[17] M. A. Saum, *Adaptive discontinuous galerkin finite element methods for second and fourth order elliptic partial differential equations*, 2006.

[18] B. Cockburn, G. E. Karniadakis, C.-W. Shu, *Discontinuous Galerkin methods: theory, computation and applications,* volume 11, Springer Science & Business Media, 2012.

[19] A. J. Perella, *A class of Petrov-Galerkin finite element methods for the numerical solution of the stationary convection-diffusion equation,* PhD thesis, Durham University, 1996.

Accepted: June 10, 2022