On the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)$-diaphony of the nets of type of Halton-Zaremba constructed over finite groups

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Abstract. In the present paper, the authors introduce an arithmetic based on finite groups with respect to arbitrary bijections. This algebraic background is used to construct the function system $\mathcal{W}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}$ of the Walsh functions over the set $G_{\mathbf{b}}$ of groups with respect to the set $\varphi_{\mathbf{b}}$ of bijections. The developed algebraic base is also used to introduce a wide class of two-dimensional nets ${}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}$ of type of Halton-Zaremba. Four concrete nets of this class are constructed and graphically illustrated. The so-called $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony is applied as a appropriate tool for studying the nets of the introduced class. An exact formula for the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony of the nets of class ${}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}$ is presented. This formula allows us to show the influence of the vector α on the exact order of the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony of these nets.

Keywords: $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$ diaphony, nets of type of Halton-Zaremba constructed over finite groups, exact formula, exact orders.

1. Introduction

Let $s \geq 1$ be a fixed integer, which will denote the dimension of the objects considered in the paper. We will remind the notion of uniformly distributed sequence. So, following Kuipers and Niederreiter [16] let $\xi = (\mathbf{x}_n)_{n\geq 0}$ be an arbitrary sequence of points in $[0,1)^s$. For an arbitrary integer $N \geq 1$ and a subinterval J of $[0,1)^s$ with a Lebesgue measure $\lambda_s(J)$ let us denote $A_N(\xi;J) = \#\{n : 0 \leq n \leq N-1, \mathbf{x}_n \in J\}$. The sequence ξ is called uniformly distributed in

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$[0,1)^s$ if the limit equality $\lim_{N\to\infty}\frac{A_N(\xi;J)}{N}=\lambda_s(J)$ holds for each subinterval J of $[0,1)^s$.

The functions of some orthonormal function systems are used to solve many problems of the theory of the uniformly distributed sequences with very big success. We will remind the definitions of the functions of some of these classes.

For an arbitrary integer k and a real x the function $e_k:\mathbb{R}\to\mathbb{C}$ is defined as $e_k(x)=e^{2\pi\mathbf{i}kx}$. For an arbitrary vector $\mathbf{k}=(k_1,\ldots,k_s)\in\mathbb{Z}^s$ the $\mathbf{k}-$th multivariate trigonometric function $e_{\mathbf{k}}:[0,1)^s\to\mathbb{C}$ is defined as $e_{\mathbf{k}}(\mathbf{x})=\prod_{j=1}^s e_{k_j}(x_j)$, $\mathbf{x}=(x_1,\ldots,x_s)\in[0,1)^s$. The set $\mathcal{T}^s=\{e_{\mathbf{k}}(\mathbf{x}):\mathbf{k}\in\mathbb{Z}^s,\ \mathbf{x}\in[0,1)^s\}$ is called trigonometric function system.

Following Chrestenson [4] we will recall the constructive principle of the Walsh functions. Let $b\ge 2$ be a fixed integer. For an arbitrary integer $k\ge 0$ and a real $x\in[0,1)$ with the $b-$adic representation $k=\sum_{i=0}^{\nu}k_ib^i$ and $x=\sum_{i=0}^{\infty}x_ib^{-i-1}$, where $k_i,x_i\in\{0,1,\ldots,b-1\}$, $k_\nu\neq 0$ and for infinitely many values of i we have $x_i\neq b-1$, the $k-$th Walsh function ${}_bwal_k:[0,1)\to\mathbb{C}$ is defined as

$$ {}_bwal_k(x)=e^{\frac{2\pi\mathbf{i}}{b}(k_0x_0+\ldots+k_\nu x_\nu)}. $$

Let us denote $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. For an arbitrary vector $\mathbf{k}=(k_1,\ldots,k_s)\in\mathbb{N}_0^s$ the $\mathbf{k}-$th multivariate Walsh function ${}_bwal_{\mathbf{k}}:[0,1)^s\to\mathbb{C}$ is defined as

$$ {}_bwal_{\mathbf{k}}(\mathbf{x})=\prod_{j=1}^s {}_bwal_{k_j}(x_j),\quad \mathbf{x}=(x_1,\ldots,x_s)\in[0,1)^s. $$

The set $\mathcal{W}(b)=\{{}_bwal_{\mathbf{k}}(\mathbf{x}):\mathbf{k}\in\mathbb{N}_0^s,\ \mathbf{x}\in[0,1)^s\}$ is called Walsh function system in base b. In the case when $b=2$ the system $\mathcal{W}(2)$ is the original system of Walsh [22] functions.

The different kinds of the diaphony are numerical measures, which show the quality of the distribution of the points of sequences and nets. So, let $\xi_N=\{\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_{N-1}\}$ be an arbitrary net composed by N points in $[0,1)^s$.

Firstly Zinterhof [25] introduced the notion of the so-called classical diaphony. So, the classical diaphony of the net ξ_N is defined as

$$ F(\mathcal{T}^s;\xi_N)=\left(\sum_{\mathbf{k}\in\mathbb{Z}^s\setminus\{\mathbf{0}\}}R^{-2}(\mathbf{k})\left|\frac{1}{N}\sum_{n=0}^{N-1}e_{\mathbf{k}}(\mathbf{x}_n)\right|^2\right)^{\frac{1}{2}}, $$

where for each vector $\mathbf{k}=(k_1,\ldots,k_s)\in\mathbb{Z}^s$ the coefficient $R(\mathbf{k})=\prod_{j=1}^s R(k_j)$ and for an arbitrary integer k

$$ R(k)=\begin{cases}1, & \text{if}\ \ k=0,\\ |k|, & \text{if}\ \ k\neq 0.\end{cases} $$

Hellekalek and Leeb [15] introduced the notion of the dyadic diaphony, which is based on using the original system $\mathcal{W}(2)$ of the Walsh function. Grozdanov and Stoilova [10] generalized the notion of the dyadic diaphony to the so-called $b-$adic diaphony. So, the $b-$adic diaphony of the net ξ_N is defined as

$$F(\mathcal{W}(b);\xi_N) = \left(\frac{1}{(b+1)^s - 1} \sum_{\mathbf{k}\in\mathbb{N}_0^s\setminus\{\mathbf{0}\}} \rho(\mathbf{k}) \left| \frac{1}{N} \sum_{n=0}^{N-1} {}_b wal_{\mathbf{k}}(\mathbf{x}_n) \right|^2 \right)^{\frac{1}{2}},$$

where for each vector $\mathbf{k} = (k_1, \dots, k_s) \in \mathbb{N}_0^s$ the coefficient $\rho(\mathbf{k}) = \prod_{j=1}^s \rho(k_j)$ and for an arbitrary non-negative integer k

$$\rho(k) = \begin{cases} 1, & \text{if } \ k = 0, \\ b^{-2g}, & \text{if } \ b^g \le k \le b^{g+1}, \quad g \ge 0, \quad g \in \mathbb{Z}. \end{cases}$$

In 1986 Proinov [18] established a general lower bound of the classical diaphony. So, for any net ξ_N composed of N points in $[0,1)^s$ the lower bound

$$F(\mathcal{T}^s;\xi_N) > \alpha(s)\frac{(\log N)^{\frac{s-1}{2}}}{N} \tag{1}$$

holds, where $\alpha(s)$ is a positive constant depending only on the dimension s. For a dimension $s = 1$ from the inequality (1) the result of Stegbuchner [20] is obtained

$$F(\mathcal{T}^s;\xi_N) \ge \frac{\pi}{\sqrt{3}} \cdot \frac{1}{N}.$$

To show the exactness of the lower bound (1) for a dimension $s = 2$ we need to present the techniques of the construction of two classical two-dimensional nets. For this purpose, let $\nu > 0$ be a fixed integer. For $0 \le i \le b^\nu - 1$ we denote $\eta_{b,\nu}(i) = \frac{i}{b^\nu}$. Following Van der Corput [21] and Halton [12] for an arbitrary integer i, $0 \le i \le b^\nu - 1$, with the $b-$adic representation $i = \sum_{j=0}^{\nu-1} i_j b^j$, where for $0 \le j \le \nu - 1$ $i_j \in \{0, 1, \dots, b-1\}$, we put $p_{b,\nu}(i) = \sum_{j=0}^{\nu-1} i_j b^{-j-1}$. Roth [19] considered the two-dimensional net $R_{b,\nu} = \{(\eta_{b,\nu}(i), p_{b,\nu}(i)) : 0 \le i \le b^\nu - 1\}$, which now is called a net of Roth. The net $R_{b,\nu}$ is also known as two-dimensional Hammersley [14] point set.

In 1969, Halton and Zaremba [13] used the original net of Van der Corput $\{p_{2,\nu}(i) = 0.i_0 i_1 \dots i_{\nu-1} : 0 \le i \le 2^\nu - 1 \; i_j \in \{0,1\}\}$ and change the digits i_j that stay in the even positions with the digit $1 - i_j$. Let us for $0 \le i \le 2^\nu - 1$ signify $z_{2,\nu}(i) = 0.(1-i_0)i_1(1-i_2)\dots$. The net $Z_{2,\nu} = \{(\eta_{2,\nu}(i), z_{2,\nu}(i)) : 0 \le i \le 2^\nu - 1\}$, which is called net of Halton-Zaremba is constructed.

In 1998 Xiao [24] proved that the classical diaphony of the net of Roth $R_{b,\nu}$ and the net of Halton-Zaremba $Z_{2,\nu}$ have an exact order $\mathcal{O}\left(\frac{\sqrt{\log N}}{N}\right)$, where respectively $N = b^\nu$ and $N = 2^\nu$.

Cristea and Pillichshammer [5] proved a general lower bound of the $b-$adic diaphony. So, for any net ξ_N composed of N points in $[0,1]$ the lower bound

$$F(\mathcal{W}(b);\xi_N) \ge C(b,s)\frac{(\log N)^{\frac{s-1}{2}}}{N} \tag{2}$$

holds, where $C(b,s)$ is a positive constant depending on the base b and the dimension s.

Grozdanov and Stoilova [11] proved the exactness of the lower bound (2) for dimension $s=2$. They proved that the $b-$adic diaphony of the net of Roth $R_{b,\nu}$ and the net of Halton-Zaremba $Z_{2,\nu}$ have an exact order $\mathcal{O}\left(\frac{\sqrt{\log N}}{N}\right)$, where respectively $N=b^\nu$ and $N=2^\nu$.

The $b-$adic diaphony is closely related with the worst-case error of the quasi-Monte Carlo integration in appropriate Hilbert spaces. Aronszajn [1] introduced the notion of a reproducing kernel for Hilbert space. So, following this approach let $H_s(K)$ be a Hilbert space with a reproducing kernel $K:[0,1)^s\to\mathbb{C}$, an inner product $<\cdot,\cdot>_{H_s(K)}$ and a norm $||\cdot||_{H_s(K)}$. We are interested to approximate the multivariate integral

$$I_s(f)=\int_{[0,1]^s} f(\mathbf{x})d\mathbf{x},\quad f\in H_s(K).$$

Let $N\geq 1$ be an arbitrary and fixed integer. We will approximate the integral $I_s(f)$ through quasi-Monte Carlo algorithm $Q_s(f;P_N)=\frac{1}{N}\sum_{n=0}^{N-1} f(\mathbf{x}_n)$, where $P_N=\{\mathbf{x}_0,\ldots,\mathbf{x}_{N-1}\}$ is a deterministic sample point set in $[0,1)^s$. The worst-case error of the integration in the space $H_s(K)$ by using the net P_N is defined as

$$e(H_s(K);P_N)=\sup_{f\in H_s(K),\ ||f||_{H_s(K)}\leq 1} |I_s(f)-Q_s(f;P_N)|.$$

Dick and Pillichshammer [6] used the Walsh functions as a tool for investigation of the worst-case error of the multivariate integration in Hilbert spaces. This error is presented in the terms of the reproducing kernel, which generates this space.

Likewise, Dick and Pillichshammer [7] introduced a special reproducing kernel Hilbert space and the worst-case error of the integration in this space and the $b-$adic diaphony of the net of the nodes of the integration are connected. In this sense, we see that the so-called low diaphony nets with very big success can be used in the practice of the quasi-Monte Carlo integration. This determines the interest to this class of nets.

The rest of the paper is organized in the following manner: In Section 2 the concept of the function system $\mathcal{W}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}$ is reminded. In Section 3 we introduce a class of nets ${}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}$ of type of Halton-Zaremba constructed over finite groups. By graphical illustrations, we show the distribution of four nets from this class. In Section 4 the concept of the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony is presented. In Section 5 an explicit formula for the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony of the nets ${}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}$ is presented. This formula allows us to show the influence of the vector α of exponential parameters to the exact orders of the considered diaphony of these nets. In Section 6 some preliminary results are presented. In Section 7 the main results of the paper are proved.

2. The function system $\mathcal{W}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}$

In 1996 Larcher, Niederreiter and W. Ch. Schmid [17] introduced the concept of the so-called Walsh function system over finite groups. So, the details are as follows: Let $m \geq 1$ be a given integer and let $\{b_1, b_2, \ldots, b_m : b_l \geq 2, 1 \leq l \leq m\}$ be a set of fixed integers. For $1 \leq l \leq m$ let $\mathbb{Z}_{b_l} = \{0, 1, \ldots, b_l - 1\}$ and the operation $\oplus_{b_l}$ be the addition modulus b_l of the elements of the set $\mathbb{Z}_{b_l}$. Then, $(\mathbb{Z}_{b_l}, \oplus_{b_l})$ is a discrete cyclic group of order b_l.

Let $G = \mathbb{Z}_{b_1} \times \ldots \times \mathbb{Z}_{b_m}$ be the Cartesian product of the sets $\mathbb{Z}_{b_1}, \ldots, \mathbb{Z}_{b_m}$. For each pair $\mathbf{g} = (g_1, \ldots, g_m)$, $\mathbf{y} = (y_1, \ldots, y_m) \in G$ by using the group operations $\oplus_{b_1}, \ldots, \oplus_{b_s}$ let the operation $\oplus_G$ be defined as $\mathbf{g} \oplus_G \mathbf{y} = (g_1 \oplus_{b_1} y_1, \ldots, g_m \oplus_{b_m} y_m)$. Then, $(G, \oplus_G)$ is a finite group of order $b = b_1 b_2 \ldots b_m$. For the given elements $\mathbf{g}, \mathbf{y} \in G$ the character function on the group G is defined as

$$\chi_{\mathbf{g}}(\mathbf{y}) = \prod_{l=1}^{m} e^{2\pi \mathbf{i} \frac{g_l y_l}{b_l}}.$$

Let us denote $\mathbb{Z}_b = \{0, 1, \ldots, b-1\}$ and let $\varphi : \mathbb{Z}_b \to G$ be an arbitrary bijection, which satisfies the condition $\varphi(0) = \mathbf{0}$.

Definition 1. *For an arbitrary integer $k \geq 0$ and a real $x \in [0,1)$ with the $b-$adic representations $k = \sum_{i=0}^{\nu} k_i b^i$ and $x = \sum_{i=0}^{\infty} x_i b^{-i-1}$, where for $i \geq 0$ $k_i, x_i \in \{0, 1, \ldots, b-1\}$ $k_\nu \neq 0$ and for infinitely many values of i $x_i \neq b-1$, the function ${}_{G,\varphi}wal_k : [0,1) \to \mathbb{C}$ is defined as ${}_{G,\varphi}wal_k(x) = \prod_{i=0}^{\nu} \chi_{\varphi(k_i)}(\varphi(x_i))$.*

The set $\mathcal{W}_{G,\varphi} = \{{}_{G,\varphi}wal_k(x) : k \in \mathbb{N}_0,\ x \in [0,1)\}$ is called Walsh function system over the group G with respect to the bijection φ.

Now, we will introduce the concept of the multidimensional function system of Walsh functions over finite groups. For this purpose, let $\mathbf{b} = (b_1, \ldots, b_s)$ be a vector of not necessarily distinct integers $b_j \geq 2$. For $1 \leq j \leq s$ let $(G_{b_j}, \oplus_{G_{b_j}})$ be an arbitrary group of order b_j constructed as above. Let us denote $\mathbb{Z}_{b_j} = \{0, 1, \ldots, b_j - 1\}$ and let $\varphi_{b_j} : \mathbb{Z}_{b_j} \to G_{b_j}$ be an arbitrary bijection, which satisfies the condition $\varphi_{b_j}(0) = \mathbf{0}$. Let $\mathcal{W}_{G_{b_j},\varphi_{b_j}} = \{{}_{G_{b_j},\varphi_{b_j}}wal_k(x) : k \in \mathbb{N}_0,\ x \in [0,1)\}$ be the corresponding Walsh function system over the group G_{b_j} with respect to the bijection φ_{b_j}.

By using the groups $G_{b_1}, \ldots, G_{b_s}$, the sets $\mathbb{Z}_{b_1}, \ldots, \mathbb{Z}_{b_s}$ and the bijections $\varphi_{b_1}, \ldots, \varphi_{b_s}$ let us introduce the next significations $G_{\mathbf{b}} = (G_{b_1}, \ldots, G_{b_s})$, $\mathbb{Z}_{\mathbf{b}} = (\mathbb{Z}_{b_1}, \ldots, \mathbb{Z}_{b_s})$ and $\varphi_{\mathbf{b}} = (\varphi_{b_1}, \ldots, \varphi_{b_s})$.

Let $\mathcal{W}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}} = \mathcal{W}_{G_{b_1},\varphi_{b_1}} \otimes \ldots \otimes \mathcal{W}_{G_{b_s},\varphi_{b_s}}$ be the tensor product of the function systems $\mathcal{W}_{G_{b_1},\varphi_{b_1}}, \ldots, \mathcal{W}_{G_{b_s},\varphi_{b_s}}$. This means that for an arbitrary vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ the $\mathbf{k}-$th Walsh function ${}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}wal_{\mathbf{k}}(\mathbf{x})$ is defined as

$${}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}wal_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{s} {}_{G_{b_j},\varphi_{b_j}}wal_{k_j}(x_j), \quad \mathbf{x} = (x_1, \ldots, x_s) \in [0,1)^s.$$

We will call the set $\mathcal{W}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}} = \{{}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}wal_{\mathbf{k}}(\mathbf{x}) : \mathbf{k} \in \mathbb{N}_0^s,\ \mathbf{x} \in [0,1)^s\}$ a multidimensional system of the Walsh functions over the set $G_{\mathbf{b}}$ of groups with respect to the set $\varphi_{\mathbf{b}}$ of bijections.

We will introduce some elements of the $\mathbf{b}-$adic arithmetic. By using the operation $\oplus_G$ over the group G and the bijection φ we will define the operation $\oplus_{\mathbb{Z}_b,\varphi} : \mathbb{Z}_b^2 \to \mathbb{Z}_b$ by the following manner: for arbitrary elements $u, v \in \mathbb{Z}_b$, we put $u \oplus_{\mathbb{Z}_b,\varphi} v = \varphi^{-1}(\varphi(u) \oplus_G \varphi(v))$. For an arbitrary element $u \in \mathbb{Z}_b$, let the element $\overline{u} \in \mathbb{Z}_b$ be such that $u \oplus_{\mathbb{Z}_b,\varphi} \overline{u} = 0$. We will prove that for arbitrary digits $p, q, r \in \mathbb{Z}_b$ the character function satisfies the equalities

$$\chi_{\varphi(p)}(\varphi(q) \oplus_G \varphi(r)) = \chi_{\varphi(p)}(\varphi(q))\chi_{\varphi(p)}(\varphi(r)) \tag{3}$$

and

$$\chi_{\varphi(p)\oplus_G\varphi(q)}(\varphi(r)) = \chi_{\varphi(p)}(\varphi(r))\chi_{\varphi(q)}(\varphi(r)). \tag{4}$$

Let us signify $\varphi(p) = (p^{(1)}, \ldots, p^{(m)})$, $\varphi(q) = (q^{(1)}, \ldots, q^{(m)})$ and $\varphi(r) = (r^{(1)}, \ldots, r^{(m)})$. Hence, we obtain that

$$\chi_{\varphi(p)}(\varphi(q) \oplus_G \varphi(r)) = \prod_{l=1}^{m} e^{2\pi\mathbf{i}\frac{p^{(l)}[q^{(l)}+r^{(l)}(mod\ b_l)]}{b_l}} = \prod_{l=1}^{m} e^{2\pi\mathbf{i}\frac{p^{(l)}(q^{(l)}+r^{(l)})}{b_l}}$$
$$= \prod_{l=1}^{m} e^{2\pi\mathbf{i}\frac{p^{(l)}q^{(l)}}{b_l}} \prod_{l=1}^{m} e^{2\pi\mathbf{i}\frac{p^{(l)}r^{(l)}}{b_l}} = \chi_{\varphi(p)}(\varphi(q))\chi_{\varphi(p)}(\varphi(r))$$

and

$$\chi_{\varphi(p)\oplus_G\varphi(q)}(\varphi(r)) = \prod_{l=1}^{m} e^{2\pi\mathbf{i}\frac{[p^{(l)}+q^{(l)}(mod\ b_l)]r^{(l)}}{b_l}} = \prod_{l=1}^{m} e^{2\pi\mathbf{i}\frac{(p^{(l)}+q^{(l)})r^{(l)}}{b_l}}$$
$$= \prod_{l=1}^{m} e^{2\pi\mathbf{i}\frac{p^{(l)}r^{(l)}}{b_l}} \prod_{l=1}^{m} e^{2\pi\mathbf{i}\frac{q^{(l)}r^{(l)}}{b_l}} = \chi_{\varphi(p)}(\varphi(r))\chi_{\varphi(q)}(\varphi(r)).$$

For arbitrary reals $x, y \in [0,1)$ with the $b-$adic representations $x = \sum_{i=0}^{\infty} x_i b^{-i-1}$ and $y = \sum_{i=0}^{\infty} y_i b^{-i-1}$, where for $i \geq 0$ $x_i, y_i \in \mathbb{Z}_b$ and for infinitely many values of i $x_i, y_i \neq b-1$, let us define the next operation

$$x \oplus_{\mathbb{Z}_b,\varphi}^{[0,1)} y = \left(\sum_{i=0}^{\infty}(x_i \oplus_{\mathbb{Z}_b,\varphi} y_i)b^{-i-1}\right)\ (mod\ 1).$$

We will prove that for an arbitrary integer $k \in \mathbb{N}_0$ and arbitrary reals $x, y \in [0,1)$ the equality holds

$${}_{G,\varphi}wal_k(x \oplus_{\mathbb{Z}_b,\varphi}^{[0,1)} y) = {}_{G,\varphi}wal_k(x)\,{}_{G,\varphi}wal_k(y). \tag{5}$$

Let k have the $b-$adic representation $k=\sum_{i=0}^{\nu}k_ib^i$, where for $0\leq i\leq\nu$ $k_i\in\{0,1,\ldots,b-1\}$, x and y be as above. Then, by using the equality (3) we obtain that

$$
\begin{aligned}
&{}_{G,\varphi}wal_k(x\oplus^{[0,1)}_{\mathbb{Z}_b,\varphi}y)\\
&=\prod_{i=0}^{\nu}\chi_{\varphi(k_i)}(\varphi(\varphi^{-1}(\varphi(x_i)\oplus_G\varphi(y_i))))=\prod_{i=0}^{\nu}\chi_{\varphi(k_i)}(\varphi(x_i)\oplus_G\varphi(y_i))\\
&=\prod_{i=0}^{\nu}\chi_{\varphi(k_i)}(\varphi(x_i))\prod_{i=0}^{\nu}\chi_{\varphi(k_i)}(\varphi(y_i))={}_{G,\varphi}wal_k(x){}_{G,\varphi}wal_k(y).
\end{aligned}
$$

For arbitrary vectors $\mathbf{x}=(x_1,\ldots,x_s)\in[0,1)^s$ and $\mathbf{y}=(y_1,\ldots,y_s)\in[0,1)^s$ to define the operation $\mathbf{x}\oplus^{[0,1)^s}_{\mathbb{Z}_{\mathbf{b}},\varphi_{\mathbf{b}}}\mathbf{y}=(x_1\oplus^{[0,1)}_{\mathbb{Z}_{b_1},\varphi_{b_1}}y_1,\ldots,x_s\oplus^{[0,1)}_{\mathbb{Z}_{b_s},\varphi_{b_s}}y_s)$. Then, the following equality holds

$$
(6)\qquad {}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}wal_{\mathbf{k}}(\mathbf{x}\oplus^{[0,1)^s}_{\mathbb{Z}_{\mathbf{b}},\varphi_{\mathbf{b}}}\mathbf{y})={}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}wal_{\mathbf{k}}(\mathbf{x}){}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}wal_{\mathbf{k}}(\mathbf{y}),\ \forall\ \mathbf{k}\in\mathbb{N}_0^s.
$$

Let $\mathbf{k}=(k_1,\ldots,k_s)\in\mathbb{N}_0^s$ be an arbitrary vector. Then, by using the equality (5) the following holds

$$
\begin{aligned}
&{}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}wal_{\mathbf{k}}(\mathbf{x}\oplus^{[0,1)^s}_{\mathbb{Z}_{\mathbf{b}},\varphi_{\mathbf{b}}}\mathbf{y})=\prod_{j=1}^{s}{}_{G_{b_j},\varphi_{b_j}}wal_{k_j}(x_j\oplus^{[0,1)}_{\mathbb{Z}_{b_j},\varphi_{b_j}}y_j)\\
&=\prod_{j=1}^{s}{}_{G_{b_j},\varphi_{b_j}}wal_{k_j}(x_j){}_{G_{b_j},\varphi_{b_j}}wal_{k_j}(y_j)\\
&=\prod_{j=1}^{s}{}_{G_{b_j},\varphi_{b_j}}wal_{k_j}(x_j)\prod_{j=1}^{s}{}_{G_{b_j},\varphi_{b_j}}wal_{k_j}(y_j)={}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}wal_{\mathbf{k}}(\mathbf{x}){}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}wal_{\mathbf{k}}(\mathbf{y}).
\end{aligned}
$$

3. Nets of type of Halton - Zaremba constructed over finite groups

To present the definition of the nets of type of Halton-Zaremba constructed over finite groups, we will apply the same algebraic basis, which we used to present the functions of the system $\mathcal{W}_{G_{\mathbf{b}},\varphi_{\mathbf{b}}}$. In this way, a process of a synchronization between the construction of the nets and the tool for their investigation will be realized.

For this purpose, let $b_1\geq 2$ and $b_2\geq 2$ be arbitrary and fixed bases and denote $\mathbf{b}=(b_1,b_2)$. Let $(\mathbb{Z}_{b_1},\oplus_{b_1})$ and $(\mathbb{Z}_{b_2},\oplus_{b_2})$ be the corresponding discrete cyclic groups of orders b_1 and b_2. Let $b=b_1b_2$ and as yet to define $G_b=\mathbb{Z}_{b_1}\times\mathbb{Z}_{b_2}$ and $\oplus_{G_b}=(\oplus_{b_1},\oplus_{b_2})$. Let $\mathbb{Z}_b=\{0,1,\ldots,b-1\}$, $\varphi_1:\mathbb{Z}_b\to G_b$ and $\varphi_2:\mathbb{Z}_b\to G_b$ be two arbitrary bijections, which satisfy the conditions $\varphi_1(0)=\mathbf{0}$, $\varphi_2(0)=\mathbf{0}$ and denote $\varphi_b=(\varphi_1,\varphi_2)$. Let $\oplus^{[0,1)}_{\mathbb{Z}_b,\varphi_1}$ and $\oplus^{[0,1)}_{\mathbb{Z}_b,\varphi_2}$ be the operations over $[0,1)$, which correspond respectively to the bijections φ_1 and φ_2.

Let $\nu\geq 1$ be an arbitrary and fixed integer. Let $\kappa=0.\kappa_0\kappa_1\ldots\kappa_{\nu-1}$ and $\mu=0.\mu_0\mu_1\ldots\mu_{\nu-1}$ be arbitrary and fixed $b-$adic rational numbers. For $0\leq$

$i \leq b^\nu - 1$ let us denote $\eta_{b,\nu}(i) = \frac{i}{b^\nu}$ and $p_{b,\nu}(i)$ be the general term of the Van der Corput sequence. Let us define the $b-$adic rational numbers

$$_{G_b,\varphi_1}\xi^{\kappa}_{b,\nu}(i) = \eta_{b,\nu}(i) \oplus^{[0,1)}_{\mathbb{Z}_b,\varphi_1} \kappa \text{ and } _{G_b,\varphi_2}\zeta^{\mu}_{b,\nu}(i) = p_{b,\nu}(i) \oplus^{[0,1)}_{\mathbb{Z}_b,\varphi_2} \mu.$$

Dimitrievska Ristovska and Grozdanov [8] introduced the next class of two-dimensional nets:

Definition 2. *For an arbitrary integer $\nu \geq 1$ and for arbitrary fixed $b-$adic rational numbers κ and μ we define the net*

$$_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu} = \left\{ \left(_{G_b,\varphi_1}\xi^{\kappa}_{b,\nu}(i), _{G_b,\varphi_2}\zeta^{\mu}_{b,\nu}(i) \right) : 0 \leq i \leq b^\nu - 1 \right\},$$

which we will call a net of type of Halton-Zaremba constructed over the group G_b with respect to the set φ_b, which corresponds to the parameters κ and μ in base b.

We will concrete the choice of the parameters κ and μ from Definition 2: Let us choose $\kappa = 0$. We will construct the digits of the parameters μ in the following manner: Let $p, q \in \mathbb{Z}_b$ be arbitrary and fixed digits. For $0 \leq j \leq \nu - 1$ we define the digits $\mu_j \in \mathbb{Z}_b$ as the solutions of the linear recurrence equation $\mu_j \equiv p \cdot j + q \ (mod\ b)$ and to put $\mu = 0.\mu_0\mu_1 \ldots \mu_{\nu-1}$. For $0 \leq i \leq b^\nu - 1$ let us denote $_{G_b,\varphi_2}\zeta^{p,q}_{b,\nu}(i) = p_{b,\nu}(i) \oplus^{[0,1)}_{\mathbb{Z}_b,\varphi_2} \mu$. In this case, we obtain the net $_{G_b,\varphi_2}Z^{p,q}_{b,\nu} = \left\{ (\eta_{b,\nu}(i), _{G_b,\varphi_2}\zeta^{p,q}_{b,\nu}(i)) : 0 \leq i \leq b^\nu - 1 \right\}$, which was introduced by Grozdanov [9].

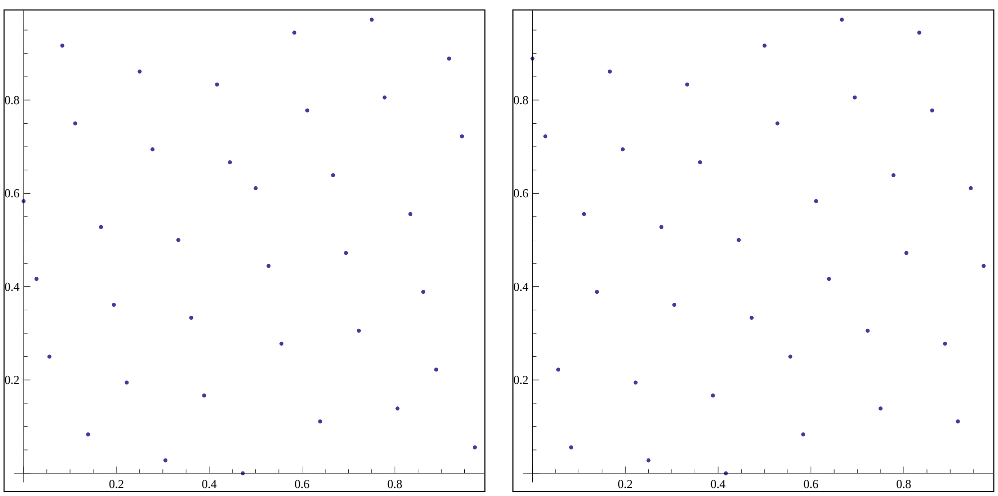
In the case when $G = \mathbb{Z}_b$ and $\varphi_2 = id$ is the identity of the set $\mathbb{Z}_b$ in itself, from the net $_{G_b,\varphi_2}Z^{p,q}_{b,\nu}$ we obtain the net $_{\mathbb{Z}_b,id}Z^{p,q}_{b,\nu}$, which was introduced by Grozdanov and Stoilova [11]. In the case when $p = 1$ and $q = 0$ from the net $_{\mathbb{Z}_b,id}Z^{p,q}_{b,\nu}$ we obtain the net $_{\mathbb{Z}_b,id}Z^{1,0}_{b,\nu}$, which was introduced by Warnock [23]. In the case when $b = 2$, $p = 1$ and $q = 1$ from the net $_{\mathbb{Z}_b,id}Z^{p,q}_{b,\nu}$ we obtain the net $_{\mathbb{Z}_2,id}Z^{1,1}_{2,\nu}$, which is the original net of Halton-Zaremba. In the case when $b = 2$, $p = 0$ and $q = 0$ from the net $_{\mathbb{Z}_b,id}Z^{p,q}_{b,\nu}$ we obtain the net $_{\mathbb{Z}_2,id}Z^{0,0}_{2,\nu}$, which is the original net of Roth [19].

We will construct and show the distributions of the points of four concrete nets $_{G_b,\varphi_b}Z^{\kappa,\mu}_{b,\nu}$ of type of Halton-Zaremba.

Example 1. The algebraic background of the first example is as follows: Let $m = 2$ and choose the bases $b_1 = 2$ and $b_2 = 3$. The discrete cyclic groups of orders b_1 and b_2 are $\mathbb{Z}_{b_1} = \{0, 1\}$ and $\mathbb{Z}_{b_2} = \{0, 1, 2\}$. We have that $b = 6$, the group G_b is $G_b = \{(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2)\}$ and $\mathbb{Z}_b = \{0, 1, 2, 3, 4, 5\}$. Let us select the bijections φ_1 and φ_2 as $\varphi_1(0) = (0, 0), \varphi_1(1) = (1, 0), \varphi_1(2) = (0, 2), \varphi_1(3) = (1, 2), \varphi_1(4) = (0, 1), \varphi_1(5) = (1, 1)$ and $\varphi_2(0) = (0, 0), \varphi_2(1) = (1, 2), \varphi_2(2) = (1, 0), \varphi_2(3) = (1, 1), \varphi_2(4) = (0, 2), \varphi_2(5) = (0, 1)$. In addition, we choose the parameters $\nu = 2$, $\kappa = 0.42$ and $\mu = 0.15$. The points of the obtained net are:

$$
\begin{aligned}
&{}_{G_6,\varphi_6}Z_{6,2}^{\kappa,\mu}\\
&= \left\{ \left(\frac{26}{36},\frac{11}{36}\right), \left(\frac{27}{36},\frac{35}{36}\right), \left(\frac{28}{36},\frac{29}{36}\right), \left(\frac{29}{36},\frac{5}{36}\right), \left(\frac{24}{36},\frac{23}{36}\right), \left(\frac{25}{36},\frac{17}{36}\right),\right.\\
&\left(\frac{32}{36},\frac{8}{36}\right), \left(\frac{33}{36},\frac{32}{36}\right), \left(\frac{34}{36},\frac{26}{36}\right), \left(\frac{35}{36},\frac{2}{36}\right), \left(\frac{30}{36},\frac{20}{36}\right), \left(\frac{31}{36},\frac{14}{36}\right),\\
&\left(\frac{2}{36},\frac{9}{36}\right), \left(\frac{3}{36},\frac{33}{36}\right), \left(\frac{4}{36},\frac{27}{36}\right), \left(\frac{5}{36},\frac{3}{36}\right), \left(\frac{0}{36},\frac{21}{36}\right), \left(\frac{1}{36},\frac{15}{36}\right),\\
&\left(\frac{8}{36},\frac{7}{36}\right), \left(\frac{9}{36},\frac{31}{36}\right), \left(\frac{10}{36},\frac{25}{36}\right), \left(\frac{11}{36},\frac{1}{36}\right), \left(\frac{6}{36},\frac{19}{36}\right), \left(\frac{7}{36},\frac{13}{36}\right),\\
&\left(\frac{14}{36},\frac{6}{36}\right), \left(\frac{15}{36},\frac{30}{36}\right), \left(\frac{16}{36},\frac{24}{36}\right), \left(\frac{17}{36},\frac{0}{36}\right), \left(\frac{12}{36},\frac{18}{36}\right), \left(\frac{13}{36},\frac{12}{36}\right),\\
&\left.\left(\frac{20}{36},\frac{10}{36}\right), \left(\frac{21}{36},\frac{34}{36}\right), \left(\frac{22}{36},\frac{28}{36}\right), \left(\frac{23}{36},\frac{4}{36}\right), \left(\frac{18}{36},\frac{22}{36}\right), \left(\frac{19}{36},\frac{16}{36}\right)\right\}.
\end{aligned}
$$

The distribution of the points of the net ${}_{G_6,\varphi_6}Z_{6,2}^{\kappa,\mu}$ is shown in Figure 1a).



a) b)

Figure 1: Nets of Example 1 and 2 ($\nu = 2, b1 = 2, b2 = 3$, different bijections φ_1, φ_2)

Example 2. To construct the second net, we will use the same group G_b and parameters $\nu = 2$, $\kappa = 0.42$ and $\mu = 0.15$. Let us choose the bijections $\varphi_1(0) = (0,0), \varphi_1(1) = (1,1), \varphi_1(2) = (1,2), \varphi_1(3) = (0,2), \varphi_1(4) = (0,1), \varphi_1(5) = (1,0)$

and $\varphi_2(0) = (0,0)$, $\varphi_2(1) = (0,2), \varphi_2(2) = (1,1), \varphi_2(3) = (1,0), \varphi_2(4) = (1,2), \varphi_2(5) = (0,1)$. The distribution of the points of the obtained net is shown in Figure 1b).

Example 3. To construct the third net, we use the same group G_b and bijections φ_1 and φ_2 as in Example 1. We choose the parameters $\nu = 4$, $\kappa = 0.2112$ and $\mu = 0.1302$. The distribution of the points of the obtained net is shown in Figure 2.

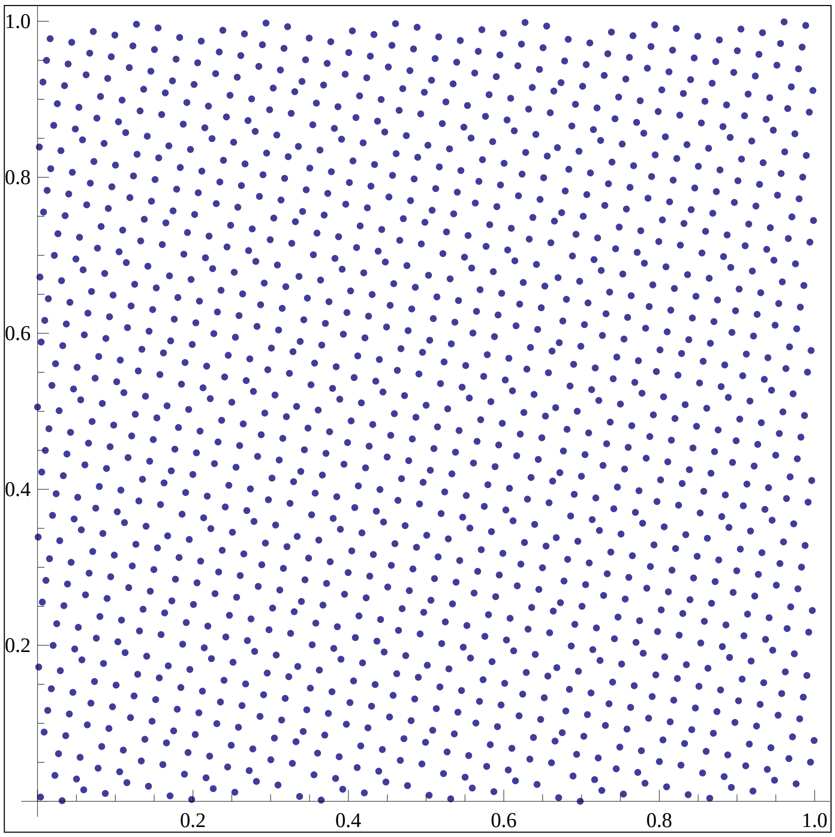


Figure 2: Net of Example 3: $\nu = 4, b1 = 2, b2 = 3$.

Example 4. The algebraic background of the fourth net is as follows: Let $m = 2$ and choose the bases $b_1 = 3$ and $b_2 = 4$. The discrete cyclic groups of orders b_1 and b_2 are $\mathbb{Z}_{b_1} = \{0,1,2\}$ and $\mathbb{Z}_{b_2} = \{0,1,2,3\}$. We have that $b = 12$, the group G_b is $G_b = \{(0,0),(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(1,3),(2,0),(2,1),(2,2),(2,3)\}$ and $\mathbb{Z}_b = \{0,1,\ldots,11\}$. Let us select the bijections $\varphi_1(0) = (0,0)$, $\varphi_1(1) = (1,0)$, $\varphi_1(2) = (0,3)$, $\varphi_1(3) = (1,2)$, $\varphi_1(4) = (0,1)$, $\varphi_1(5) = (2,3)$, $\varphi_1(6) = (0,2)$, $\varphi_1(7) = (2,2)$, $\varphi_1(8) = (1,3)$, $\varphi_1(9) = (1,1)$, $\varphi_1(10) = (2,0)$, $\varphi_1(11) = (2,1)$ and $\varphi_2(0) = (0,0)$, $\varphi_2(1) = (1,3)$, $\varphi_2(2) = (1,0)$, $\varphi_2(3) = (2,1)$,

$\varphi_2(4) = (0,3)$, $\varphi_2(5) = (2,3)$, $\varphi_2(6) = (2,0)$, $\varphi_2(7) = (1,2)$, $\varphi_2(8) = (1,1)$, $\varphi_2(9) = (0,2)$, $\varphi_2(10) = (0,1)$, $\varphi_2(11) = (2,2)$. We choose the parameters $\nu = 2$, $\kappa = 0.42$ and $\mu = 0.15$. The distribution of the points of the obtained net is shown in Figure 3.

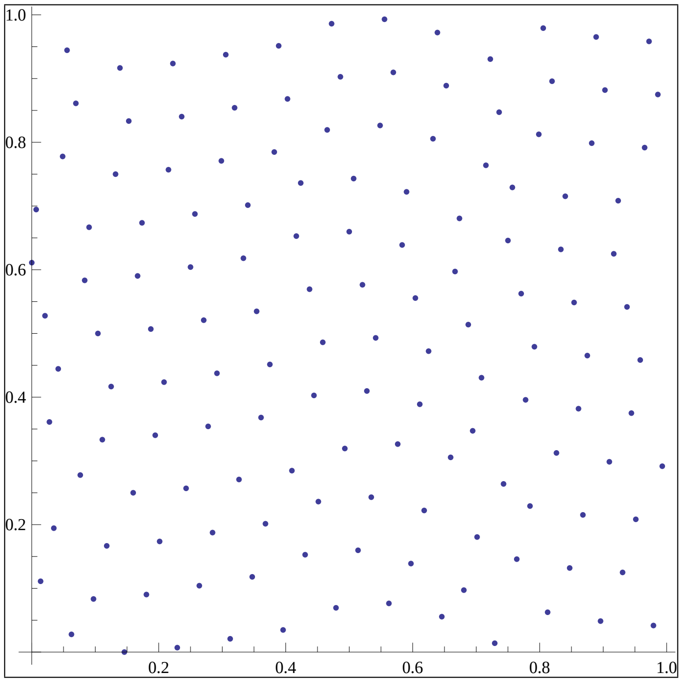


Figure 3: Net of Example 4: $m = 2, b1 = 3, b2 = 4$.

We will present the program code in the mathematical package *Mathematica*, which can compute the coordinates and visualize the points of an arbitrary net of type of Halton-Zaremba.

```
(*Program code for constructing nets *)
e = Input[e];m = Input[m]; (*vectors Eta and Mu*)
points = {};
b1 = Input[b1];b2 = Input[b2];
ni = Input[ni];b = b1*b2;
phi1 = Input[phi1];
phi2 = Input[phi2];
Do[i = IntegerDigits[i1, b]; k = ni - 1;
    While[k > 0,
        If[i1 < b^k, PrependTo[i, 0]]; k = k - 1];
    apc = ord = 0;
    Do[  cif1 = phi1[[i[[j]] + 1]];
         cif2 = phi1[[e[[j]] + 1]];
```

```
          cif = {Mod[cif1[[1]] + cif2[[1]], b1],
          Mod[cif1[[2]] + cif2[[2]], b2]};
          cifra = Position[phi1, cif][[1]][[1]] - 1;
          apc = apc + cifra/b^j;
          cif1 = phi2[[i[[ni - j + 1]] + 1]];
          cif2 = phi2[[m[[j]] + 1]];
          cif = {Mod[cif1[[1]] + cif2[[1]], b1],
          Mod[cif1[[2]] + cif2[[2]], b2]};
       cifra = Position[phi2, cif][[1]][[1]] - 1;
          ord = ord + cifra/b^j,
    {j, 1, ni}];
    AppendTo[points, {apc, ord}],
{i1, 0, b^ni - 1}];
ListPlot[points,AspectRatio->Automatic]
```

4. The $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony

In the previous section, we presented one wide class of two-dimensional nets constructed over finite groups with respect to arbitrary bijections. We need of appropriate analytical tool for studying the quality of the distribution of the points of these nets. In our case, it is important to realize a process of a synchronisation between the technique for construction of the nets and the tool for their investigation.

The different kinds of the diaphony are numerical measures for studying the irregularity of the distribution of sequences and nets. The construction of the diaphony is always connected with some complete orthonormal function system. Concrete for studying sequences and nets constructed over finite groups with respect to arbitrary bijections, the suitable version of the diaphony is the one, which is based on the system of Walsh functions constructed also over the same finite groups. For us, this is the motivation to use the so-called $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony as a tool for studying of the nets of the class ${}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}$.

To define the concept of the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony we need to present some preliminary notations. Let the considered sets of bases and bijections be $\mathbf{b} = (b,\ldots,b)$ and $\varphi = (\varphi,\ldots,\varphi)$. Let $\mathcal{W}_{G_{\mathbf{b}},\varphi} = \{{}_{G_{\mathbf{b}},\varphi}wal_{\mathbf{k}}(\mathbf{x}) : \mathbf{k} \in \mathbb{N}_0^s,\ \mathbf{x} \in [0,1)^s\}$ be the defined in previous section system of Walsh functions over the group $G_{\mathbf{b}}$ with respect to the bijection φ.

For arbitrary integers $b \geq 2$, $k \geq 0$ and a real $\alpha > 1$ we introduce the coefficient

$$\rho(\alpha;b;k) = \begin{cases} 1, & \text{if } k = 0, \\ b^{-\alpha\cdot g}, & \text{if } b^g \leq k < b^{g+1},\ g \geq 0,\ g \in \mathbb{Z}. \end{cases}$$

Let $\alpha = (\alpha_1, \ldots, \alpha_s)$, where for $1 \le j \le s$ $\alpha_j > 1$, be a given vector of real numbers. For an arbitrary vector $\mathbf{k} = (k_1, \ldots, k_s) \in \mathbb{N}_0^s$ we define the coefficient

$$R(\alpha; \mathbf{b}; \mathbf{k}) = \prod_{j=1}^{s} \rho(\alpha_j; b; k_j). \tag{7}$$

Let us signify $C(\alpha; \mathbf{b}) = \sum_{\mathbf{k}\in\mathbb{N}_0^s\setminus\{\mathbf{0}\}} R(\alpha; \mathbf{b}; \mathbf{k})$. So, the equality holds

$$C(\alpha; \mathbf{b}) = -1 + \prod_{j=1}^{s} \left[1 + (b-1)\frac{b^{\alpha_j}}{b^{\alpha_j} - b}\right]. \tag{8}$$

The notion of $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony is a partial case of more general kind of the diaphony, called hybrid weighted diaphony, which was introduced by Baycheva and Grozdanov [2]. So, following this concept we will present the next definition:

Definition 3. *Let $\xi = (\mathbf{x}_n)_{n\ge 0}$ be an arbitrary sequence of points in $[0,1)^s$. For each integer $N \ge 1$ the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony of the first N elements of the sequence ξ is defined as*

$$F_N(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;\xi) = \left(\frac{1}{C(\alpha;\mathbf{b})}\sum_{\mathbf{k}\in\mathbb{N}_0^s\setminus\{\mathbf{0}\}} R(\alpha;\mathbf{b};\mathbf{k})\left|\frac{1}{N}\sum_{n=0}^{N-1} {}_{G_{\mathbf{b}},\varphi}wal_{\mathbf{k}}(\mathbf{x}_n)\right|^2\right)^{\frac{1}{2}},$$

where the coefficients $R(\alpha;\mathbf{b};\mathbf{k})$ and the constant $C(\alpha;\mathbf{b})$ are defined respectively by the equalities (7) *and* (8)*.*

Following Baycheva and Grozdanov [2], see also [3], it is a well known fact that the sequence ξ is uniformly distributed in $[0,1)^s$ if and only if the next limit equality $\lim_{N\to\infty} F_N(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;\xi) = 0$ holds for each vector α, as above.

To the authors is unknown a lower bound of the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony of an arbitrary net as the one presented in the equality (2) and which is related with the $b-$adic diaphony.

5. On the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony of the nets of type of Halton-Zaremba

In the next theorem we will give an explicit formula for the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony of an arbitrary net ${}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}$ of type of Halton-Zaremba.

Theorem 1. *Let ${}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}$ be an arbitrary net of type of Halton-Zaremba. For each integer $\nu \ge 1$ the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony of the net ${}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}$ satisfies the* *equality*

$$F^2(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})$$
$$= \frac{1}{C(\alpha;b)}\left\{\frac{(b-1)b^{\alpha_2}(b^{\alpha_2}-1)}{b^{\alpha_2}-b}\cdot\frac{1}{b^{\alpha_2\nu}}\sum_{g=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g}\right.$$
$$\left.+(b-1)\frac{b^{\alpha_1}}{b^{\alpha_1}-b}\left[1+(b-1)\frac{b^{\alpha_2}}{b^{\alpha_2}-b}\right]\frac{1}{b^{\alpha_1\nu}}+(b-1)\frac{b^{\alpha_2}}{b^{\alpha_2}-b}\cdot\frac{1}{b^{\alpha_2\nu}}\right\},$$

where $C(\alpha;b)=\frac{(b-1)b^{\alpha_1}}{b^{\alpha_1}-b}+\frac{(b-1)b^{\alpha_2}}{b^{\alpha_2}-b}+(b-1)^2\frac{b^{\alpha_1+\alpha_2}}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)}$.

Corollary 1. *Let the conditions of Theorem 1 be realized. Let us assume that* $\alpha_1=\alpha_2=\alpha>1$. *Then, the following statements follow:*

(*i*) *For each integer* $\nu>0$ *the* $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$*diaphony of the net* ${}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}$ *satisfies the equality*

$$F^2(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})=\frac{b^\alpha-1}{(b-1)\frac{b^\alpha}{b^\alpha-b}+2}\cdot\frac{\nu}{b^{\alpha\nu}}+\frac{1}{b^{\alpha\nu}};$$

(*ii*) *Let us signify* $N=b^\nu$. *Then, the limit equality holds*

$$\lim_{\substack{\nu\to\infty\\ N=b^\nu}}\frac{N^{\frac{\alpha}{2}}\cdot F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})}{\sqrt{\log N}}=\sqrt{\frac{b^\alpha-1}{\left[(b-1)\frac{b^\alpha}{b^\alpha-b}+2\right]\log b}}.$$

(*iii*) *Let* $1<\alpha<2$. *Then, there exists a number* ε *such that* $0<\varepsilon<\frac{1}{2}$, *for which the inclusion* $F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})\in\mathcal{O}\left(\frac{\sqrt{\log N}}{N^{1-\varepsilon}}\right)$ *holds;*

(*iv*) *Let* $\alpha=2$. *Then, the inclusion* $F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})\in\mathcal{O}\left(\frac{\sqrt{\log N}}{N}\right)$ *holds;*

(*v*) *Let* $\alpha=2$. *Then, the limit equality holds*

$$\lim_{\substack{\nu\to\infty\\ N=b^\nu}}\frac{N\cdot F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})}{\sqrt{\log N}}=\sqrt{\frac{b^2-1}{(b+2)\log b}}.$$

(*vi*) *Let* $\alpha>2$. *Then, there exists a positive number* ε *such that the inclusion holds*

$$F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})\in\mathcal{O}\left(\frac{\sqrt{\log N}}{N^{1+\varepsilon}}\right).$$

Corollary 2. *Let the conditions of Theorem 1 be realized. Let us assume that* $\alpha_1>\alpha_2>1$. *Then, the following statements follow:*

(i) For each integer $\nu > 0$ the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony of the net ${}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}$ satisfies the equality

$$\begin{aligned}
&b^{\alpha_2\nu}\cdot F^2(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})\\
&=\frac{1}{C(\alpha;b)}\left\{(b-1)\frac{b^{\alpha_2}}{b^{\alpha_2}-b}\left[\frac{b^{\alpha_1}(b^{\alpha_2}-1)}{b^{\alpha_1}-b^{\alpha_2}}+1\right]\right.\\
&\left.+\left[\frac{(b-1)b^{\alpha_1+\alpha_2}(b\cdot b^{\alpha_1}+b^{\alpha_2}-b^{\alpha_1+\alpha_2}-b)}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)(b^{\alpha_1}-b^{\alpha_2})}+(b-1)\frac{b^{\alpha_1}}{b^{\alpha_1}-b}\right]\frac{1}{b^{(\alpha_1-\alpha_2)\nu}}\right\},
\end{aligned}$$

where the constant $C(\alpha;b)$ was defined in the condition of Theorem 1;

(ii) Let us signify $N=b^\nu$. Then, the limit equality holds

$$\begin{aligned}
&\lim_{\substack{\nu\to\infty\\ N=b^\nu}} N^{\frac{\alpha_2}{2}}\cdot F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})\\
&=\sqrt{\frac{b^{\alpha_2}(b^{\alpha_1}-b)[b^{\alpha_1}-b^{\alpha_2}+b^{\alpha_1}(b^{\alpha_2}-1)]}{(b^{\alpha_1}-b^{\alpha_2})[b^{\alpha_1}(b^{\alpha_2}-b)+b^{\alpha_2}(b^{\alpha_1}-b)+(b-1)b^{\alpha_1+\alpha_2}]}}.
\end{aligned}$$

(iii) Let $1<\alpha_2<2$. Then, there exists a number ε such that $0<\varepsilon<\frac{1}{2}$, for which the inclusion $F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})\in\mathcal{O}\left(\frac{1}{N^{1-\varepsilon}}\right)$ holds;

(iv) Let $\alpha_2=2$. Then, the inclusion $F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})\in\mathcal{O}\left(\frac{1}{N}\right)$ holds;

(v) Let $\alpha_2>2$. Then, there exists a number $\varepsilon>0$ such that the inclusion holds

$$F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})\in\mathcal{O}\left(\frac{1}{N^{1+\varepsilon}}\right).$$

The results of Theorem 1 and Corollaries 1 and 2 were announced by authors in [8]. Here we will develop the complete proofs of these statements.

6. Preliminary results

In this section, we will present some preliminary statements, which will be essentially used to prove the main results of the paper. The following lemmas hold:

Lemma 1. *Let $b\ge 2$ be a fixed integer, G_b be a finite group of order b and $\varphi:\mathbb{Z}_b\to G_b$ be an arbitrary bijection. For arbitrary integers $\nu>0$ and $k\ge 1$ we define the function*

$$\delta_{b^\nu}(k)=\begin{cases}1, & \text{if } k\equiv 0\ (mod\ b^\nu),\\ 0, & \text{if } k\not\equiv 0\ (mod\ b^\nu).\end{cases}$$

Then, the equalities hold

$$\sum_{i=0}^{b^\nu-1}{}_{G_b,\varphi}wal_k(\eta_{b,\nu}(i))=\sum_{i=0}^{b^\nu-1}{}_{G_b,\varphi}wal_k(p_{b,\nu}(i))=b^\nu\cdot\delta_{b^\nu}(k).$$

Proof. For the integer k and an arbitrary integer i, $0 \leq i \leq b^{\nu} - 1$, we will use the $b-$adic representations $k = \sum_{j=0}^{\infty} k_j b^j$ and $i = \sum_{j=0}^{\nu-1} i_j b^j$. Then, we have that $\eta_{b,\nu}(i) = 0.i_{\nu-1} i_{\nu-2} \dots i_0$ and ${}_{G_{b,\varphi}}wal_k(\eta_{b,\nu}(i)) = \prod_{j=0}^{\nu-1} \chi_{\varphi(k_j)}(\varphi(i_{\nu-1-j}))$. Hence, we obtain that

$$\sum_{i=0}^{b^{\nu}-1} {}_{G_{b,\varphi}}wal_k(\eta_{b,\nu}(i)) = \sum_{i_{\nu-1}=0}^{b-1} \chi_{\varphi(k_0)}(\varphi(i_{\nu-1})) \dots \sum_{i_0=0}^{b-1} \chi_{\varphi(k_{\nu-1})}(\varphi(i_0)). \tag{9}$$

Let us assume that $k \equiv 0(mod\ b^{\nu})$. Then, we have that $k_0 = k_1 = \dots = k_{\nu-1} = 0$ and from the equality (9) we obtain that $\sum_{i=0}^{b^{\nu}-1} {}_{G,\varphi}wal_k(\eta_{b,\nu}(i)) = b^{\nu}$.

Let us assume that $k \not\equiv 0(mod\ b^{\nu})$. Then, there exists at least one index δ, $0 \leq \delta \leq \nu - 1$ such that $k_{\delta} \neq 0$. In this case, the corresponding sum $\sum_{i_{\nu-1-\delta}=0}^{b-1} \chi_{\varphi(k_{\delta})}(\varphi(i_{\nu-1-\delta})) = 0$ and from the equality (9) we obtain that $\sum_{i=0}^{b^{\nu}-1} {}_{G,\varphi}wal_k(\eta_{b,\nu}(i)) = 0$.

The second equality of the statement of the Lemma can be proved by similar manner. □

Lemma 2. *Let the conditions* $(C1)$ *and* $(C2)$ *be fulfilled. Then, the following holds:*

(i) *For arbitrary integers* $0 \leq g \leq g_1 \leq \nu - 1$ *we define the set*

$$A(g_1; g) = \left\{ k_1 : k_1 = \sum_{j=g}^{g_1} k_j^{(1)} b^j, g \leq j \leq g_1, k_j^{(1)} \in \{0, 1, \dots, b-1\} \text{ and } k_g^{(1)}, k_{g_1}^{(1)} \neq 0 \right\}.$$

For each integer $k_1 \in A(g_1; g)$ *we define the integer* $k_1^* = \sum_{j=g}^{g_1} \overline{k}_j^{(1)} b^{\nu-1-j}$. *Then, for all integers* $0 \leq g_2 \leq \nu - 1$ *and* $b^{g_2} \leq k_2 \leq b^{g_2+1} - 1$ *the equalities hold*

$$\sum_{i=0}^{b^{\nu}-1} {}_{G_{b,\varphi}}wal_{k_1}(\eta_{b,\nu}(i)) {}_{G_{b,\varphi}}wal_{k_2}(p_{b,\nu}(i)) = \begin{cases} b^{\nu}, & \text{if } k_2 = k_1^*, \\ 0, & \text{if } k_2 \neq k_1^*. \end{cases}$$

In the case when $k_2 = k_1^*$, *we have that* $g_2 = \nu - 1 - g$;

(ii) *Let the integers* g_1 *and* g_2 *such that* $0 \leq g_1 \leq \nu - 1 < g_2$ *be arbitrary. An arbitrary integer* k_1 *such that* $b^{g_1} \leq k_1 \leq b^{g_1+1} - 1$ *we present in the form* $k_1 = \sum_{j=0}^{\nu-1} k_j^{(1)} b^j$. *An arbitrary integer* k_2 *such that* $b^{g_2} \leq k_2 \leq b^{g_2+1} - 1$ *we present in the form* $k_2 = \sum_{j=0}^{g_2} k_j^{(2)} b^j$. *For each integer* k_1, *as above, we define the set*

$$A(k_1) = \left\{ k_2 = \sum_{j=0}^{g_2} k_j^{(2)} b^j : k_0^{(2)} = \overline{k}_{\nu-1}^{(1)}, k_1^{(2)} = \overline{k}_{\nu-2}^{(1)}, \dots, k_{\nu-1}^{(2)} = \overline{k}_0^{(1)} \right.$$
$$\left. \text{and the digits } k_{\nu}^{(2)}, k_{\nu+1}^{(2)}, \dots, k_{g_2}^{(2)} \text{ are arbitrary} \right\}.$$

Then, the equalities hold

$$\left|\sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i))_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i))\right| = \begin{cases} b^\nu, & \text{if } k_2 \in A(k_1), \\ 0, & \text{if } k_2 \notin A(k_1); \end{cases}$$

(iii) Let the integers g_2 and g_1 such that $0 \le g_2 \le \nu - 1 < g_1$ be arbitrary. An arbitrary integer k_2 such that $b^{g_2} \le k_2 \le b^{g_2+1} - 1$ we present in the form $k_2 = \sum_{j=0}^{\nu-1} k_j^{(2)} b^j$. An arbitrary integer k_1 such that $b^{g_1} \le k_1 \le b^{g_1+1} - 1$ we present in the form $k_1 = \sum_{j=0}^{g_1} k_j^{(1)} b^j$. For each integer k_2, as above, we define the set

$$B(k_2) = \left\{ k_1 = \sum_{j=0}^{g_1} k_j^{(1)} b^j : k_0^{(1)} = \overline{k}_{\nu-1}^{(2)}, k_1^{(1)} = \overline{k}_{\nu-2}^{(2)}, \dots, k_{\nu-1}^{(1)} = \overline{k}_0^{(2)} \right.$$
$$\left. \text{and the digits } k_\nu^{(1)}, k_{\nu+1}^{(1)}, \dots, k_{g_1}^{(1)} \text{ are arbitrary} \right\}.$$

Then, the equalities hold

$$\left|\sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i))_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i))\right| = \begin{cases} b^\nu, & \text{if } k_1 \in B(k_2), \\ 0, & \text{if } k_1 \notin B(k_2); \end{cases}$$

(iv) Let the integers g_1 and g_2 such that $g_1 \ge \nu$ and $g_2 \ge \nu$ be arbitrary. Arbitrary integers k_1 and k_2 such that $b^{g_1} \le k_1 \le b^{g_1+1} - 1$ and $b^{g_2} \le k_2 \le b^{g_2+1} - 1$ we present in the form $k_1 = \sum_{j=0}^{g_1} k_j^{(1)} b^j$ and $k_2 = \sum_{j=0}^{g_2} k_j^{(2)} b^j$. For each integer k_1, as above, we define the set

$$C(k_1) = \left\{ k_2 = \sum_{j=0}^{g_2} k_j^{(2)} b^j : k_0^{(2)} = \overline{k}_{\nu-1}^{(1)}, k_1^{(2)} = \overline{k}_{\nu-2}^{(1)}, \dots, k_{\nu-1}^{(2)} = \overline{k}_0^{(1)} \right.$$
$$\left. \text{and the digits } k_\nu^{(2)}, k_{\nu+1}^{(2)}, \dots, k_{g_2}^{(2)} \text{ are arbitrary} \right\}.$$

Then, the equalities hold

$$\left|\sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i))_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i))\right| = \begin{cases} b^\nu, & \text{if } k_2 \in C(k_1), \\ 0, & \text{if } k_2 \notin C(k_1). \end{cases}$$

Proof. For an arbitrary integer i, $0 \le i \le b^\nu - 1$, with the $b-$adic representation $i = \sum_{j=0}^{\nu-1} i_j b^j$ we have that $\eta_{b,\nu}(i) = 0.i_{\nu-1}i_{\nu-2}\dots i_0$ and $p_{b,\nu}(i) = 0.i_0 i_1 \dots i_{\nu-1}$.

(i) For each integer $k_1 \in A(g_1; g)$ we have that

$$(10) \quad {}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i)) = \prod_{j=g}^{g_1} \chi_{\varphi(k_j^{(1)})}(\varphi(i_{\nu-1-j})) = \prod_{j=\nu-1-g_1}^{\nu-1-g} \chi_{\varphi(k_{\nu-1-j}^{(1)})}(\varphi(i_j)).$$

Let an arbitrary integer k_2 such that $b^{g_2} \leq k_2 \leq b^{g_2+1}-1$ have the $b-$adic representation $k_2 = \sum_{j=0}^{\nu-1} k_j^{(2)} b^j$ with the assumption that for $g_2+1 \leq j \leq \nu-1$ the equalities $k_j^{(2)} = 0$ hold. Hence, we have that

$$
\begin{aligned}
{}_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i)) &= \prod_{j=0}^{\nu-1} \chi_{\varphi(k_j^{(2)})}(\varphi(i_j)) \\
&= \prod_{j=0}^{\nu-2-g_1} \chi_{\varphi(k_j^{(2)})}(\varphi(i_j)). \prod_{j=\nu-1-g_1}^{\nu-1-g} \chi_{\varphi(k_j^{(2)})}(\varphi(i_j)). \prod_{j=\nu-g}^{\nu-1} \chi_{\varphi(k_j^{(2)})}(\varphi(i_j)).
\end{aligned}
\tag{11}
$$

Then, from the equalities (4), (10) and (11) we obtain that

$$
\begin{aligned}
&\sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i))_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i)) \\
&= \prod_{j=0}^{\nu-2-g_1} \sum_{i_j=0}^{b-1} \chi_{\varphi(k_j^{(2)})}(\varphi(i_j)) \\
&\times \prod_{j=\nu-1-g_1}^{\nu-1-g} \sum_{i_j=0}^{b-1} \chi_{\varphi(k_{\nu-1-j}^{(1)}) \oplus_{G_b} \varphi(k_j^{(2)})}(\varphi(i_j)) \prod_{j=\nu-g}^{\nu-1} \sum_{i_j=0}^{b-1} \chi_{\varphi(k_j^{(2)})}(\varphi(i_j)).
\end{aligned}
\tag{12}
$$

Let us assume that $k_2 = k_1^*$. This means the following: For $0 \leq j \leq \nu-2-g_1$ we have that $k_j^{(2)} = 0$. For $\nu-1-g_1 \leq j \leq \nu-1-g$ we have that $k_j^{(2)} = \overline{k}_{\nu-1-j}^{(1)}$ and hence, for each i_j, $0 \leq i_j \leq b-1$, the equality $\chi_{\varphi(k_{\nu-1-j}^{(1)}) \oplus_{G_b} \varphi(k_j^{(2)})}(\varphi(i_j)) = 1$ holds. For $\nu - g \leq j \leq \nu-1$ we have that $k_j^{(2)} = 0$. Then, from the equality (12) we obtain that

$$
\sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i))_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i)) = b^\nu.
$$

The condition $k_2 \neq k_1^*$ means that there exists at least one index δ, $0 \leq \delta \leq \nu-2-g_1$, such that $k_\delta^{(2)} \neq 0$, or there exists at least one index κ, $\nu-1-g_1 \leq \kappa \leq \nu-1-g$, such that $k_\kappa^{(2)} \neq \overline{k}_{\nu-1-\kappa}^{(1)}$, or there exists at least one index τ, $\nu - g \leq \tau \leq \nu-1$, such that $k_\tau^{(2)} \neq 0$. In the first case, the corresponding sum $\sum_{i_\delta=0}^{b-1} \chi_{\varphi(k_\delta^{(2)})}(\varphi(i_\delta)) = 0$, in the second case $\sum_{i_\kappa=0}^{b-1} \chi_{\varphi(k_{\nu-1-\kappa}^{(1)}) \oplus_{G_b} \varphi(k_\kappa^{(2)})}(\varphi(i_\kappa)) = 0$ and in the third case $\sum_{i_\tau=0}^{b-1} \chi_{\varphi(k_\tau^{(2)})}(\varphi(i_\tau)) = 0$. According to the equality (12), we obtain that $\sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i))_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i)) = 0$.

The another statements of Lemma 2 can be proved by using similar techniques. □

7. Proofs of the main results

Proof of Theorem 1. According to Definition 3, and by using the equality (5) for the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$diaphony of the net ${}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}$, we have that

$$
\begin{aligned}
&F^2(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}) = \frac{1}{C(\alpha;b)} \sum_{(k_1,k_2)\in\mathbb{N}_0^2\setminus\{\mathbf{0}\}} R(\alpha;\mathbf{b};(k_1,k_2)) \\
&\times |{}_{G_b,\varphi}wal_{k_1}(\kappa)|^2 |{}_{G_b,\varphi}wal_{k_2}(\mu)|^2 \\
&\times \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i))\,{}_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i)) \right|^2 \\
&= \frac{1}{C(\alpha;b)} \left\{ \sum_{k=1}^{\infty} \rho(\alpha_1;b;k) \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_k(\eta_{b,\nu}(i)) \right|^2 \right. \\
&+ \sum_{k=1}^{\infty} \rho(\alpha_2;b;k) \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_k(p_{b,\nu}(i)) \right|^2 \\
&+ \left[\sum_{g_1=0}^{\nu-1} \sum_{k_1=b^{g_1}}^{b^{g_1+1}-1} \sum_{g_2=0}^{\nu-1} \sum_{k_2=b^{g_2}}^{b^{g_2+1}-1} + \sum_{g_1=0}^{\nu-1} \sum_{k_1=b^{g_1}}^{b^{g_1+1}-1} \sum_{g_2=\nu}^{\infty} \sum_{k_2=b^{g_2}}^{b^{g_2+1}-1} \right. \\
&+ \left. \sum_{g_1=\nu}^{\infty} \sum_{k_1=b^{g_1}}^{b^{g_1+1}-1} \sum_{g_2=0}^{\nu-1} \sum_{k_2=b^{g_2}}^{b^{g_2+1}-1} + \sum_{g_1=\nu}^{\infty} \sum_{k_1=b^{g_1}}^{b^{g_1+1}-1} \sum_{g_2=\nu}^{\infty} \sum_{k_2=b^{g_2}}^{b^{g_2+1}-1} \right] \\
&\left. \times\ R(\alpha;\mathbf{b};(k_1,k_2)) \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i))\,{}_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i)) \right|^2 \right\}
\end{aligned}
$$

$$
= \frac{1}{C(\alpha;b)} (\Sigma_1 + \Sigma_2 + \Sigma_3 + \Sigma_4 + \Sigma_5 + \Sigma_6). \tag{13}
$$

We will calculate the sums in the equality (13). For the sum Σ_1, we have the following: In Lemma 1 for each integer $k \geq 1$ was shown the exact value of the trigonometric sum $\sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_k(\eta_{b,\nu}(i))$. By using this result, we obtain that

$$
\begin{aligned}
\Sigma_1 &= \sum_{k=1}^{\infty} \rho(\alpha_1;b;k) \left| \frac{1}{b^\nu} \sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_k(\eta_{b,\nu}(i)) \right|^2 = \sum_{k=1}^{\infty} \rho(\alpha_1;b;k).\delta_{b^\nu}(k) \\
&= \sum_{\substack{k=1 \\ k\equiv 0 (mod\ b^\nu)}}^{\infty} \rho(\alpha_1;b;k) = \sum_{k_1=1}^{\infty} \rho(\alpha_1;b;k_1 b^\nu) = \sum_{g_1=0}^{\infty} \sum_{k_1=b^{g_1}}^{b^{g_1+1}-1} \rho(\alpha_1;b;k_1 b^\nu) \\
&= \sum_{g_1=0}^{\infty} \sum_{k_1=b^{g_1}}^{b^{g_1+1}-1} b^{-\alpha_1(g_1+\nu)} = b^{-\alpha_1 \nu} \sum_{g_1=0}^{\infty} b^{-\alpha_1 g_1} \sum_{k_1=b^{g_1}}^{b^{g_1+1}-1} 1
\end{aligned}
$$

$$= (b-1)b^{-\alpha_1\nu}\sum_{g_1=0}^{\infty} b^{(1-\alpha_1)g_1} = (b-1)\frac{b^{\alpha_1}}{b^{\alpha_1}-b}\cdot\frac{1}{b^{\alpha_1\nu}}. \tag{14}$$

By using the same technique, we can prove that

$$\Sigma_2 = \sum_{k=1}^{\infty}\rho(\alpha_2;b;k)\left|\frac{1}{b^\nu}\sum_{i=0}^{b^\nu-1}{}_{G_{b,\varphi}}wal_k(p_{b,\nu}(i))\right|^2 = (b-1)\frac{b^{\alpha_2}}{b^{\alpha_2}-b}\cdot\frac{1}{b^{\alpha_2\nu}}. \tag{15}$$

To calculate the sum Σ_3, we will use the introduced in Lemma 2 (i) sets $A(g_1;g)$ and obtain that

$$\Sigma_3 = \sum_{g_1=0}^{\nu-1} b^{-\alpha_1 g_1}\sum_{g=0}^{g_1}\sum_{k_1\in A(g_1;g)}\sum_{g_2=0}^{\nu-1} b^{-\alpha_2 g_2}\sum_{k_2=b^{g_2}}^{b^{g_2+1}-1}$$
$$\times\left|\frac{1}{b^\nu}\sum_{i=0}^{b^\nu-1}{}_{G_{b,\varphi}}wal_{k_1}(\eta_{b,\nu}(i))_{G_{b,\varphi}}wal_{k_2}(p_{b,\nu}(i))\right|^2.$$

By using Lemma 2 (i), we have that only in the case when $g_2 = \nu-1-g$ and $k_2 = k_1^*$ the trigonometric sum $\sum_{i=0}^{b^\nu-1}{}_{G_{b,\varphi}}wal_{k_1}(\eta_{b,\nu}(i))_{G_{b,\varphi}}wal_{k_2}(p_{b,\nu}(i))$ has a value b^ν and in the another cases - a value 0. In this way, we obtain that

$$\Sigma_3 = \sum_{g_1=0}^{\nu-1} b^{-\alpha_1 g_1}\sum_{g=0}^{g_1}\sum_{k_1\in A(g_1;g)} b^{-\alpha_2(\nu-1-g)} = \frac{b^{\alpha_2}}{b^{\alpha_2\nu}}\sum_{g_1=0}^{\nu-1} b^{-\alpha_1 g_1}\sum_{g=0}^{g_1} b^{\alpha_2 g}\sum_{k_1\in A(g_1;g)} 1.$$

For arbitrary integers $0\le g\le g_1\le \nu-1$ the set $A(g_1;g)$ has a cardinality

$$|A(g_1;g)| = \begin{cases}(b-1)^2 b^{g_1-g-1}, & \text{if } \le g\le g_1-1,\\ b-1, & \text{if } g=g_1.\end{cases}$$

According to the above two statements, for the sum Σ_3, we will use the following presentation

$$\Sigma_3 = \frac{b^{\alpha_2}}{b^{\alpha_2\nu}}\left[\sum_{g_1=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1}\sum_{k\in A(g_1;g_1)} 1 + \sum_{g_1=1}^{\nu-1} b^{-\alpha_1 g_1}\sum_{g=0}^{g_1-1} b^{\alpha_2 g}\sum_{k_1\in A(g_1;g)} 1\right]$$
$$= \frac{b^{\alpha_2}}{b^{\alpha_2\nu}}\left[(b-1)\sum_{g_1=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1} + \frac{(b-1)^2}{b}\sum_{g_1=1}^{\nu-1} b^{(1-\alpha_1)g_1}\sum_{g=0}^{g_1-1} b^{(\alpha_2-1)g}\right]$$
$$= \frac{b^{\alpha_2}}{b^{\alpha_2\nu}}\left\{(b-1)\sum_{g_1=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1} + \frac{(b-1)^2}{b^{\alpha_2}-b}\sum_{g_1=1}^{\nu-1} b^{(1-\alpha_1)g_1}\left[b^{(\alpha_2-1)g_1}-1\right]\right\}$$
$$= \frac{b^{\alpha_2}}{b^{\alpha_2\nu}}\left\{(b-1)\sum_{g_1=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1} + \frac{(b-1)^2}{b^{\alpha_2}-b}\left[\sum_{g_1=1}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1} - \sum_{g_1=1}^{\nu-1} b^{(1-\alpha_1)g_1}\right]\right\}$$

$$= \frac{b^{\alpha_2}}{b^{\alpha_2\nu}}\left\{(b-1)\sum_{g_1=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1} + \frac{(b-1)^2}{b^{\alpha_2}-b}\left[\sum_{g_1=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1} - 1\right]\right.$$

$$\left. -\frac{(b-1)^2}{b^{\alpha_2}-b}\sum_{g_1=1}^{\nu-1} b^{(1-\alpha_1)g_1}\right\}$$

$$= \frac{b^{\alpha_2}}{b^{\alpha_2\nu}}\left\{(b-1)\sum_{g_1=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1} + \frac{(b-1)^2}{b^{\alpha_2}-b}\sum_{g_1=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1}\right.$$

$$\left. -\frac{(b-1)^2}{b^{\alpha_2}-b} - \frac{(b-1)^2}{b^{\alpha_2}-b}\sum_{g_1=1}^{\nu-1} b^{(1-\alpha_1)g_1}\right\}$$

$$= \frac{b^{\alpha_2}}{b^{\alpha_2\nu}}\left\{\frac{(b-1)(b^{\alpha_2}-1)}{b^{\alpha_2}-b}\sum_{g_1=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1} - \frac{(b-1)^2}{b^{\alpha_2}-b}\sum_{g_1=0}^{\nu-1} b^{(1-\alpha_1)g_1}\right\}$$

$$= \frac{b^{\alpha_2}}{b^{\alpha_2\nu}}\left\{\frac{(b-1)(b^{\alpha_2}-1)}{b^{\alpha_2}-b}\sum_{g_1=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g_1} + \frac{(b-1)^2 b^{\alpha_1}}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)}\left[b^{(1-\alpha_1)\nu}-1\right]\right\}$$

$$= \frac{(b-1)b^{\alpha_2}(b^{\alpha_2}-1)}{b^{\alpha_2}-b}\cdot\frac{1}{b^{\alpha_2\nu}}\sum_{g=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g}$$

$$+\frac{(b-1)^2 b^{\alpha_1+\alpha_2}}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)}\cdot\frac{1}{b^{(\alpha_1+\alpha_2-1)\nu}} - \frac{(b-1)^2 b^{\alpha_1+\alpha_2}}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)}\cdot\frac{1}{b^{\alpha_2\nu}}. \tag{16}$$

We will calculate the sum Σ_4. For this purpose, let the integers $0 \le g_1 \le \nu-1$, $b^{g_1} \le k_1 \le b^{g_1+1}-1$, and $g_2 \ge \nu$ be fixed. We will use the introduced in Lemma 2 (ii) sets $A(k_1)$. Hence for each integer $b^{g_2} \le k_2 \le b^{g_2+1}-1$ the modulus of the trigonometric sum $\left|\sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i))_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i))\right|$ will accept a value b^ν exactly $(b-1)b^{g_2-\nu}$ times. This is based on the fact that the digits $k^{(2)}_\nu, k^{(2)}_{\nu+1}, \ldots, k^{(2)}_{g_2}$ can be arbitrary. In this way, we obtain that

$$\Sigma_4 = \sum_{g_1=0}^{\nu-1} b^{-\alpha_1 g_1}\sum_{k_1=b^{g_1}}^{b^{g_1+1}-1}\sum_{g_2=\nu}^{\infty} b^{-\alpha_2 g_2}.(b-1)b^{g_2-\nu}$$

$$= \frac{(b-1)^2}{b^\nu}\sum_{g_1=0}^{\nu-1} b^{(1-\alpha_1)g_1}\sum_{g_2=\nu}^{\infty} b^{(1-\alpha_2)g_2}$$

$$= (b-1)^2\frac{b^{\alpha_2}}{b^{\alpha_2}-b}\cdot\frac{1}{b^\nu}\cdot\frac{1}{b^{(\alpha_2-1)\nu}}\sum_{g_1=0}^{\nu-1} b^{(1-\alpha_1)g_1}$$

$$= (b-1)^2\frac{b^{\alpha_2}}{b^{\alpha_2}-b}\cdot\frac{1}{b^{\alpha_2\nu}}\left[\frac{b^{\alpha_1}}{b^{\alpha_1}-b} - \frac{b^{\alpha_1}}{b^{\alpha_1}-b}\cdot\frac{1}{b^{(\alpha_1-1)\nu}}\right]$$

$$= (b-1)^2 \frac{b^{\alpha_1+\alpha_2}}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)} \cdot \frac{1}{b^{\alpha_2 \nu}}$$

$$- (b-1)^2 \frac{b^{\alpha_1+\alpha_2}}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)} \cdot \frac{1}{b^{(\alpha_1+\alpha_2-1)\nu}}. \tag{17}$$

To calculate the sum Σ_5, we can use the same techniques as above and obtain that

$$\Sigma_5 = (b-1)^2 \frac{b^{\alpha_1+\alpha_2}}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)} \cdot \frac{1}{b^{\alpha_1 \nu}}$$

$$- (b-1)^2 \frac{b^{\alpha_1+\alpha_2}}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)} \cdot \frac{1}{b^{(\alpha_1+\alpha_2-1)\nu}}. \tag{18}$$

It is evident the symmetry between the results obtained in the equalities (17) and (18).

We will calculate the sum Σ_6. For this purpose, let the integers $g_1 \geq \nu$, $b^{g_1} \leq k_1 \leq b^{g_1+1}-1$, and $g_2 \geq \nu$ be fixed. We will use the introduced in Lemma 2 (iv) sets $C(k_1)$. Hence, for each integer $b^{g_2} \leq k_2 \leq b^{g_2+1}-1$ the modulus of the trigonometric sum $\left|\sum_{i=0}^{b^\nu-1} {}_{G_b,\varphi}wal_{k_1}(\eta_{b,\nu}(i))_{G_b,\varphi}wal_{k_2}(p_{b,\nu}(i))\right|$ will accept a value b^ν exactly $(b-1)b^{g_2-\nu}$ times. This is based on the fact that the digits $k_\nu^{(2)}, k_{\nu+1}^{(2)}, \dots, k_{g_2}^{(2)}$ can be arbitrary. In this way, we obtain that

$$\Sigma_6 = \sum_{g_1=\nu}^{\infty} b^{-\alpha_1 g_1} \sum_{k_1=b^{g_1}}^{b^{g_1+1}-1} \sum_{g_2=\nu}^{\infty} b^{-\alpha_2 g_2}.(b-1)b^{g_2-\nu}$$

$$= (b-1)^2 \frac{1}{b^\nu} \sum_{g_1=\nu}^{\infty} b^{(1-\alpha_1)g_1} \sum_{g_2=\nu}^{\infty} b^{(1-\alpha_2)g_2}$$

$$= (b-1)^2 \frac{1}{b^\nu} \cdot \frac{b^{\alpha_1}}{b^{\alpha_1}-b} \cdot \frac{1}{b^{(\alpha_1-1)\nu}} \cdot \frac{b^{\alpha_2}}{b^{\alpha_2}-b} \cdot \frac{1}{b^{(\alpha_2-1)\nu}}$$

$$= (b-1)^2 \frac{b^{\alpha_1+\alpha_2}}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)} \cdot \frac{1}{b^{(\alpha_1+\alpha_2-1)\nu}}. \tag{19}$$

From the equalities (13), (14), (15), (16), (17), (18) and (19) we obtain that the $(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha)-$ diaphony of the net ${}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}$ satisfies the equality

$$F^2(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}) = \frac{1}{C(\alpha;b)} \left\{ \frac{(b-1)b^{\alpha_2}(b^{\alpha_2}-1)}{b^{\alpha_2}-b} \cdot \frac{1}{b^{\alpha_2 \nu}} \sum_{g=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g} \right.$$

$$\left. +(b-1)\frac{b^{\alpha_1}}{b^{\alpha_1}-b}\left[1+(b-1)\frac{b^{\alpha_2}}{b^{\alpha_2}-b}\right]\frac{1}{b^{\alpha_1 \nu}} + (b-1)\frac{b^{\alpha_2}}{b^{\alpha_2}-b} \cdot \frac{1}{b^{\alpha_2 \nu}} \right\}$$

with the introduced in the condition of the theorem constant $C(\alpha;b)$. Theorem 1 is finally proved.

Proof of Corollary 1. (i) According to Theorem 1, in the case when $\alpha_1 = \alpha_2 = \alpha$ we obtain that

$$F^2(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}) = \frac{b^\alpha - 1}{(b-1)\frac{b^\alpha}{b^\alpha - b} + 2}\cdot\frac{\nu}{b^{\alpha\nu}} + \frac{1}{b^{\alpha\nu}}.$$

(ii) From the above expression we obtain that

$$\frac{b^{\alpha\nu}\cdot F^2(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})}{\nu} = \frac{b^\alpha - 1}{(b-1)\frac{b^\alpha}{b^\alpha - b} + 2} + \frac{1}{\nu}$$

and hence, the limit equality holds

$$\lim_{\nu\to\infty}\frac{b^{\frac{\alpha}{2}\nu}\cdot F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})}{\sqrt{\nu}} = \sqrt{\frac{b^\alpha - 1}{(b-1)\frac{b^\alpha}{b^\alpha - b} + 2}}.$$

We put $N = b^\nu$ and find that $\nu = \frac{\log N}{\log b}$. From the above limit equality we obtain that

$$\lim_{\substack{\nu\to\infty\\ N=b^\nu}}\frac{N^{\frac{\alpha}{2}}\cdot F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})}{\sqrt{\log N}} = \sqrt{\frac{b^\alpha - 1}{\left[(b-1)\frac{b^\alpha}{b^\alpha - b} + 2\right]\log b}}. \tag{20}$$

(iii) Let us assume that $1 < \alpha < 2$. Then, there exists a number $0 < \varepsilon < \frac{1}{2}$ such that $\frac{\alpha}{2} = 1 - \varepsilon$. The equality (20) gives us that

$$F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}) \in \mathcal{O}\left(\frac{\sqrt{\log N}}{N^{1-\varepsilon}}\right).$$

(iv) When $\alpha = 2$ the equality (20) shows that

$$F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}) \in \mathcal{O}\left(\frac{\sqrt{\log N}}{N}\right).$$

(v) Let us in the equality (20) put $\alpha = 2$ and obtain the limit equality

$$\lim_{\substack{\nu\to\infty\\ N=b^\nu}}\frac{N\cdot F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu})}{\sqrt{\log N}} = \sqrt{\frac{b^2 - 1}{(b+2)\log b}}.$$

(vi) Let us assume that $\alpha > 2$. Then, there exists a number $\varepsilon > 0$ that $\frac{\alpha}{2} = 1 + \varepsilon$. The equality (20) shows that the inclusion

$$F(\mathcal{W}_{G_{\mathbf{b}},\varphi};\alpha;{}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}) \in \mathcal{O}\left(\frac{\sqrt{\log N}}{N^{1+\varepsilon}}\right)$$

holds.

Corollary 1 is finally proved.

Proof of Corollary 2. (i) The condition $\alpha_1 > \alpha_2$ allows us to calculate the value of the sum $\sum_{g=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g}$. So, the equality holds

$$\sum_{g=0}^{\nu-1} b^{(\alpha_2-\alpha_1)g} = \frac{b^{\alpha_1}}{b^{\alpha_1}-b^{\alpha_2}} - \frac{b^{\alpha_1}}{b^{\alpha_1}-b^{\alpha_2}} \cdot \frac{1}{b^{(\alpha_1-\alpha_2)\nu}}.$$

According to Theorem 1, in the case when $\alpha_1 > \alpha_2$ the presentation holds

$$\begin{aligned} & b^{\alpha_2\nu} \cdot F^2(\mathcal{W}_{G_{\mathbf{b}},\varphi}; \alpha; {}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}) \\ & = \frac{1}{C(\alpha;b)} \left\{ (b-1)\frac{b^{\alpha_2}}{b^{\alpha_2}-b} \left[\frac{b^{\alpha_1}(b^{\alpha_2}-1)}{b^{\alpha_1}-b^{\alpha_2}} + 1 \right] \right. \\ & \left. + \left[\frac{(b-1)b^{\alpha_1+\alpha_2}(b\cdot b^{\alpha_1}+b^{\alpha_2}-b^{\alpha_1+\alpha_2}-b)}{(b^{\alpha_1}-b)(b^{\alpha_2}-b)(b^{\alpha_1}-b^{\alpha_2})} + (b-1)\frac{b^{\alpha_1}}{b^{\alpha_1}-b} \right] \frac{1}{b^{(\alpha_1-\alpha_2)\nu}} \right\}. \end{aligned}$$

(ii) From the above equality we obtain the limit equality

$$\begin{aligned} & \lim_{\substack{\nu\to\infty \\ N=b^\nu}} N^{\frac{\alpha_2}{2}} \cdot F(\mathcal{W}_{G_{\mathbf{b}},\varphi}; \alpha; {}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}) \\ & = \sqrt{\frac{b^{\alpha_2}(b^{\alpha_1}-b)[(b^{\alpha_1}-b^{\alpha_2})+b^{\alpha_1}(b^{\alpha_2}-1)]}{(b^{\alpha_1}-b^{\alpha_2})[b^{\alpha_1}(b^{\alpha_2}-b)+b^{\alpha_2}(b^{\alpha_1}-b)+(b-1)b^{\alpha_1+\alpha_2}]}}. \end{aligned} \tag{21}$$

(iii) Let us assume that $1 < \alpha_2 < 2$. Then, there exists a number $0 < \varepsilon < \frac{1}{2}$ such that $\frac{\alpha_2}{2} = 1 - \varepsilon$. The equality (21) gives us that

$$F(\mathcal{W}_{G_{\mathbf{b}},\varphi}; \alpha; {}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}) \in \mathcal{O}\left(\frac{1}{N^{1-\varepsilon}}\right).$$

(iv) Let $\alpha = 2$. From the equality (21) we find that

$$F(\mathcal{W}_{G_{\mathbf{b}},\varphi}; \alpha; {}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}) \in \mathcal{O}\left(\frac{1}{N}\right).$$

(v) Let us assume that $\alpha_2 > 2$. Then, there exists a number $\varepsilon > 0$ such that $\frac{\alpha_2}{2} = 1 + \varepsilon$.

The equality (21) shows us that the inclusion

$$F(\mathcal{W}_{G_{\mathbf{b}},\varphi}; \alpha; {}_{G_b,\varphi_b}Z_{b,\nu}^{\kappa,\mu}) \in \mathcal{O}\left(\frac{1}{N^{1+\varepsilon}}\right)$$

holds.

Corollary 2 is finally proved.

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