Some properties of regular topology on $C(X, Y)$

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Abstract. The recently introduced regular topology for the function space $C(X, Y)$ has been explored up to some metrizability and various countability and completeness properties. The main aim of this paper is to explore the regular topology on the function space $C(X, Y)$ in which we study submetrizability and extend various properties equivalent to the metrizability of the space $C_r(X, Y)$. We also study number of maps corresponding to the space $C_r(X, Y)$ and prove that the regular topology on the space $C(X, Y)$ is strong when X is taken discrete. Furthermore, we study various separation axioms on the space $C_r(X, Y)$, where we prove that the function space $C_r(X)$ is normal by taking X to be countable, compactly generated compact space and prove certain equivalent conditions to various separation axioms on the space $C_r(X, Y)$.

Keywords: function space, regular topology, G_δ set, submetrizability, induced map, pseudocompact, separation axioms.

1. Introduction

The function space $C(X, Y)$ symbolizes the space of continuous functions from a space X to a space Y. This space has been topologized in numerous ways and those topologies include the innate topologies such as point-open topology, compact-open topology and uniform topology. However, more stronger topologies than that of the uniform topology such as the fine topology (also known as m-topology) and the graph topology have also been studied. The fine topology on $C(X) = C(X, \mathbb{R})$ along with the topological properties was studied by Hewitt [4]. Moreover, the basis elements for fine topology on $C(X, Y)$ where X is a Tychonoff space and (Y, d) a metric space are of the fashion: $B(f, \epsilon) = \{g \in C(X, Y) | d(f(x), g(x)) < \epsilon(x), \forall x \in X\}$, where $f \in C(X, Y)$ and ϵ is a positive unit of the ring $C(X)$. Later, the topological properties corre-

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sponding to this topology have also been discussed in [11]. The space $C(X,Y)$ equipped with fine topology is proved to be submetrizable in [11].

Iberklied et al. in [15] introduced a more stronger topology than the fine topology on the space $C(X)$ and named it as the regular topology or the r-topology. This topology was defined in a manner that the positive unit in the basis elements of fine topology is replaced by a positive regular element of the ring $C(X)$. That is the basis elements for the regular topology on the space $C(X)$ are of the fashion: $R(f,r) = \{g \in C(X) \colon |f(x) - g(x)| < r(x), \forall x \in coz(r)\}$, where $f \in C(X)$, r is a positive regular element (non-zero divisor) of the ring $C(X)$ and $coz(r) = \{x \in X \colon r(x) \neq 0\}$. The space $C(X)$ equipped with the regular topology is represented as $C_r(X)$. Afterwards, Azarpanah et al. in [5] investigated compactness, connectedness and countability of this topology on the space $C(X)$. However, no study has been done on the submetrizability, separation axioms with respect to the regular topology on $C(X)$ and no map has been studied corresponding to the regular topology on the space $C(X)$.

Later, Jindal et al. [1] explored this regular topology on a more general space $C(X,Y)$, where X is Tychonoff and Y is a metric space with non-trivial path. They used the same idea as before to define the basis element for the regular topology on $C(X,Y)$ as: $R(f,r) = \{g \in C(X,Y) | d(f(x), g(x)) < r(x), \forall x \in coz(r)\}$, where where $f \in C(X,Y)$, r is a positive regular element (non-zero divisor) of the ring $C(X)$. The space $C(X,Y)$ endowed with regular topology is represented as $C_r(X,Y)$. Moreover, they studied various topological properties like metrizability, countability and several completeness properties. Despite all, the submetrizability was not studied on the space $C_r(X,Y)$, no separation axiom has been investigated for the space $C_r(X,Y)$ and no map with respect to this topology was studied. However, the submetrizability property has been studied for various function space topologies in [12], [14], [2].

The main concern of our work is to investigate submetrizability for the function space $C_r(X,Y)$, to investigate certain separation axioms and various kinds of maps on the space $C_r(X,Y)$, where X is a Tychonoff space and Y a metric space with a non-trivial path. In the first section, we demonstrate that the space $C_r(X,Y)$ is submetrizabe along with some equivalent conditions to its submetrizability. Moreover, we stretch the listicle of equivalent properties to its metrizability by replacing the metric space Y with a normed linear space with supremum norm. With this, we also see how by taking Y as a normed linear space makes the function space $C_r(X,Y)$ into a topological group.

In the second section, we study various maps such as composition function, induced map and embedding with respect to the regular topology on $C(X,Y)$. Specifically, we show how one function space can be embedded into other and derive a necessary condition when the regular topology on $C(X,Y)$ can be categorized as a strong topology.

Finally, in last portion we examine several separation axioms for the space $C_r(X,Y)$ such as Hausdorffness and regularity and provide some equivalent characterizations with respect to other function space topologies.

Moreover, the conventions that we use throughout this paper are: The space X will always represent a Hausdorff completely regular space (we will acknnowledge if it has an extra structure). The set of positive regular elements(non-zero divisors) of the ring $C(X)$ is symbolized by $r^+(X)$ and the multiplicative units of the same ring are symbolized by $U^+(X)$. The function space $C(X)$ and $C(X,Y)$ equipped with the regular topology are represented as $C_r(X)$ and $C_r(X,Y)$, respectively. The operation $\leq$ is used to represent the strength of two comparative topologies, which means the one on LHS is weaker than the one on RHS.

2. Pre-requisites

Definition 2.1.

1. *Let $g \in C(X)$, then $Z(g) = \{x \in X : g(x) = 0\}$ denotes the zero set of g and $coz(g) = \{x \in X : g(x) \neq 0\}$, is the set-theoretic complement of $Z(g)$.*
2. *Topologically, the regular elements of the ring $C(X)$ are characterized as : Let $g \in C(X)$, then it is said to be the regular element of $C(X)$ if and only if $Int_X(Z(g)) = \phi$ if and only if $coz(g)$ is dense subset of X.*
3. *A space Z is said to be pseudocompact if $f(Z)$ is bounded subset of $\mathbb{R}$, $\forall$ $f \in C(X)$, that is, for every $f \in C(X)$ there exists a natural number N for which $|f(z)| \leq N$ $\forall$ $z \in Z$.*

Definition 2.2. *In [15], an almost P-space is defined as the space where each nonempty G_δ-set has a nonempty interior. Moreover, in terms of elements of the ring $C(X)$, a space X is said to be an almost P-space if the regular elements coincide with the multiplicative units of ring $C(X)$.*

Theorem 2.1 (Theorem 2.1, [1])**.** *A space X is said to be an almost P-space if it satisfies anyone of the following conditions :*

1. *Every non-empty zero set of X has a non-empty interior.*
2. *Every non-empty G_δ-set of X has a non-empty interior.*
3. *Every zero set in X is a regular-closed set.*
4. *Every G_δ-set has an interior dense in itself.*

Theorem 2.2 (Theorem 1.8, [15])**.** *For a space X, the following are equivalent:*

1. *$C_r(X) = C_m(X)$.*
2. *X is an almost P-space.*
3. *$r^+(X) = U^+(X)$.*

Theorem 2.3 (Theorem 1.9, [15])**.** *For a space X, the following are equivalent:*

1. *$C_r(X) = C_u(X)$*
2. *X is pseudocompact, almost P-space.*

3. Submetrizability

In this section, we are going to investigate when the space $C_r(X, Y)$ is submetrizable. Moreover, we discuss how the submetrizability of space $C_r(X)$ can be characterized in terms of other weaker properties.

Definition 3.1. *A completely regular Hausdorff space (X, τ) is called submetrizable if it admits a weaker metrizable topology, equivalently, if there exists a continuous injection $f : X \to Y$, where Y is a metric space.*

Theorem 3.1. *For a space X and a Tychonoff space Y, the space $C_r(X, Y)$ is Tychonoff.*

Proof. Suppose Y is a Tychonoff space, implies Y is uniformizable. Consequently, $C_r(X, Y)$ is uniformizable [1]. Which means $C_r(X, Y)$ is Tychonoff. □

Theorem 3.2. *For a space X and a metric space (Y, d), the space $C_r(X, Y)$ is always submetrizable.*

Proof. As we know that the regular topology on $C(X, Y)$ is stronger than the fine topology on it [1]. Consequently, we can write $C_d(X, Y) \leq C_r(X, Y)$, and since $C_d(X, Y)$ is always metrizable (Corollary 2.1, [11]). Therefore, the space $C_r(X, Y)$ is submetrizable. □

Definition 3.2 (Definition 2.2, [11])**.** *A topological space Y is called a space of countable pseudocharacter if every point in Y is a G_δ-set (countable intersection of open sets) in Y. Such spaces are also called as E_0-spaces. Moreover, in a submetrizable space, every point is a G_δ-set. So, the submetrizable spaces are E_0-spaces. The study regarding E_0-spaces and submetrizable spaces can be found in [3] and [6], respectively.*

Corollary 3.1. *The space $C_r(X, Y)$ is of countable pseudocharacter.*

Remark 3.1 (Remark 5.2 in [12])**.**

1. If a space is having G_δ-diagonal, that is for a space X, if the set $\{(x, x) : x \in X\}$ is a G_δ-set in the product space $X \times X$, then each element of X is a G_δ-set. Note that every metrizable space has a zero-set diagonal which implies it has a regular G_δ-diagonal implies it has a G_δ-diagonal. Consequently, every submetrizable space has a zero-set diagonal.

2. In submetrizable spaces, all compact sets, pseudocompact sets, countably compact sets and singleton sets are G_δ-sets.

Next, we see various properties which are equivalent to the submetrizability of space $C_r(X)$. The above remark leads us to the following theorem:

Theorem 3.3. *For a space X, we have the following equivalent properties:*

1. *$C_r(X)$ is submetrizable.*
2. *$C_r(X)$ has a zero set diagonal.*
3. *$C_r(X)$ has a regular G_δ-diagonal.*
4. *$C_r(X)$ has a G_δ-diagonal.*
5. *Each singleton set in $C(X)$ is G_δ in $C_r(X)$.*
6. *$\{0_X\}$ is a G_δ in $C_r(X)$.*
7. *X is separable*
8. *$C_p(X)$ is submetrizable.*

Proof. Since (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) follows from the above discussion. (4) $\Rightarrow$ (5) $\Rightarrow$ (6) are immediate.
(6) $\Rightarrow$ (7) Suppose $\{0_X\}$ is G_δ in $C_r(X)$, then there exists a countable family $\mathfrak{N}$ of open sets in $C_r(X)$ so that $\{0_X\} = \bigcap \mathfrak{N}$.

Now, assume that $\mathfrak{N}$ has elements of the form $B(f_1, r_1), \cdots \cap B(f_k, r_k) \cap B(0_X, r_m), \cdots \cap B(0_X, r_n)$, where $f_i \in C(X), r_j \in r^+(X), 0_X$ is a constant function and $1 < i < k$ and $1 < j < n$.

Now, for each $U = B(f_1, r_1), \cdots \cap B(f_k, r_k) \cap B(0_X, r_m), \cdots \cap B(0_X, r_n) \in \mathfrak{N}$, fix $x_j \in coz(r_j)$ and put $H(U) = \{y_1, \cdots y_m, x_1, \cdots, x_n\}$. Let $A = \{H(U) : U \in \mathfrak{N}\}$. Clearly, A is countable. Suppose $Cl(A) \neq X$, so $\exists\ x_0 \in X - Cl(A)$. Since X is a completely regular space so $\exists\ f \in C(X)$ such that $f(x_0) = 1, f(y) = 0 \forall y \in cl(A)$. This implies $f \in U$ for each $U \in \mathfrak{N}$. So, $f = 0_X$, but $f(x_0) = 1$. Thus, $cl(A) = X$. Hence, X is separable.

(7) $\Leftrightarrow$ (8) is well known.

(8) $\Rightarrow$ (1) Since $C_p(X) \leq C_r(X)$. □

In the next result, we stretch the list of equivalent characterizations of metrizability of $C_r(X,Y)$. Infact, we see how X being pseudocompact, almost P-space acts also as the necessary and sufficient condition for the space $C_r(X,Y)$ to be countably tight, radial and pseudoradial.

Theorem 3.4. *For a space X and a metric space (Y,d) with a non-trivial path, we have the following equivalent conditions:*

1. *X is pseudocompact, almost P-space.*
2. *$C_d(X,Y) = C_r(X,Y)$.*
3. *$C_r(X,Y)$ is metrizable.*
4. *$C_r(X,Y)$ is first countable.*
5. *$C_r(X,Y)$ is of pointwise countable type.*

6. $C_r(X,Y)$ *is an r-space.*

7. $C_r(X,Y)$ *is an M-space.*

8. $C_r(X,Y)$ *is an p-space.*

9. $C_r(X,Y)$ *is an q-space.*

10. $C_r(X,Y)$ *is a Frechet space.*

11. $C_r(X,Y)$ *is a Sequential space.*

12. $C_r(X,Y)$ *is a k-space.*

13. $C_r(X,Y)$ *is countably tight.*

14. $C_r(X,Y)$ *is radial.*

15. $C_r(X,Y)$ *is pseudoradial.*

Proof. The equivalent conditions from (1) upto (9) are true as proved in (Theorem 2.7, [1]).

And since (4) $\Rightarrow$ (10) $\Rightarrow$ (11) $\Rightarrow$ (12) are well known.

(12) $\Rightarrow$ (13) It supports because a regular k-space having points G_δ is countably tight. However, let's prove it by contradiction. Suppose a regular k-space Z with points G_δ is not countably tight, then there exists a subset S of Z in such manner that the set $H = \{\bar{P} \colon P \subseteq S$ and P is countable $\} \subsetneq \bar{S}$. Since H contains S and H is not closed. Therefore, there exists a compact subset C of Z in such a way that $H \cap C$ is not closed in C. In addition, every compact space where singleton sets are G_δ is first countable. Thus, there exists a sequence (x_n) in $H \cap C$ converging to some $x \in C \backslash H$.

Now, $\forall\, n \in N$, $\exists$ a countable $P_n \subseteq S$ so that $x_n \in \bar{P_n}$. Hence, $x \in \overline{\bigcup_{n\in N} P_n}$. Since $\bigcup_{n\in N} P_n$ is countable in S, $x \in H$. Which is a contradiction.

Now, (13) $\Rightarrow$ (1) Suppose X is not an almost P-space. Then, we can find a non-empty zero set say S in X which has empty interior. Let $r \in C(X)$ such that $Z(r) = S$. Since $Z(r) = Z(|r|)$, then we can assume $r \geq 0$. Consequently, $r \in r^+(X)$. As $C_r(X,Y)$ is countably tight, so we can consider a countable subset $\{g_n \colon n \in N\}$.

Now, choose $e \in Z(r)$. Since Y contains a non-trivial path, so we can find $t_0 \in Y \setminus \{g_n(e) \colon n \in N\}$. Let g_0 be a constant function in $C_r(X,Y)$ taking values t_0. Then, $R(g_0, r)$ is a non-empty open set in $C_r(X,Y)$ that does not intersect $\{g_n \colon n \in N\}$. Which is not true. Thus, X is an almost P-space.

Hence, by (Theorem 2.2, [1]), $C_f(X,Y) = C_r(X,Y)$. Thus, $C_f(X,Y)$ is also countably tight. But, the (Theorem 3.3, [8]) implies that X is pseudocompact. Which finishes the proof (13) $\Rightarrow$ (1).

Clearly, (10) $\Rightarrow$ (14) $\Rightarrow$ (15). We show that (15) $\Rightarrow$ (13) by contradiction. Consider a nonclosed subset N of $C_r(X,Y)$. Then, there exists a cardinal k

and a k-sequence in N, say $(g_\sigma)_{\sigma<k}$ in such a way that the sequence converges to some $g \in N$. We lay claim to the fact that there is an $\aleph_0$-subsequence that converges to g. If this is shown, it will declare that $C_r(X,Y)$ is a sequential space.

For every natural number n, we can choose an ordinal $\sigma_n < k$ so that $\sigma_n > \sigma_{n-1}$ and for every $\sigma_n < \tau < k, g_\tau \in B_g(g, 1/n)$. The sequence (σ_n) converges to k. Otherwise there is an ordinal $\tau < k$ such that $\sigma_n < \tau$ for each n, hence $g = g_\tau \in N$; a contradiction. Next, for any $r \in r^+(X)$, there is an ordinal σ such that for every $\sigma < \tau < k$, we have $g_\tau \in B_g(g,r)$. Since (σ_n) converges to k, there is an n such that $\sigma < \sigma_m < k, \forall m \geq n$. Hence, $g_{\sigma_m} \in B_g(g,r)$ for each $m \geq n$. Thus, $g_{\sigma_m} \forall m \geq n$ converges to g. □

Example 3.1. Let $X = [0, \omega_1)$ and $Y = \mathbb{R}$, the the space $C_r([0,\omega_1))$ is submetrizable. Since the space $[0,\omega_1)$ is countably compact [Example 2.2, [11]] implies X is pseudocompact. The space $C_f([0,\omega_1))$ is metrizable. Also the space $[0,\omega_1)$ is not an almost P-space. Therefore, we have $C_f([0,\omega_1)) \neq C_r([0,\omega_1))$. Hence, the space $C_r([0,\omega_1))$ is submetrizable.

Example 3.2. For a real line $\mathbb{R}$, let $\beta\mathbb{R}$ denotes its Stone-Cech compactification. Let $X = \beta\mathbb{R} - \mathbb{R}$, then X is an almost P-space [10] and since $\mathbb{R}$ is locally compact, so it is open in $\beta\mathbb{R}$, and $\beta\mathbb{R} - \mathbb{R}$ is therefore compact, thus pseudocompact. Then, we have $C_d(\beta\mathbb{R} - \mathbb{R}) = C_r(\beta\mathbb{R} - \mathbb{R})$, implies $C_r(\beta\mathbb{R} - \mathbb{R})$ is metrizable and hence submetrizable.

In the upcoming result, we see how by taking Y as a normed linear space with supremum norm, one can further stretch the list of characterizations equivalent to metrizability of the space $C_r(X,Y)$. Before that we require the below results to prove the main theorem.

Theorem 3.5. *For a space X and a normed linear space $(Y, \|.\|_\infty)$ with supremum norm, the function space $C_r(X,Y)$ is a topological group under pointwise addition.*

Proof. Clearly, under pointwise addition, $C_r(X,Y)$ is a group.

Now, it is sufficient to prove that the group operations are continuous. Suppose $s\colon C_r(X,Y) \times C_r(X,Y) \to C_r(X,Y)$ be defined as $s(g_1,g_2) = g_1 + g_2, \forall g_1, g_2 \in C(X,Y)$. Consider a basic neighborhood $B(g_1+g_2, r)$ of g_1+g_2 in $C_r(X,Y)$, where r is the regular element of ring $C(X)$. Take $\epsilon_1 = r(x)/3 = \epsilon_2, x \in coz(r)$, and observe the neighborhood $B(g_1,\epsilon_1) \times B(g_2,\epsilon_2)$ of (g_1,g_2) in $C_r(X,Y) \times C_r(X,Y)$. Suppose $(h_1,h_2) \in B(g_1,\epsilon_1) \times B(g_2,\epsilon_2)$. Then, for $x \in coz(r)$,

$$\begin{aligned}\|(g_1+g_2)(x) - (h_1+h_2)(x)\| &\leq \|g_1(x) - h_1(x)\| + \|g_2(x) - h_2(x)\| \\ &< \epsilon_1(x) + \epsilon_2(x) < r(x)\end{aligned}$$

Then, $s(B(g_1,\epsilon_1) \times B(g_2,\epsilon_2)) \subseteq B(g_1+g_2, r)$. Therefore, s is continuous.

Now, let $I\colon C_r(X,Y) \to C_r(X,Y)$ defined by $I(f) = -f$ for any $f \in C(X,Y)$, where $(-f)(x) = -f(x) \in Y$. Observe the neighborhood $B(-f,r)$ of $-f$. Therefore, $I(B(f,r)) = B(-f,r)$. Thus, I is continuous. Hence, $C_r(X,Y)$ is a topological group. □

Since we have shown that the function space $C_r(X,Y)$ is topological group, for a space X and a normed linear space $(Y, \|.\|_\infty)$. Thus, it is a homogeneous space [11]. However, a space A is termed to be a homogeneous space if for each pair of points $a, b \in A$, there exists a homeomorphism of A onto itself that carries a to b. Further, to prove next result, we first require the following two lemmas:

Lemma 3.1 (Lemma 2.1, [11])**.** *Let D be a dense subset of a space X and $x \in D$. Then, x has a countable local π-base in D if and only if x has a countable local π-base in X.*

Lemma 3.2 (Lemma 2.3, [11])**.** *Let D be a dense subset of a space X and C be a compact subset D. Then, C has countable character in D if and only if C has countable character in X.*

Theorem 3.6. *For a space X and a normed linear space $(Y, \|.\|_\infty)$, the space $C_r(X,Y)$ has a countable π-character if and only if $C_r(X,Y)$ has a dense subspace having countable π-character.*

Proof. Consider a dense subspace C of $C_r(X,Y)$ having a countable π-character. Take $f \in C$ to be arbitrary. Because f has a countable local π-base in C, then by the (Lemma 3.1) f has a countable local π-base in $C_r(X,Y)$. Therefore, there exists a sequence $\{O_n \colon n \in \mathbb{N}\}$ of open sets in $C_r(X,Y)$ in such a manner that whenever O is an open set carrying f, $O_n \subseteq O$ for some n. Take an arbitrary $g \in C_r(X,Y)$. As $C_r(X,Y)$ is a homogeneous space, thus there exists a homeomorphism $h\colon C_r(X,Y) \to C_r(X,Y)$ defined by $h(f) = g$. Therefore, $\{h(O_n) \colon n \in \mathbb{N}\}$ is a sequence of open sets in $C_r(X,Y)$. Let P be an open set with $g \in P$. Therefore, $f \in h^{-1}(P)$ and there exists n such that $O_n \subseteq f^{-1}(P)$. As a consequence, g has a countable local π-base in $C_r(X,Y)$. Hence, $C_r(X,Y)$ has a countable π-character. Clearly, the converse follows. □

Theorem 3.7. *For a space X and a normed linear space $(Y, \|.\|_\infty)$, the space $C_r(X,Y)$ is of pointwise countable type if and only if $C_r(X,Y)$ has a dense subspace of pointwise countable type.*

Proof. Consider a dense subspace C of $C_r(X,Y)$ that is of pointwise countable type. Let $f \in C$ and $g \in C(X,Y)$. Since $C_r(X,Y)$ is homogeneous, so there exists a homeomorphism $H\colon C_r(X,Y) \to C_r(X,Y)$ so that $H(f) = g$. Since C is a dense subspace of $C_r(X,Y)$, so there exists a compact subset, say K so that $f \in K$ and is of pointwise countable character in C. Thus, by above (Lemma 3.2), K has countable character in $C_r(X,Y)$. Therefore, $H(K)$ is a compact subset of $C_r(X,Y)$ having countable character in $C_r(X,Y)$, and $g \in H(K)$. Hence, $C_r(X,Y)$ is of pointwise countable type. The converse is immediate. □

Theorem 3.8. *For a space X and a normed linear space $(Y, \|.\|_\infty)$, we have the following equivalences :*

1. *X is pseudocompact, almost P-space.*
2. *$C_d(X,Y) = C_r(X,Y)$.*
3. *$C_r(X,Y)$ is metrizable.*
4. *$C_r(X,Y)$ is of pointwise countable type.*
5. *$C_r(X,Y)$ has a dense subset which is of pointwise countable type.*
6. *$C_r(X,Y)$ is countably tight.*
7. *$C_r(X,Y)$ is first countable.*
8. *$C_r(X,Y)$ has a countable π-character.*
9. *$C_r(X,Y)$ has a dense subspace of countable π-character.*
10. *$C_r(X,Y)$ is normed linear space.*
11. *$C_r(X,Y)$ is topological vector space.*

Proof. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) are true (Theorem 2.7, [1]).

(4) $\Leftrightarrow$ (5) is proved in above (Theorem 3.7).

(1) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) are true as proved in (Theorem 3.4).

(7) $\Rightarrow$ (8). Since $C_r(X,Y)$ is a topological group and a topological group is first countable if and only if it has countable π-character.

(8) $\Leftrightarrow$ (9) is proved in above (Theorem 3.6).

(1) $\Rightarrow$ (10) Suppose X is pseudocompact and almost P-space then $C_r(X,Y) = C_d(X,Y)$ (Theorem 2.7, [1]). But when X is pseudocompact, then $C_d(X,Y)$ is a normed linear space under the supremum norm $||.||_\infty$ defined by $\|f\|_\infty = sup\{\|f(x)\| : x \in X\}$. Thus, the space $C_r(X,Y)$ is a normed linear space.

(10) $\Rightarrow$ (11) is immediate.

(11) $\Rightarrow$ (1) Suppose X is not an almost P-space, then there exists a non-empty zero set say A which has empty interior in X. Let $s \in C(X)$ be in such a way that $Z(s) = A$. As $Z(s) = Z(|s|)$, thus $s \in r^+(X)$. Without the loss of generality, we can assume s in such a way that there $\nexists$ any $\delta > 0$ so that $\delta < s(x), \forall x \in coz(s)$. Consider a non-zero element y_0 and define $f_{y_0} : X \to Y$ as $f_{y_0}(x) = y_0, \forall x \in X$. We prove that the scaler multiplication is not continuous at $(0, f_{y_0}) \in \mathbb{R} \times C_r(X,Y)$. Consider a basic neighborhood $B(0_X, s)$ of 0_X in $C_r(X,Y)$ where $0_X(x) = 0, \forall x \in X$.

Now, consider a basic neighborhood $(-\epsilon, \epsilon) \times B(f_{y_0}, r)$ of $(0, f_{y_0})$ in $\mathbb{R} \times C_r(X,Y)$, where $\epsilon > 0$ and $r \in r^+(X)$. Then, for any non-zero $\alpha \in (-\epsilon, \epsilon)$, αf_{y_0} does not belong to $B(0_X, s), \forall x \in coz(s)$. Because then $||\alpha f_{y_0}(x)\| = |\alpha| ||y_0|| < s(x), \forall x \in coz(s)$. But this contradicts our choice of $s \in r^+(X)$. So, if X is not an almost P-space, then $C_r(X, Y)$ is not a topological vector space. In other words, $C_r(X, Y)$ being topological vector space implies X is an almost P-space.

But X being almost P-space implies that $C_r(X, Y) = C_f(X, Y)$ (Theorem 2.2, [1]). Therefore, $C_f(X, Y)$ is a topological vector space. However, (Theorem 2.2, [11]) shows that $C_f(X, Y)$ is topological vector space if and only if X is pseudocompact. This finishes the proof that (11) ⇒ (1). □

4. Some special maps

In this section, we will be discussing various maps that can be drawn over or from the space $C_r(X, Y)$ which includes composition function, induced map and embedding. In function spaces, the function $i\colon Y \to C(X, Y)$ defined as $i(t) = c_t$, where c_t is a constant map is an injection [7]. However, in particular, the function $i\colon \mathbb{R} \to C(X, \mathbb{R})$ defined as $i(t) = c_t$, where $c_t \ \forall\ t \in \mathbb{R}$ is a constant map is an injection [7].

Definition 4.1 (Composition function)**.** *Suppose X, Y and $\mathbb{R}$ are spaces, a composition function $\phi\colon C_r(X, Y) \times C_r(Y, \mathbb{R}) \to C_r(X, \mathbb{R})$ is defined by $\phi(f, g) = g \circ f$, $f \in C_r(X, Y)$, $g \in C_r(Y, \mathbb{R})$*

Definition 4.2 (Induced map)**.** *Suppose $g \in C_r(Y, \mathbb{R})$, then an induced map $g_*\colon C_r(X, Y) \to C_r(X, \mathbb{R})$ is defined by $g_*(f) = \phi(f, g) = g \circ f$, $f \in C_r(X, Y)$. In particular, for $g \in C_r(X, Y)$, then an induced map for the function space $C(X)$ is defined as $g_*\colon C_r(Y) \to C_r(X)$ with $g_*(f) = \phi(f, g) = g \circ f$, $f \in C_r(Y)$.*

An induced map is formed by fixing one of the components of composition function. Note that the induced maps preserve composition as : $(g \circ f)_* = g_* \circ f_*$.

Theorem 4.1. *Let $g \in C_r(Y, \mathbb{R})$, then g is one-to-one if and only if $g_* : C_r(X, Y) \to C_r(X, \mathbb{R})$ is one-to-one.*

Proof. Let g is one-to-one. To prove $g_*\colon C_r(X, Y) \to C_r(X, \mathbb{R})$ is one-to-one. Let's consider f_1, $f_2 \in C_r(X, Y)$ and let $g_*(f_1) = g_*(f_2)$. This implies $\phi(f_1, g) = \phi(f_2, g)$. Which implies $g \circ f_1 = g \circ f_2$. Then, $g(f_1) = g(f_2)$. Implies $f_1 = f_2$. Therefore, $g_*\colon C_r(X, Y) \to C_r(X, \mathbb{R})$ is one-to-one.

Conversely, let g_* is one-to-one. To prove $g \in C(Y, \mathbb{R})$ is one-to-one. For this, consider $x_1, x_2 \in Y$ and let $g(x_1) = g(x_2)$. This implies $g_*(g(x_1)) = g_*(g(x_2))$. Which implies $\phi(g(x_1), g) = \phi(g(x_2), g)$. Then, $\phi(g, g) = \phi(g, g)$. Then, we can write $g^{-1}(g(x_1)) = g^{-1}(g(x_2))$. Implies $x_1 = x_2$. Therefore, g is one-to-one. □

Theorem 4.2. *Let $g \in C_r(Y, \mathbb{R})$ and $g_*\colon C_r(X, Y) \to C_r(X, \mathbb{R})$ is onto then g is onto.*

Proof. Let g_* is onto, then by definition there exists $f_1 \in C(X, \mathbb{R})$ such that $f_1 = g_*(g_1)$, $\forall\ g_1 \in C(X, Y)$. This implies $f_1 = \phi(g_1, g)$, which implies $f_1 = g \circ g_1$. Then, $f_1 = g(g_1)$. Thus, g is onto. □

Definition 4.3. *A function f from a non-empty set A to a topological space B is said to be an almost onto map if $f(A)$ is dense in B.*

Theorem 4.3 (Theorem 2.2.6 (a), [7])**.** *Let $g \in C(X,Y)$, then the induced map $g_*\colon C(Y) \to C(X)$ is one-one if and only if g is almost onto.*

Theorem 4.4. *For a Tychonoff space X and a metric space (Y,d), and let $g \in C_r(Y,\mathbb{R})$, then the induced map $g_*\colon C_r(X,Y) \to C_r(X,\mathbb{R})$ defined as $g_*(f) = \phi(f,g) = g \circ f$, $f \in C_r(X,Y)$ is continuous.*

Proof. Let $B(f,r)$ be a basic open subset of $C_r(X)$, where r is a non-negative regular element of the ring $C(X)$ and $B(f,r) = \{h \in C(X)\colon |f(x) - h(x)| < r(x), \forall x \in coz(r)\}$.

Now, we will show that $g_*^{-1}[B(f,r)]$ is open in $C_r(X,Y)$. So, for this, let $h \in g_*^{-1}[B(f,r)]$ and we will show it is an interior point of $g_*^{-1}[B(f,r)]$.

For every $x \in coz(r)$, we know from the definition that

$$|g(h(x)) - f(x)| < r(x) \Rightarrow g(h(x)) \in B_{r(x)}(f(x))$$

Since $B_{r(x)}(f(x)$ is open, we can thus find another regular element $\acute{r} \in C(X)$ so that

$$B_{\acute{r}(x)}(g(h(x))) \subseteq B_{r(x)}(f(x)) \tag{1}$$

Then, as g is continuous so by the continuity of g at x, $\exists\, \delta$ a non-negative regular element of ring $C(X)$ such that

$$\forall y \in coz(\delta)\colon d_Y(h(x),y) < \delta(x) \Rightarrow g(y) \in B_{\acute{r}(x)}(g(h(x))) \tag{2}$$

Now, if $\acute{h} \in B(h,\delta)$, from (2) we can conclude that

$$\forall x \in coz(\acute{r})\colon g(\acute{h}(x)) \in B_{\acute{r}(x)}(g(h(x)))$$

Thus, from (1) it is evident that $g_*(\acute{h}) \in B(f,r)$. Therefore, $B(h,\delta) \subseteq g_*^{-1}[B(f,r)]$ as required. □

Corollary 4.1. *For a space X, let $g \in C_r(X,Y)$ for some space Y, then the induced map $g_*\colon C_r(Y) \to C_r(X)$ is continuous.*

Theorem 4.5. *For a space X and a metric space (Y,d), the map $\phi\colon Y \to C_r(X,Y)$ where $\phi(y) = \bar{y}$ and $\bar{y}$ is a constant map in $C_r(X,Y)$, is an embedding.*

Proof. Since, ϕ is one-one and the basis elements for regular topology on $C(X,Y)$ are of the form $B(f,r)$ where $f \in C(X,Y)$, r is a non-negative regular element of the ring $C(X)$, and

$$B(f,r) = \{g \in C(X,Y)\colon d(f(x),g(x)) < r(x), \forall x \in coz(r)\}$$

Now, as ϕ maps $y \in Y$ to $\phi(y) \in C_r(X,Y)$ defined by $\phi(y)(x) = \bar{y}(x) \forall x \in X$ is continuous.

Suppose $y_n \to y_0$ in (Y,d), it is enough to show sequential continuity, as Y is a first countable space. Then, it is clear that $\phi(y_n) \to \phi(y_0)$ such that if $B(\phi(y_0), r)$ is a basic neighborhood of $\phi(y_0)$ then by convergence, there is some N such that $n \geq N$ implies $d(y_n, y_0) < r(x), \forall x \in coz(r)$. Then, also $n \geq N$ implies $\phi(y_n) \in B(\phi(y_0), r)$.

Thus, ϕ is an embedding and we have $\phi[B(y,r)] \cap \phi[Y] = B(\phi(y), r) \cap \phi[Y]$ so ϕ maps open sets to open sets in $\phi(y)$. □

Corollary 4.2. *For a space X and a real line $\mathbb{R}$, the map $\phi \colon \mathbb{R} \to C_r(X)$ where $\phi(y) = \bar{y}$ and $\bar{y}$ is a constant map in $C_r(X)$ is an embedding.*

Now, we provide a scenario in which a function space can be embedded into another function space with regular topology.

Theorem 4.6. *Suppose that the space Y is a continuous image of the space X. Then, $C_r(Y)$ can be embedded into $C_r(X)$.*

Proof. Let $s \colon X \to Y$ be a continuous surjection, i.e. s is a continuous function from X onto Y. Define the map $\psi \colon C_r(Y) \to C_r(X)$ by $\psi(f) = f \circ s$ for all $f \in C_r(Y)$. We show that ψ is a homeomorphism from $C_r(Y)$ into $C_r(X)$.

First we show ψ is a one-to-one map. Let $f, g \in C_r(Y)$ with $f \neq g$ such that $\psi(f) \neq \psi(g)$. Then, there exists $y \in Y \colon f(y) \neq g(y)$. Choose some $x \in X \colon s(x) = y$. Which means $f \circ s \neq g \circ s$. Implies that $f(s(x)) \neq g(s(x)) \Rightarrow f(y) \neq g(y)$.

Next, we show that ψ is continuous. Let $f \in C_r(Y)$ and $B(g, r_i) = \{q \in C_r(X) \colon |q(x_i) - g(x_i)| < r_i(x_i), x_i \in Coz(r_i)\}$, where $x_i \in X$ and $r_i \in r^+(X)$. Next, for each i, $f(s(x_i)) \in B(g, r_i)$.

Now, consider $R(h, l_i) = \{p \in C_r(Y) \colon |p(s(x_i)) - h(s(x_i))| < l_i(x_i), x_i \in Coz(l_i)\}$. Clearly $f \in R(h,l)$. It follows that $\psi R(h, l_i) \subset B(g, r_i)$. Since for each $p \in R(h, l_i)$, it is clear that $\psi(p) = p \circ s \in B(g, r_i)$.

Now, we prove that $\psi^{-1} \colon \psi(C_r(Y)) \to C_r(Y)$ is continuous. Let $\psi(f) = f \circ s \in \psi(C_r(Y)), f \in C_r(Y)$. Let G be an open set with $\psi^{-1}(f \circ s) = f \in G$ such that $G(g, r_i) = \{p \in C_r(Y) \colon |g(y_i) - p(y_i)| < r_i(y_i), y_i \in Coz(r_i)\}$. Choose $x_1, x_2, \dots x_m$ such that $s(x_i) = y_i \forall i$. We have $f(s(x_i)) \in G(g, r_i) \forall i$. Define an open set $H(h, l_i) = \{q \in \psi(C_r(Y)) \subset C_r(X), \forall i$ such that $|h(x_i) - q(x_i)| < l_i(x_i)\}$. Clearly, $f \circ s \in H$. Note that $\psi^{-1}(H) \subset G$. To see this, let $p \circ s \in H$, where $p \in C_r(Y)$. Implies $p(s(x_i)) = p(y_i)$. It follows that ψ^{-1} is continuous. □

Now, we define restriction map. Suppose A is a subset of B, then the restriction map is defined as: $\pi_A \colon C(B) \to C(A)$ as $\pi_A(f) = f_{|A}$.

Theorem 4.7. *For an arbitrary subspace Y of a space X, the map $\pi_Y \colon C_r(X) \to C_r(Y)$ is continuous.*

Proof. Let $B(f,r) = \{g \in C(Y)\colon |f(y) - g(y)| < r(y), y \in coz(r)\}$ be an open set in $C_r(Y)$. We need to prove that $\pi_Y^{-1}(B(f,r))$ is open in $C_r(X)$. We have $\pi_Y^{-1}(B(f,r)) = \{g \in C(X)\colon |\pi_Y(g)(y) - f(y)| < r(y), y \in coz(r)\}$ $= \{g \in C(X)\colon |g_{|Y}(y) - f(y)| < r(y)\}$ which is open in $C_r(X)$. Hence, the map $\pi_Y\colon C_r(X) \to C_r(Y)$ is continuous. □

Theorem 4.8. *The map $\pi_Y\colon C_r(X) \to C_r(Y)$ is one-to-one if and only if Y is dense in X.*

Proof. Suppose Y is dense in X, we will show that $\pi_Y\colon C_r(X) \to C_r(Y)$ is one-to-one. Let $f, g \in C_r(X)$. Then, due to the continuity of these functions and $\bar{Y} = X$, it implies that if $f \neq g$ then $f_{|Y} \neq g_{|Y} \Rightarrow \pi_Y(f) \neq \pi_Y(g)$. Hence, π_Y is one-to-one.

Conversely, suppose that π_Y is one-to-one. We will show that Y is dense in X by contradiction. Assume that Y is not dense in X and let $f, g \in C_r(X)$. Then, $f \neq g$ does not imply that $f_{|Y} \neq g_{|Y}$. Thus, we can have $f_{|Y} = g_{|Y} \Rightarrow \pi_Y(f) = \pi_Y(g)$, which is a contradiction to π_Y being one-to-one. Hence, Y is dense in X. □

Theorem 4.9. *For a dense subspace Y of a space X, the map $\pi_Y\colon C_r(X) \to C_r(Y)$ is an embedding.*

Proof. Since the map π_Y is one-to-one and continuous. Then, we only need to prove that it is an open map onto $\pi_Y(C_r(X))$. For this let $B(f,r)$ be an open set in $C_r(X)$.

Now, we will show that $\pi_Y(B(f,r)) = B(f_{|Y}, r) \cap \pi_Y(C_r(X))$. Let $h \in \pi_Y(B(f,r))$, then by definition $|h(y) - \pi_Y(f)(y)| < r(y), y \in coz(r) \Rightarrow |h(y) - f_{|Y}(y)| < r(y)$. This implies $h \in B(f_{|Y}) \cap \pi_Y(C_r(X))$. Therefore, $\pi_Y(B(f,r)) \subset B(f_{|Y}, r) \cap \pi_Y(C_r(X))$.

Next, let $h \in B(f_Y, r) \cap \pi_Y(C_r(X))$. Then, $|h(y) - f_{|Y}(y)| < r(y), y \in coz(r)$ $\Rightarrow |h(y) - \pi_Y(f)(y)| < r(y)$. Therefore, $h \in \pi_Y(B(f,r))$ and thus $B(f_{|Y}, r) \cap \pi_Y(C_r(X)) \subset \pi_Y(B(f,r))$. Hence, π_Y is an embedding and $\pi_Y(C_r(X))$ can be treated as a subspace of $C_r(Y)$. □

Theorem 4.10. *For a space X, if Y is a subspace of X and $\pi_Y\colon C_r(X) \to C_r(Y)$ is defined as $\pi_Y(f) = f_{|Y}$. Then, $C_r(Y) = \pi_Y(C_r(X))$.*

Proof. Since $\pi_Y(C_r(X)) \subset C_r(Y)$, we will show that $C_r(Y) \subset \pi_Y(C_r(X))$. So, for this, let $g \in C_r(Y)$ and $B(g,r)$ be a basic neighborhood of g in $C_r(Y)$. Define a function $f\colon X \to \mathbb{R}$ as:

$$f(x) = \begin{cases} 0, & x \in X \ coz(r), \\ g(y), & x \in coz(r). \end{cases}$$

Consequently, $f \in C_r(X)$ and $\pi_Y(f) \in B(g,r)$. Thus, $C_r(Y) \subset \pi_Y(C_r(X))$. Hence, $C_r(Y) = \pi_Y(C_r(X))$. □

In the next result, we show that the regular topology on the space $C(X,Y)$ is strong based on the result that was investigated in [9] as: A topology on $C(X,Y)$ is said to be strong if and only if it makes the evaluation map $e\colon C(X,Y) \times X \to Y$ as $(f,x) \mapsto f(x)$ continuous.

Theorem 4.11. *For a discrete space X and a metric space (Y,d), the regular topology on $C(X,Y)$ is strong.*

Proof. To prove that the regular topology on $C(X,Y)$ is strong, it is sufficient to prove that the evaluation map $e\colon C_r(X,Y) \times X \to Y$ defined as $(f,x) \mapsto f(x)$ is continuous.

Given a point (f,x) in $C_r(X,Y) \times X$ and an open set $B(f(x),\epsilon), \epsilon > 0$ about the image point $e(f,x) = f(x)$, we wish to find an open set about (f,x) that e maps into $B(f(x),\epsilon)$. Let $B(f,r)$ be an open set in $C_r(X,Y)$ such that $B(f,r) = \{g \in C(X,Y)\colon d(f(x),g(x)) < r(x), x \in coz(r)\}$. Since $coz(r)$ is dense in X and X has a discrete topology, then for all $x \in X$, there exists a neighborhood of x. As a consequence, there exists an open set say U in X such that $B(f,r) \times U$ is open in $C_r(X,Y) \times X$ that maps (f,x) to $f(x)$ in Y. Thus, if $(g,a) \in B(f,r) \times U$, then $e(g,a) = g(a)$. □

5. Separation axioms

In this section, we are going to discuss about various separation axioms corresponding to the function space $C_r(X,Y)$ such as Hausdorffness, regularity and normality.

Theorem 5.1. *For a space X, if Y is T_0 or T_1, then the space $C_r(X,Y)$ is T_0 or T_1, respectively.*

Proof. Suppose Y is T_0 or T_1. Then, the space Y^X is T_0 or T_1, respectively in the Tychonoff topology. Since $C_p(X,Y)$ is a subspace of Y^X, implies $C_p(X,Y)$ is T_0 or T_1. As $C_p(X,Y) \leq C_r(X,Y)$ and hence $C_r(X,Y)$ is T_0 or T_1, respectively. □

Theorem 5.2. *For a space X, if Y is Hausdorff, then the space $C_r(X,Y)$ is also Hausdorff.*

Proof. Suppose Y is Hausdorff, then the space Y^X is Hausdorff in the Tychonoff topology. Since $C_p(X,Y)$ is a subspace of Y^X, implies $C_p(X,Y)$ is Hausdorff. As $C_p(X,Y) \leq C_r(X,Y)$, hence $C_r(X,Y)$ is Hausdorff. □

Theorem 5.3. *For a space X, if Y is a completely regular space, then the space $C_r(X,Y)$ is also completely regular.*

Proof. Since every uniformizable space is completely regular. However, we can prove it as : Suppose Y is completely regular, then the space Y^X is completely regular in the Tychonoff topology. Since $C_p(X,Y)$ is a subspace of Y^X, implies

$C_p(X,Y)$ is completely regular. As $C_p(X,Y) \leq C_r(X,Y)$, hence $C_r(X,Y)$ is completely regular. □

Theorem 5.4. *For a space X, if Y is a regular space, then the space $C_r(X,Y)$ is also regular.*

Proof. Suppose Y is regular, then the space Y^X is regular in the Tychonoff topology. Since $C_p(X,Y)$ is a subspace of Y^X, implies $C_p(X,Y)$ is regular. As $C_p(X,Y) \leq C_r(X,Y)$ and hence $C_r(X,Y)$ is regular. □

Theorem 5.5. *For a pseudocompact and almost P-space X and a metric space (Y,d), the space $C_r(X,Y)$ is normal.*

Proof. Since the space $C_r(X,Y)$ is metrizable if and only if X is pseudocompact and almost P-space. Also we know that all metrizable spaces are normal (Theorem 3.20, [13]). Hence, the space $C_r(X,Y)$ is normal. □

Theorem 5.6. *For a countable, compactly generated, compact space X, the space $C_r(X)$ is normal.*

Proof. Suppose X is a compactly generated compact space, then $C_k(X) = C_r(X)$ and thus $C_r(X)$ is closed in $\mathbb{R}^X$. Since X is countable, and we know that $\mathbb{R}^X$ is normal if and only if X is countable. Thus, we get $\mathbb{R}^X$ is normal. However, $C_r(X)$ being closed subset of $\mathbb{R}^X$ is also normal. □

Corollary 5.1. *For a discrete space X, the space $C_r(X)$ is normal if and only if X is countable.*

Theorem 5.7. *For a pseudocompact and almost P-space X and a metric space (Y,d), the space $C_r(X,Y)$ is completely normal.*

Proof. Since the space $C_r(X,Y)$ is metrizable if and only if X is pseudocompact and almost P-space. Also, metrizable spaces are completely normal (Chapter 4, [13]). Hence, the space $C_r(X,Y)$ is completely normal. □

Theorem 5.8. *For a pseudocompact and almost P-space X, the space $C_r(X,Y)$ is perfectly normal Hausdorff.*

Proof. Since the space $C_r(X,Y)$ is metrizable if and only if X is pseudocompact and almost P-space. As we know that all metrizable spaces are perfectly normal Hausdorff. Hence, the proof. □

Corollary 5.2. *For a pseudocompact and almost P-space, the space $C_r(X,Y)$ is completely normal Hausdorff.*

Proof. All perfectly normal Hausdorff spaces are completely normal Hausdorff.
□

Theorem 5.9. *For Tychonoff spaces X and Y, the space $C_r(X,Y)$ is regular Hausdorff and completely Hausdorff.*

Proof. Since the space $C_r(X,Y)$ is a Tychonoff space, so as every Tychonoff space is regular Hausdorff and completely Hausdorff. Which proves the theorem. □

Theorem 5.10. *For a space X and a metric space (Y,d), the following are equivalent:*

1. *Y is T_1 (respectively T_0);*
2. *$C_p(X,Y)$ is T_1 (respectively T_0);*
3. *$C_k(X,Y)$ is T_1 (respectively T_0);*
4. *$C_f(X,Y)$ is T_1 (respectively T_0);*
5. *$C_r(X,Y)$ is T_1 (respectively T_0).*

Proof. If Y is T_0, T_1, then Y^X with Tychonoff topology is T_0, T_1, respectively. Since $C_p(X,Y)$ is a subspace of Y^X is T_0, T_1, respectively. Moreover, $C_p(X,Y) \leq C_k(X,Y) \leq C_f(X,Y) \leq C_r(X,Y)$, then $C_k(X,Y)$, $C_f(X,Y)$ and $C_r(X,Y)$ are T_0, T_1, respectively.

(2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are immediate.

(5) $\Rightarrow$ (1) Now, if $C_r(X,Y)$ is T_0 or T_1. Since $\phi\colon Y \to C_r(X,Y)$ is an embedding, and therefore Y can be treated as subspace. Consequently, Y is T_0, T_1, respectively. □

Theorem 5.11. *For a space X and a metric space (Y,d), the following are equivalent:*

1. *Y is T_2 (respectively $T_3, T_{3(1/2)}$);*
2. *$C_p(X,Y)$ is T_2 (respectively $T_3, T_{3(1/2)}$);*
3. *$C_k(X,Y)$ is T_2 (respectively $T_3, T_{3(1/2)}$);*
4. *$C_r(X,Y)$ is T_2 (respectively $T_3, T_{3(1/2)}$).*

Proof. If Y is T_2 (respectively, $T_3, T_{3(1/2)}$), then Y^X with Tychonoff topology is T_2 (respectively, $T_3, T_{3(1/2)}$). Since $C_p(X,Y)$ is a subspace of Y^X is T_2 (respectively, $T_3, T_{3(1/2)}$). Moreover, $C_p(X,Y) \leq C_k(X,Y) \leq C_r(X,Y)$, then $C_k(X,Y)$ and $C_r(X,Y)$ are T_2 (respectively, $T_3, T_{3(1/2)}$).

(2) $\Rightarrow$ (3) is immediate.

(3) $\Rightarrow$ (4) Suppose $C_k(X,Y)$ is T_2 (respectively, $T_3, T_{3(1/2)}$). Since $C_k(X,Y) \leq C_r(X,Y)$, then $C_r(X,Y)$ is T_2 (respectively, $T_3, T_{3(1/2)}$).

(4) $\Rightarrow$ (1) Now, if $C_r(X,Y)$ is T_2 (respectively, $T_3, T_{3(1/2)}$). Since $\phi\colon Y \to C_r(X,Y)$ is an embedding, and therefore Y can be treated as subspace. Consequently, Y is T_2 (respectively, $T_3, T_{3(1/2)}$). □

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Accepted: June 10, 2022