A note on the p-length of a p-soluble group

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Abstract. Suppose that the finite group $G = AB$ is a mutually permutable product of two p-soluble subgroups A and B. By use of several invariant parameters of A and B, we present some bounds of the p-length of G. Some known results are improved.

Keywords: p-soluble, p-length, lower p-series, mutually permutable product.

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1. Introduction

All groups considered are finite. Let G be a group, we denote by $\pi(G)$ the set of all prime divisors of $|G|$. Let $p \in \pi(G)$, by G_p, we mean a Sylow p-subgroup of G. The other notations and terminologies used in this note are standard, as in [1, 2].

The p-length of a p-soluble group is an important invariant parameter. Many scholars have investigated on this invariant parameter, the readers can refer to [3]-[6] for instances. Therefore, the celebrated Hall-Higman theorem has established basic theorem on the p-length of a p-soluble group G, showing that the p-length of G is bounded above by the nilpotent class and the minimal number of generators of G_p and the p-rank of G [3].

In general, a product of two p-soluble subgroups need not be p-soluble. However, if the group G is a mutually permutable product of two p-soluble subgroups, then G is still a p-soluble group [7]. Recall that the product $G = AB$ of the subgroups A and B of a group G is called a mutually permutable product of A and B if $AU = UA$ for any subgroup U of B and $BV = VB$ for any subgroup V of B [7]. Cossey and Li in [6] investigated the p-length of a mutually permutable product of two p-soluble groups and obtained the following result:

Theorem 1.1 ([6, Theorem 1.1]). *Suppose that $G = AB$ is a mutually permutable product of two p-soluble subgroups A and B, where p is a prime in $\pi(G)$. If $l_p(A) \leq k$ and $l_p(B) \leq k$, then $l_p(G) \leq k + 1$.*

In the note, we continue the study on the p-length of a mutually permutable product of two p-soluble groups. By use of several invariant parameters of A and B, we will improve the above results as follows.

Theorem 1.2. *Suppose that $G = AB$ is a mutually permutable product of two p-soluble subgroups A and B, where p is a prime in $\pi(G)$. Then*

(1) $l_p(G) \leq \max\{c(A_p), c(B_p)\}$;
(2) $l_p(G) \leq \max\{d(A_p), d(B_p)\}$;
(3) $l_p(G) \leq \max\{l_p(A), l_p(B)\} + 1$;
(4) $l_p(G) \leq \max\{r_p(A), r_p(B)\} + 1$.

Note that, $\max\{l_p(A), l_p(B)\} \leq l_p(G)$, we get the following corollary:

Corollary 1.1. *Suppose that $G = AB$ is a mutually permutable product of two p-soluble subgroups A and B, where p is a prime in $\pi(G)$. Then, either $l_p(G) = \max\{l_p(A), l_p(B)\}$ or $l_p(G) = \max\{l_p(A), l_p(B)\} + 1$.*

2. Preliminaries

Let π be a set of primes and let G be a group. As well-known, $O^\pi(G)$ is defined to be the intersection of all normal subgroups N of G such that G/N is a π-group. Hence, $G/O^\pi(G)$ is the maximal π-factor group of G ([8, IX, 1.1]). Following [6], we invoke the following definition way of p-length of a p-soluble group.

If p is a prime, the lower p-series of G is

$$G \geq O^{p'}(G) \geq O^{p',p}(G) \geq O^{p',p,p'}(G) \geq \cdots .$$

If G is p-soluble, the last term of the lower p-series is 1 and if the lower p-series of G is

$$G = G_0 \geq G_1 \geq \cdots \geq G_s = 1,$$

then the p-length of G is the number of non-trivial p-groups in the set

$$\{G/G_1, G_1/G_2, \ldots, G_{s-1}/G_s\}.$$

Lemma 2.1 ([7, Theorem 4.1.15])**.** *Let the group G be the product of the mutually permutable subgroups A and B. If A and B are p-soluble, then G is p-soluble.*

Lemma 2.2 ([7, Lemma 4.1.10])**.** *Let the group G be the product of the mutually permutable subgroups A and B. If N is a normal subgroup of G, then G/N is a mutually permutable product of AN/N and BN/N.*

Lemma 2.3 ([7, Theorem 4.3.11])**.** *Let the non-trivial group G be the product of mutually permutable subgroups A and B. Then $A_G B_G$ is not trivial.*

Lemma 2.4 ([7, Lemma 4.3.3])**.** *Let the group G be the product of the mutually permutable subgroups A and B. Then:*

(1) *If N is a minimal normal subgroup of G, then $\{A \cap N, B \cap N\} \subseteq \{N, 1\}$.*

(2) *If N is a minimal normal subgroup of G contained in A and $B \cap N = 1$, then $N \leq C_G(A)$ or $N \leq C_G(B)$. If furthermore N is not cyclic, then $N \leq C_G(B)$.*

Lemma 2.5 ([7, Corollary 4.1.25])**.** *Let the group G be the product of the mutually permutable subgroups A and B. Then A' and B' are subnormal in G.*

3. Proof of Theorem 1.2

Proof. It is clear that (3) implies (4) by Hall-Higman theorem on the p-length of p-soluble groups. Hence, we only need to prove (1), (2) and (3).

Let G be a counter-example of minimal order. We proceed in steps.

Step 1. G is p-soluble.

This follows from Lemma 2.1.

Step 2. $N = O_p(G)$ is unique minimal normal and complemented in G and $N = C_G(N)$.

Let N be a minimal normal subgroup of G. We consider $\overline{G} = G/N$ together with $\overline{A} = AN/N$ and $\overline{B} = BN/N$. It is clear that $\overline{A}_p = A_pN/N$ and $\overline{B}_p = B_pN/N$ is respectively a Sylow p-subgroup of $\overline{A}$ and $\overline{B}$. By Lemma 2.2, $\overline{G}$ is the mutually product of two p-soluble subgroups $\overline{A}$ and $\overline{B}$, hence $\overline{G}$ satisfies the hypotheses of the theorem. For (1), the choice of G implies that

$$l_p(\overline{G}) \leq \max\{c(\overline{A}_p), c(\overline{B}_p)\} \leq \max\{c(A_p), c(B_p)\}.$$

If N_1 is minimal normal in G with $N_1 \neq N$, then we also have

$$l_p(G/N_1) \leq \max\{c(A_p), c(B_p)\}.$$

It follows that

$$l_p(G) \leq \max\{l_p(G/N), l_p(G/N_1)\} \leq \max\{c(A_p), c(B_p)\},$$

a contradiction. Therefore N is the unique minimal normal subgroup of G. Moreover, if $N \leq O_{p'}(G)$ or $N \leq \Phi(G)$, then

$$l_p(G) = l_p(\overline{G}) \leq \max\{c(A_p), c(B_p)\},$$

contradicting to the choice of G. Hence, $O_{p'}(G) = \Phi(G) = 1$ and $N = O_p(G)$, Step 1 follows. Similarly, we can prove Step 1 for (2) and (3).

Step 3. $N \leq A \cap B$.

Since $A_G B_G \neq 1$ by Lemma 2.3, we may assume $N \leq A$ by Step 1. If $N \nleq B$, then $N \cap B = 1$ by Lemma 2.4(1). If N is cyclic, then $N = C_G(N) \in \mathrm{Syl}_p(G)$, hence $l_p(G) = 1$, a contradiction. Thus, N is not cyclic and $N \leq C_G(B)$ by Lemma 2.4(2). Furthermore, $B \leq C_G(N) = N \leq A$ and so $G = AB = A$, Theorem 1.2 holds by Hall-Higman theorem. This shows $N \leq A \cap B$.

Step 4. If $N \leq M \leq G$, then $O_{p'}(M) = 1$.

Since $O_{p'}(M) \leq C_M(N) \leq C_G(N) = N$, we have $O_{p'}(M) = 1$.

Step 5. Finishing the proof.

For convenience, write $\overline{G} = G/N$, $\overline{A} = A/N$ and $\overline{B} = B/N$. We know that $\overline{G}$ satisfies the hypotheses of the theorem. Now, we prove by distinguishing three invariant parameters.

(1) By Step 2 and 3, $Z(A_p) \leq C_A(N) = N$, hence $c(\overline{A}_p) \leq c(A_p) - 1$. Similarly, $c(\overline{B}_p) \leq c(B_p) - 1$. The minimality of G implies that

$$l_p(\overline{G}) \leq \max\{c(\overline{A}_p), c(\overline{B}_p)\} \leq \max\{c(A_p) - 1, c(B_p) - 1\}.$$

Thus, $l_p(G) \leq \max\{c(A_p), c(B_p)\}$. This is the final contradiction.

(2) Since N is complemented in G, $N \nleq \Phi(A_p)$, i.e., $N \cap \Phi(A_p) < N$. Now, that A_p is a p-group, we have $\Phi(\overline{A}_p) = \Phi(A_p)N/N$ and so

$$\overline{A}_p/\Phi(\overline{A}_p) = (A_p/N)/(\Phi(A_p)N/N) \cong A_p/(\Phi(A_p)N).$$

Furthermore,

$$|\overline{A}_p/\Phi(\overline{A}_p)| = |A_p/(\Phi(A_p)N)| = |A_p/\Phi(A_p)|/|N/(N \cap \Phi(A_p))| < |A_p/\Phi(A_p)|.$$

This implies that $d(\overline{A}_p) \leq d(A_p) - 1$. Similarly, $d(\overline{B}_p) \leq d(B_p) - 1$.

The choice of G implies that

$$l_p(\overline{G}) \leq \max\{d(\overline{A}_p), d(\overline{B}_p)\} \leq \max\{d(A_p)-1, d(B_p)-1\}.$$

Thus, $l_p(G) \leq \max\{d(A_p), d(B_p)\}$. This is the final contradiction.

(3) Firstly, we have

Claim 1. $\max\{l_p(A), l_p(B)\} > 1$.

Suppose otherwise, $\max\{l_p(A), l_p(B)\} = 1$. Then

(i) $\overline{A} = \overline{A}_p \times \overline{A}_{p'}$ and $\overline{B} = \overline{B}_p \times \overline{B}_{p'}$.

Since $l_p(A) \leq 1$, by Step 4, we can write $A = [A_p]A_{p'}$. Since $[A_p, A_{p'}] \trianglelefteq \langle A_p, A_{p'}\rangle = A$ and $[A_p, A_{p'}] \leq [A, A] = A'$, we have $[A_p, A_{p'}] \trianglelefteq A'$. Noticing that A' is subnormal in G by Lemma 2.5, $[A_p, A_{p'}]$ is a subnormal p-subgroup of G. Hence, $[A_p, A_{p'}] \leq O_p(G) = N$ and consequently, $\overline{A} = \overline{A}_p \times \overline{A}_{p'}$.

Similarly, $\overline{B} = \overline{B}_p \times \overline{B}_{p'}$.

(ii) Both $\overline{A}_p$ and $\overline{B}_p$ are abelian groups.

By (i), $(\overline{A})' = (\overline{A}_p)' \times (\overline{B}_p)'$. Note that $(\overline{A})'$ is subnormal in $\overline{G}$ by Lemma 2.2 and 2.5, $(\overline{A}_p)'$ is a subnormal p-subgroup of $\overline{G}$. Hence, $(\overline{A}_p)' \leq O_p(\overline{G}) = 1$, that is, $\overline{A}_p$ is abelian.

Similarly, $\overline{B}_p$ is also abelian.

(iii) Finishing the proof of Claim 1.

In view of (ii) and the result of (1), $l_p(\overline{G}) \leq \max\{c(\overline{A}_p), c(\overline{B}_p)\} \leq 1$. Hence, $l_p(G) \leq 2$, a contradiction.

Now, we may assume that $\max\{l_p(A), l_p(B)\} = l_p(A) > 1$. Furthermore, we have

Claim 2. $1 < O_p(A') \leq N$.

Since $l_p(A) > 1$, $A' \neq 1$. But $O_{p'}(A') \leq O_{p'}(A)$, hence $O_{p'}(A') = 1$ by Step 4. Because A' is subnormal in G, $O_p(A')$ is subnormal in G. Thereby $1 < O_p(A') \leq N$.

Claim 3. Let O be the last non-trivial term of the lower p-series of A. Then $O \leq O_p(A')$.

Since $O_{p'}(A) = 1$, O is a p-group and $O \leq A_p$. On the other hand, since $l_p(A) > 1$,

$$O \leq O^{p',p}(A) = O^p(O^{p'}(A)) \leq O^p(A).$$

Consequently, $O \leq A_pA' \cap O^p(A)A'$. Noticing that

$$A/A' = A_pA'/A' \times O^p(A/A') = A_pA'/A' \times O^p(A)A'/A',$$

we have $A_pA' \cap O^p(A)A' = A'$. Hence, $O \leq A'$ and $O \leq O_p(A')$.

Claim 4. $\max\{l_p(\overline{A}), l_p(\overline{B})\} \leq l_p(A) - 1$.

By Claim 2 and 3, we have $O \leq N$. Clearly, $l_p(\overline{A}) \leq l_p(A/O) \leq l_p(A) - 1$. Similarly, $l_p(\overline{B}) \leq l_p(A) - 1$ if $l_p(B) = l_p(A)$. Of course, $l_p(\overline{B}) \leq l_p(B) \leq l_p(A) - 1$ if $l_p(B) < l_p(A)$. Thus, Claim 4 follows.

Finally, since $l_p(\overline{G}) \leq \max\{l_p(\overline{A}), l_p(\overline{B})\} + 1 \leq l_p(A)$, we obtain

$$l_p(G) \leq l_p(A) + 1 = \max\{l_p(A), l_p(B)\} + 1.$$

This is the final contradiction and the proof is complete. □

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