Normal structure and the modulus of weak uniform rotundity in Banach spaces

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Abstract. In this paper, we present some sufficient conditions for which a Banach space X has normal structure in term of the modulus of weak uniform rotundity $\delta_X(\epsilon, f)$, the Domínguez-Benavides coefficient $R(1, X)$ and the coefficient of weak orthogonality $\omega(X)$. Some known results are improved and strengthened.
Keywords: the modulus of weak uniform rotundity, Domínguez-Benavides coefficient, coefficient of weak orthogonality, normal structure.

1. Introduction

Let X be a Banach space, and $S_X = \{x \in X : \|x\| = 1\}$, $B_X = \{x \in X : \|x\| \leq 1\}$ denote the unit sphere and the unit ball of the Banach space X, respectively. For $x \in S_X$, let $\nabla_x \subset S_{X^*}$ be the set of norm 1 supporting functionals of S_X at x, that is $f \in \nabla_x \Leftrightarrow \langle f, x\rangle = 1$, where X^* stands for the dual space of X.

Definition 1.1. *The bounded convex subset C of a Banach space X is said to have normal structure, if for every convex subset H of C that contains more than one point, there exists a point $x_0 \in H$ such that*

$$\sup\{\|x_0 - y\| : y \in H\} < \sup\{\|x - y\| : x, y \in H\}.$$

The Banach space X is said to have weak normal structure, if every weakly compact convex subset of X that contains more than one point has normal structure. In reflexive spaces, both weak normal structure and normal structure coincide. A Banach space X is said to have uniform normal structure, if there

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exists $0 < c < 1$ such that for any closed bounded convex subset H of X that contains more than one point, there exists $x_0 \in H$ such that

$$\sup\{\|x_0 - y\| : y \in H\} < c \sup\{\|x - y\| : x, y \in H\}.$$

Let C be a nonempty bounded closed convex subset of a Banach space X, a mapping $T : C \to C$ is said to be nonexpansive provided the inequality

$$\|Tx - Ty\| \leq \|x - y\|$$

holds for every $x, y \in C$. A Banach space X is said to have the fixed point property if every nonexpansive mapping $T : C \to C$ has a fixed point.

Weak normal structure, normal structure and uniform normal structure are important in the metric fixed point theory for nonexpansive mapping. It was proved by Kirk [7] that if X has normal structure, then Banach space X has fixed point property. Since then, many mathematicians have investigated many various geometrical properties of Banach spaces implying weak normal structure, normal structure or uniform normal structure. A possible approach to look for some geometric properties in term of some geometric constants which imply weak normal structure, normal structure or uniform normal structure. Among the geometric constants, the modulus of weak uniform rotundity $\delta_X(\epsilon, f)$ plays an important role in the description of various geometric structures.

Definition 1.2. *The modulus of weak uniform rotundity is the function* $\delta_X(\epsilon, f) : [0, 2] \times S_{X^*} \to [0, 1]$ *defined in the following way ([10]):*

$$\delta_X(\epsilon, f) = \inf\left\{\{1\} \cup \{1 - \frac{\|x + y\|}{2} : x, y \in S_X, |\langle f, x - y\rangle| \geq \epsilon\}\right\},$$

where $0 \leq \epsilon \leq 2$ *and* $f \in S_{X^*}$*. The space* X *is weakly uniformly rotund if* $\delta_X(\epsilon, f) > 0$*, whenever* $0 < \epsilon \leq 2$ *and* $f \in S_{X^*}$*. For any* $f \in S_{X^*}$*,* $\delta_X(\epsilon, f)$ *is a continuous function in* $0 \leq \epsilon < 2$ *and* $\frac{\delta_X(\epsilon, f)}{\epsilon}$ *is increasing in* $(0, 2]$*.*

Rencently, Gao [3] studies the modulus of weak uniform rotundity extensively, and get some various geometrical properties and some sufficient conditions for normal structure as follows:

(i) If $\delta_X(\epsilon, f) > \frac{1}{2} - \frac{\epsilon}{4}, 0 \leq \epsilon < 2$ for all $f \in S_{X^*}$, then X is uniform nonsquare.

(ii) If $\delta_X(1, f) > 0$ for all $f \in S_{X^*}$, then X has uniform normal structure.

(iii) If $\delta_X(\epsilon, f) > \frac{1}{2} - \frac{\epsilon}{4}, 0 \leq \epsilon < 2$ for all $f \in S_{X^*}$, then X has uniform normal structure.

The purpose of this paper is to obtain some classes of Banach spaces with normal structure, which involves the modulus of weak uniform rotundity $\delta_X(\epsilon, f)$, the Domínguez-Benavides coefficient $R(1, X)$ and the coefficient of weak orthogonality $\omega(X)$. Moreover, these results are strictly wider than the previous Gao's results.

2. Preliminaries

Firstly, let us recall some basic facts about ultrapowers. A filter $\mathcal{F}$ on the set $\mathbb{N}$ of natural numbers is called to be an ultrafilter if it is maximal with respect to set inclusion. The ultrafilter is called trivial if it is of the form $A : A \subset \mathbb{N}, i_0 \in A$ for some fixed $i_0 \in \mathbb{N}$, otherwise, it is called nontrivial. The sequence $\{x_n\}$ in X converges to x with respect to $\mathcal{F}$, denoted by $\lim_{\mathcal{F}} x_i = x$ if for each neighborhood U of x, $\{i \in \mathbb{N} : x_i \in U\} \in \mathcal{F}$. Let $l_\infty(X)$ denote the subspace of the product space $\Pi_{n\in\mathbb{N}} X$ equipped with the norm

$$\|(x_n)\| := \sup_{n\in\mathbb{N}} \|x_n\| < \infty.$$

Let $\mathcal{U}$ be an ultrafilter on $\mathbb{N}$ and

$$N_{\mathcal{U}} = \{(x_n) \in l_\infty(X) : \lim_{\mathcal{U}} \|x_n\| = 0\}.$$

The ultrapower of X, denoted by $\widetilde{X}$ is the quotient space $l_\infty(X)/N_{\mathcal{U}}$ equipped with the quotient norm. $(x_n)_{\mathcal{U}}$ to denote the elements of the ultrapower. It follows from the definition of the quotient norm that

$$\|(x_n)_{\mathcal{U}}\| = \lim_{\mathcal{U}} \|x_n\|.$$

If $\mathcal{U}$ is nontrivial, then X can be embedded into $\widetilde{X}$ isometrically ([6]).

In what follows, some coefficients are introduced, which will be used in the following sections.

Definition 2.1. *The following Domínguez-Benavides coefficient was introduced in [2]:*

$$R(1, X) = \sup\big\{ \liminf_{n\to\infty}\{\|x_n + x\|\}\big\},$$

where the supremum is taken over all $x \in X$ with $\|x\| \leq 1$ and all weakly null sequences $\{x_n\}$ in B_X such that

$$D[(x_n)] := \limsup_{n\to\infty} \limsup_{m\to\infty} \|x_n - x_m\| \leq 1.$$

It is clear that $1 \leq R(1, X) \leq 2$. Some geometric conditions sufficient for normal structure in term of Domínguez-Benavides coefficient have been studied in [11], [12], [13].

Definition 2.2. *The coefficient of weak orthogonality of X was introduced by Sims in [9]:*

$$\omega(X) = \sup\{\lambda > 0 : \lambda \cdot \liminf_{n\to\infty} \|x_n + x\| \leq \liminf_{n\to\infty} \|x_n - x\|\},$$

where the supremum is taken over all the weakly null sequence (x_n) in X and all elements x of X. It is known that $\frac{1}{3} \leq \omega(X) \leq 1$ and $\omega(X) = \omega(X^*)$ in the reflexive Banach spaces (see [5], [8]).

3. Main results

Lemma 3.1 ([4]). *Let X be a Banach space without weak normal structure, then there exists a weakly null sequence $\{x_n\}_{n=1}^{\infty} \subseteq S_X$ such that*

$$\lim_n \|x_n - x\| = 1 \quad for\ all \quad x \in co\{x_n\}_{n=1}^{\infty}.$$

Theorem 3.2. *Let X be a Banach space with $\delta_X(1+\epsilon, f) > g(\epsilon)$ for all $f \in S_{X^*}$ and $0 \le \epsilon \le 1$, then X has weak normal structure, where the function $g(\epsilon)$ is defined as*

$$g(\epsilon) := \begin{cases} \frac{(R(1,X)-1)\epsilon}{2}, & 0 \le \epsilon \le \frac{1}{R(1,X)}, \\ \frac{1}{2}\left(1 - \frac{1-\epsilon}{R(1,X)-1}\right), & \frac{1}{R(1,X)} < \epsilon \le 1. \end{cases}$$

Proof. Suppose that X does not have weak normal structure, by the Lemma 3.1, there exists a weakly null sequence $\{x_n\}_{n=1}^{\infty}$ in S_X such that

$$\lim_n \|x_n - x\| = 1 \quad for\ all \quad x \in co\{x_n\}_{n=1}^{\infty}.$$

Take $\{f_n\} \subset S_{X^*}$ such that $f_n \in \nabla_{x_n}$ for all $n \in \mathbb{N}$. By the reflexivity of X^*, without loss of generality, we can assume that $f_n \xrightarrow{w^*} f$ for some $f \in B_{X^*}$. If necessary by passing to a subsequence, we can choose a subsequence of $\{x_n\}_{n=1}^{\infty}$, denoted again by $\{x_n\}_{n=1}^{\infty}$, such that

$$(1) \qquad \lim_n \|x_{n+1} - x_n\| = 1, \quad |(f_{n+1} - f)(x_n)| < \frac{1}{n}, \quad f_n(x_{n+1}) < \frac{1}{n},$$

for all $n \in \mathbb{N}$ and it follows that

$$\lim_n f_{n+1}(x_n) = \lim_n (f_{n+1} - f)(x_n) + f(x_n) = 0.$$

Note that the sequence $\{x_n\}$ is weakly null and verifies $D[\{x_n\}] = 1$. It follows from the definition of $R(1, X)$, then

$$\liminf_n \|x_{n+1} + x_n\| \le R(1, X).$$

Therefore, we can choose a subsequence $\{x_n\}$ such that

$$(2) \qquad \|x_{n+1} + x_n\| \le R(1, X).$$

Denote that $R := R(1, X)$ and consider two cases for $\epsilon \in [0, 1]$.

Firstly, if $\epsilon \in [0, \frac{1}{R}]$, take

$$\tilde{x} = (x_{n+1} - x_n)_{\mathcal{U}}, \tilde{y} = \{[1 - (R-1)\epsilon]x_{n+1} + \epsilon x_n\}_{\mathcal{U}} \text{ and } \tilde{f} = (-f_n)_{\mathcal{U}}.$$

By the 1 and 2, then

$$\|\tilde{f}\| = \tilde{f}(\tilde{x}) = \|\tilde{x}\| = 1.$$

and

$$\begin{aligned}\|\tilde{y}\| &= \Big\|[1-(R-1)\epsilon]x_{n+1}+\epsilon x_n\Big\| \\ &= \Big\|\epsilon(x_n+x_{n+1})+(1-R\epsilon)x_{n+1}\Big\| \\ &\leq R\epsilon+(1-R\epsilon)=1.\end{aligned}$$

Therefore, we have

$$\begin{aligned}\tilde{f}(\tilde{x}-\tilde{y}) &= \lim_{\mathcal{U}}(-f_n)\Big((R-1)\epsilon x_{n+1}-(1+\epsilon)x_n\Big) \\ &= 1+\epsilon,\end{aligned}$$

$$\begin{aligned}\|\tilde{x}+\tilde{y}\| &= \lim_{\mathcal{U}}\Big\|[2-(R-1)\epsilon]x_{n+1}-(1-\epsilon)x_n\Big\| \\ &\geq \lim_{\mathcal{U}}(f_{n+1})\Big([2-(R-1)\epsilon]x_{n+1}-(1-\epsilon)x_n\Big) \\ &= 2-(R-1)\epsilon.\end{aligned}$$

From the definition of $\delta_X(\epsilon,f)$, then

$$\delta_X(1+\epsilon,f)=\delta_{\widetilde{X}}(1+\epsilon,f)\leq\frac{(R-1)\epsilon}{2},$$

which is a contradiction.

Secondly, if $\epsilon\in(\frac{1}{R},1]$, in this case $R>1$, other $\epsilon>1$. Let

$$\tilde{x}=(x_{n+1}-x_n)_{\mathcal{U}},\tilde{y}=\{[1-(R-1)\epsilon']x_n+\epsilon' x_{n+1}\}_{\mathcal{U}},\text{ and }\tilde{f}=(-f_n)_{\mathcal{U}},$$

where $\epsilon'=\frac{1-\epsilon}{R-1}\in[0,\frac{1}{R})$. It follows from the first case, then

$$\|\tilde{f}\|=\|\tilde{x}\|=1\text{ and }\|\tilde{y}\|\leq 1,$$

$$\begin{aligned}\tilde{f}(\tilde{x}-\tilde{y}) &= \lim_{\mathcal{U}}(-f_n)\Big((1-\epsilon')x_{n+1}-[2-(R-1)\epsilon']x_n\Big) \\ &= 2-(R-1)\epsilon',\end{aligned}$$

$$\begin{aligned}\|\tilde{x}+\tilde{y}\| &= \lim_{\mathcal{U}}\|(1+\epsilon')x_{n+1}-(R-1)\epsilon' x_n\| \\ &\geq \lim_{\mathcal{U}}(f_{n+1})\Big((1+\epsilon')x_{n+1}-(R-1)\epsilon' x_n\Big) \\ &= 1+\epsilon'.\end{aligned}$$

From the definition of $\delta_X(\epsilon, f)$, then

$$\delta_X(2-(R-1)\epsilon', f) = \delta_{\widetilde{X}}(2-(R-1)\epsilon', f) \leq \frac{1-\epsilon'}{2},$$

which is equivalent to

$$\delta_X(1+\epsilon, f) = \delta_{\widetilde{X}}(1+\epsilon, f) \leq \frac{1}{2}\Big(1 - \frac{1-\epsilon}{R-1}\Big).$$

This is a contradiction. □

In fact, take $\epsilon = 0$ in Theorem 3.2, we can easily get the following result in [3].

Corollary 3.3. *Let X be a Banach space with*

$$\delta_X(1, f) > 0 \quad for \quad all \quad f \in S_{X^*},$$

then X has weak normal structure.

Remark 3.4. *(i) Theorem 3.2 strengthens the result of Gao: $\delta_X(1, f) > 0$ for all $f \in S_{X^*} \Longrightarrow X$ has normal structure, which gives the precise sufficient condition for the normal structure, whenever $1 \leq 1+\epsilon \leq 2$.*

(ii) It is note that Corollary 3.3 is sharp in the sense that there is a Banach space X such that $\delta_X(1, f) = 0$, X fails to have normal structure. Indeed, we consider the Bynum space $\ell_{p,\infty}$, which is the space ℓ_p $(1 < p < +\infty)$ with the norm

$$\|x\|_{p,\infty} = \max\{\|x^+\|, \|x^-\|\},$$

where x^+ is the positive part of x, defined as $x^+(i) = \max\{x(i), 0\}$ and $x^- = x^+ - x$. It is known that $\ell_{p,\infty}$ is a super-reflexive space that fails normal structure(see [1]), therefore $\delta_X(1, f) = 0$. This example shows that the condition in Corollary 3.3 is the best possible.

In the proof of following Theorem 3.5, we will get a property $\mathcal{P}$ that implying the uniform normal structure of a Banach space and also implying uniform normal structure of its dual. The proof is in the following fashion, suppose X^* fails to have uniform normal structure, then $\tilde{X}^*$ fails to have normal structure [7]. If X is super-reflexive, applying Lemma 3.1 yields vectors in $(\tilde{\tilde{X}}^*)^* = \tilde{\tilde{X}}$ that are used to show $\tilde{\tilde{X}} \notin \mathcal{P}$, which in turn implies $X \notin \mathcal{P}$ (The notation $X \notin \mathcal{P}$ will mean that a Banach space X does not satisfy the property $\mathcal{P}$.) Thus, in order to prove the property $\mathcal{P}$ implying X^* has uniform normal structure, we only need to show that if $X = X^{**}$ fails to have uniform normal structure, then $(\tilde{X})^* = (\tilde{\tilde{X}})^*$ fails to satisfy the property $\mathcal{P}$ by Lemma 3.1.

Theorem 3.5. *Let X be a Banach space such that*

$$\delta_X(1+\omega(X), f) > \frac{1-\omega(X)}{2}$$

for all $f \in S_{X^*}$, then X and X^* have uniform normal structure.

Proof. Since $\frac{1}{3} \leq \omega(X) \leq 1$, it is easy to check that

$$\delta_X(1+\omega(X), f) > \frac{1-\omega(X)}{2} \geq \frac{1}{2} - \frac{1+\omega(X)}{4} = \frac{1-\omega(X)}{4},$$

then X is uniformly nonsquare by the result (i) of Gao, it suffices to prove that X has weak normal structure whenever

$$\delta_X(1+\omega(X), f) > \frac{1-\omega(X)}{2}.$$

Firstly, repeating the arguments in the proof of Theorem 3.2, take $\tilde{x} = (x_{n+1} - x_n)_{\mathcal{U}}$, $\tilde{y} = [\omega(X)(x_{n+1}+x_n)]_{\mathcal{U}}$, and $\tilde{f} = (-f_n)_{\mathcal{U}}$. By the definition of $\omega(X)$ and Lemma 3.1, then

$$\|\tilde{f}\| = \tilde{f}(\tilde{x}) = \|\tilde{x}\| = 1,$$

$$\|\tilde{y}\| = \|[\omega(X)(x_{n+1}+x_n)]_{\mathcal{U}}\| \leq \|(x_{n+1}-x_n)_{\mathcal{U}}\| = 1,$$

$$\begin{aligned}\tilde{f}(\tilde{x}-\tilde{y}) &= \lim_{\mathcal{U}}(-f_n)\Big((1-\omega(X))x_{n+1} - (1+\omega(X))x_n\Big) \\ &= 1+\omega(X),\end{aligned}$$

$$\begin{aligned}\|\tilde{x}+\tilde{y}\| &= \lim_{\mathcal{U}} \|(1+\omega(X))x_{n+1} - (1-\omega(X))x_n\| \\ &\geq \lim_{\mathcal{U}}(f_{n+1})\Big((1+\omega(X))x_{n+1} - (1-\omega(X))x_n\Big) \\ &= 1+\omega(X).\end{aligned}$$

The definition of $\delta_X(\epsilon, f)$ implies that

$$\delta_X(1+\omega(X), f) = \delta_{\tilde{X}}(1+\omega(X), f) \leq \frac{1-\omega(X)}{2}.$$

This is a contradiction. Consequently, if $\delta_X(1+\omega(X), f) > \frac{1-\omega(X)}{2}$, then X has weak normal structure.

On the other hand, suppose that X^{**} does not have weak normal structure,

$$\tilde{x^*} = (f_n)_{\mathcal{U}}, \tilde{y^*} = [\omega(X^*)(f_{n+1} - f_{n+2})]_{\mathcal{U}} \text{ and } \tilde{x^{**}} = (x_n - x_{n+1})_{\mathcal{U}}.$$

By the definition of $\omega(X^*)$ and Lemma 3.1, then

$$\|\tilde{x^*}\| = \|(f_n)_{\mathcal{U}}\| = 1 \text{ and } \langle \tilde{x^*}, \tilde{x^{**}} \rangle = \langle f_n, x_n - x_{n+1} \rangle = 1,$$

and

$$\|\tilde{y^*}\| = \omega(X^*)\|(f_{n+1} - f_{n+2})_{\mathcal{U}}\| \leq 1.$$

Moreover, we have

$$\begin{aligned} \langle \tilde{x^*} - \tilde{y^*}, \tilde{x^{**}} \rangle &= \lim_{\mathcal{U}} \langle f_n - \omega(X^*) f_{n+1} + \omega(X^*) f_{n+2}, x_n - x_{n+1} \rangle \\ &= 1 + \omega(X^*), \end{aligned}$$

$$\begin{aligned} \|\tilde{x^*} + \tilde{y^*}\| &= \lim_{\mathcal{U}} \|f_n + \omega(X^*) f_{n+1} - \omega(X^*) f_{n+2}\| \\ &\geq \lim_{\mathcal{U}} \langle f_n + \omega(X^*) f_{n+1} - \omega(X^*) f_{n+2}, x_n - x_{n+2} \rangle \\ &= 1 + \omega(X^*). \end{aligned}$$

From the definition of $\delta_{X^*}(\epsilon, f)$ and $\omega(X) = \omega(X^*)$, then

$$\delta_{X^*}(1 + \omega(X^*), f) = \delta_{(X^*)_{\mathcal{U}}}(1 + \omega(X), f) = \delta_{(X_{\mathcal{U}})^*}(1 + \omega(X), f) > \frac{1 - \omega(X)}{2}.$$

□

Remark 3.6.

(i) It is well known that $\frac{1}{3} \leq \omega(X) \leq 1$, then $\frac{4}{3} \leq 1 + \omega(X) \leq 2$, therefore the condition in Theorem 3.5 implies the uniform normal structure of Banach spaces are shown to imply uniform normal structure of their dual spaces as well, which are complementary to the Gao's results, whenever $\frac{4}{3} \leq \epsilon \leq 2$.

(ii) Consider the Hilbert space H, it is well known that $\delta_H(\epsilon, f) = 1 - \frac{\sqrt{2(2-\epsilon)}}{2}$ for all $f \in S_{X^*}$ and $0 \leq \epsilon \leq 2$, $R(1, X) = 1$ and $\omega(X) = 1$, it is easy to check that $\delta_H(\epsilon, f) > 0 = \frac{(R(1,X)-1)\epsilon}{2} = \frac{1-\omega(X)}{2}$, then X has weak normal structure from Theorem 3.2 or Theorem 3.5.

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