Extended GE-filters in weak eGE-algebras

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Abstract. A broader concept than eGE algebra, called weak eGE algebra, is introduced, and related properties are studied. The concept of transitive and tightly (weak) eGE algebra is also considered and some properties are discussed. A weak eGE-algebra with additional conditions is used to give a way to create a GE-algebra. Extended GE filters are described in the last section. The concept of eGE-filters and upper sets is introduced and associated properties are investigated. Conditions for a superset of E in a weak eGE-algebra $(X, *, E)$ to be an eGE-filter are provided. Also, conditions for the upper set to become an eGE-filter are discussed. The characterization of the eGE-filter is established.
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1. Introduction

The concept of Hilbert algebra was introduced in early 50-ties by L. Henkin and T. Skolem for some investigations of implication in intuitionistic and other non-classical logics. In 60-ties, these algebras were studied especially by A. Horn and A. Diego [7] from algebraic point of view. Hilbert algebras are a valuable tool for some algebraic logic investigations as they can be regarded as fragments of any propositional logic that contains a logical connective implication ($\to$)

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and the constant 1 that is assumed to be the logical meaning "true". Many researchers have done a significant amount of work on Hilbert algebras [4, 5, 6, 8, 9, 10, 12, 13, 14]. As a generalization of Hilbert algebras, R.K. Bandaru et al. [1] introduced the notion of GE-algebras. They studied the various properties and filter theory of GE-algebras [2, 11, 15]. Bandaru et al. [3] introduced the notion of eGE-algebra as a generalization of GE-algebra and investigated its properties. We observed that there is a condition that do not play a remarkable role in that paper [3]. Algebraic structures with conditions that play no many role will inevitably narrow their objects, so they can weaken the value of their use. Everyone knows that the wider the object for a new algebraic structure, the wider the application. Therefore, it is necessary to increase the value of use by expanding the object of algebraic structures except for the conditions in which the role is insignificant. From this point of view, we would like to introduce a more generalized concept by deleting conditions that do not play an important role.

In this manuscript, we introduce more general version than eGE-algebras, so called weak eGE-algebra, and investigate its properties. This can generalize the several results of paper [3], and allows some of the results of the paper [3] to be classified as corollaries. We provide a condition for a weak eGE-algebra to be an eGE-algebra. We consider the concepts of a transitive and tightly (weak) eGE-algebra, and discuss some properties. Using a weak eGE-algebra with additional conditions, we provide a way to create a GE-algebra. The last section describes the expanded GE filters. We introduce the concepts of eGE-filters and upper sets and investigate their associated properties. We provide conditions for a superset of E in a weak eGE-algebra $(X, *, E)$ to be an eGE-filter. We provide conditions for the upper set to become an eGE-filter. We establish the characterization of eGE filters.

2. Preliminaries

Definition 2.1 ([1]). *By a GE-algebra we mean a nonempty set X with a constant 1 and a binary operation "$*$" satisfying the following axioms:*

(*GE*1) $u * u = 1$,

(*GE*2) $1 * u = u$,

(*GE*3) $u * (v * w) = u * (v * (u * w))$,

for all $u, v, w \in X$.

Definition 2.2 ([3]). *Let E be a nonempty subset of a set X. By a extended GE-algebra (briefly, eGE-algebra) we mean a structure $(X, *, E)$ in which $*$ is a binary operation on X satisfying the condition (GE3) and*

(*eGE*1) $(\forall x \in X)\ (x * x \in E)$, *(eGE2)* $(\forall x \in X)\ (x * E \subseteq E)$,

(eGE3) $(\forall x \in X)\ (E * x = \{x\})$,

where $E * x := \{a * x \mid a \in E\}$ *and* $x * E := \{x * a \mid a \in E\}$.

In an eGE-algebra $(X, *, E)$, we define a binary operation "$\leq_e$" as follows:

$$(\forall x, y \in X)(x \leq_e y \iff x * y \in E). \tag{1}$$

It could be noted that the binary operation "$\leq_e$" is reflexive, but it is neither antisymmetric nor transitive.

Proposition 2.1 ([3])**.** *Every eGE-algebra* $(X, *, E)$ *satisfies:*

$$(\forall x, y \in X)(x * (x * y) = x * y), \tag{2}$$

$$(\forall x, y, z \in X)(y * z \in E \Rightarrow x * (y * z) \in E), \tag{3}$$

$$(\forall x, y, z \in X)(x \leq_e y * z \Rightarrow y \leq_e x * z). \tag{4}$$

Definition 2.3 ([3])**.** *Let* $(X, *, E)$ *be a (weak) eGE-algebra. A subset* F *of* X *is called an extended GE-filter (briefly, eGE-filter) of* $(X, *, E)$ *if* F *is a superset of* E *which satisfies the next condition*

$$(\forall x, y \in X)(x * y \in F,\ x \in F \Rightarrow y \in F). \tag{5}$$

Lemma 2.1 ([3])**.** *Every eGE-filter* F *of an eGE-algebra* $(X, *, E)$ *satisfies:*

$$(\forall x, y \in X)(x \in F,\ x \leq_e y \Rightarrow y \in F). \tag{6}$$

3. Weak extended GE-algebras

Definition 3.1. *Let* E *be a nonempty subset of a set* X *and let "$*$" be a binary operation on* X*. A structure* $(X, *, E)$ *is called a weak extended GE-algebra (briefly, weak eGE-algebra) if it satisfies the following three conditions (GE3), (eGE1) and (eGE3).*

It is obvious that every eGE-algebra is a weak eGE-algebra, but the converse is not true in general as shown in the following example.

Example 3.2. Let $X = \{a, b, c, d\}$ be a set with the Cayley table which is given in Table 1.

Then, $(X, *, E)$ with $E = \{b, c\}$ is a weak eGE-algebra. But it is not an eGE-algebra since $d * E = \{a, c\} \nsubseteq E$.

It is clear that if $E = \{1\}$, then the weak eGE-algebra $(X, *, E)$ is only a GE-algebra, and vice versa. If $|E| \geq 2$, then the weak eGE-algebra $(X, *, E)$ may not be a GE-algebra as seen in the following example. Hence, we know that the weak eGE-algebra is an extension of a GE-algebra.

Table 1: Cayley table for the binary operation “$*$”

$*$	a	b	c	d
a	b	b	c	c
b	a	b	c	d
c	a	b	c	d
d	a	a	c	c

Example 3.3. Consider the weak eGE-algebra $(X, *, E)$ which is given in Example 3.2. We can see that there is no element to play a constant role and we can check that (GE1) and (GE2) are not true. Hence, $(X, *, E)$ is not a GE-algebra.

Proposition 3.1. *Every weak eGE-algebra $(X, *, E)$ satisfies:*

$$(\forall x, y \in X)(x * (x * y) = x * y). \tag{7}$$

Proof. For every $x, y \in X$, we have

$$x * (x * y) = x * ((x * x) * (x * y)) = x * ((x * x) * y) = x * y$$

by (GE3), (eGE1) and (eGE3). $\square$

Definition 3.4. *If $(X, *, E)$ is a (weak) eGE-algebra in which $(X, *, 1)$ is a GE-algebra, we say that $(X, *, E)$ is a tightly (weak) eGE-algebra.*

It is clear that every (weak) eGE-algebra $(X, *, E)$ is a tightly (weak) eGE-algebra if and only if $E = \{1\}$.

In the example below, we can see that if $(X, *, E)$ is a (weak) eGE-algebra satisfying $1 \in E$ and $|E| \geq 2$, $(X, *, E)$ may not be a tightly (weak) eGE-algebra.

Example 3.5. 1. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table which is given in Table 2. Then $(X, *, E)$ with $E = \{0, 1\}$ is an eGE-algebra. But it is

Table 2: Cayley table for the binary operation “$*$”

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	1	2	3	4
2	0	1	0	0	0
3	0	1	0	0	0
4	0	1	0	0	0

not a tightly eGE-algebra since $(X, *, 1)$ fails to satisfy (GE1), i.e., $0*0 = 0 \neq 1$.

2. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table which is given in Table 3.

Table 3: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	1	2	3	4
2	0	0	0	0	0
3	0	0	0	0	0
4	0	0	0	0	0

Then $(X, *, E)$ with $E = \{0, 1\}$ is a weak eGE-algebra. But it is not a tightly weak eGE-algebra since $(X, *, 1)$ fails to satisfy (GE1), i.e., $0 * 0 = 0 \neq 1$.

Proposition 3.2. *If $(X, *, E)$ is a weak eGE-algebra, then E is closed under the binary operation "$*$".*

Proof. Let $x, y \in E$. Then $x * y = y \in E$ and $y * x = x \in E$ by (eGE3). Hence, E is closed under "$*$" □

Question 3.1. *Let B be a subset of X such that $E \subseteq B$.*
1. If $(X, *, E)$ is a weak eGE-algebra, then is $(X, *, B)$ a weak eGE-algebra.
2. If $(X, *, B)$ is a weak eGE-algebra, then is $(X, *, E)$ a weak eGE-algebra.

The next example give a negative answer to Question 3.1.

Example 3.6. 1. Let $X = \{a, b, c, d, e\}$ be a set with the Cayley table which is given in Table 4.

Table 4: Cayley table for the binary operation "$*$"

$*$	a	b	c	d	e
a	c	b	c	c	b
b	d	d	d	d	d
c	a	b	c	d	e
d	a	b	c	d	e
e	a	c	c	d	e

Then $(X, *, E)$ with $E = \{c, d\}$ is a weak eGE-algebra. But $(X, *, B)$ with $B = \{c, d, e\}$ is not a weak eGE-algebra since $B * b = \{b, c\} \neq \{b\}$. Also $(X, *, B)$ with $B = \{c, d\}$ is a weak eGE-algebra. But $(X, *, E)$ with $E = \{c\}$ is not a weak eGE-algebra since $b * b = d \notin E$.

Remark 3.7. Let $\{E, B\}$ be a partition of X. If $(X, *, E)$ is a (weak) eGE-algebra, then $(X, *, B)$ can never be a (weak) eGE-algebra.

The following example describes Remark 3.7.

Example 3.8. In Example 3.6, if we take $E = \{c, d\}$ and $B = \{a, b, e\}$, then $\{E, B\}$ is a partition of X. We can observe that $(X, *, E)$ is a weak eGE-algebra, but $(X, *, B)$ is not a weak eGE-algebra since $B * b = \{b, c, d\} \neq \{b\}$.

By Remark 3.7, we know that if $(X, *, E)$ is a weak eGE-algebra, then $(X, *, X \setminus E)$ is not a weak eGE-algebra.

Theorem 3.2. *Every weak eGE-algebra $(X, *, E)$ with $E = \{1\}$ satisfies the condition (eGE2).*

Proof. It is straightforward. □

Question 3.3. *Let $(X, *, E)$ be a weak eGE-algebra. If E contains the constant 1, then does (eGE2) hold?*

The next example give a negative answer to Question 3.3.

Example 3.9. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table which is given in Table 5.

Table 5: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	1	1	3	3	1
1	0	1	2	3	4
2	0	1	2	3	4
3	0	1	0	1	1
4	0	1	0	1	1

Then $(X, *, E)$ with $E = \{1, 2\}$ is a weak eGE-algebra. But it does not satisfy (eGE2) since $2 * E = \{0, 2\} \nsubseteq E$.

Note that two facts below are equivalent in an eGE-algebra $(X, *, E)$ (see [3]).

$$(\forall x, y, z \in X)(x * y \leq_e (z * x) * (z * y)). \tag{8}$$

$$(\forall x, y, z \in X)(x * y \leq_e (y * z) * (x * z)). \tag{9}$$

In the next example, we can verify that (8) and (9) are not equivalent in a weak eGE-algebra.

Example 3.10. Let $X = \{0, a, b, c, d, e, f\}$ be a set with the Cayley table which is given in Table 6.

Then $(X, *, E)$ with $E = \{a, d\}$ is a weak eGE-algebra. But (8) and (9) are not equivalent. In fact, $(0 * b) * ((c * 0) * (c * b)) = a * (0 * b) = a * a = a \in E$, that is, $(0 * b) \leq_e (c * 0) * (c * b)$. But $(0 * b) * ((b * c) * (0 * c)) = a * (f * a) = a * e = e \notin E$, i.e., $(0 * b) \leq_e (b * c) * (0 * c)$ does not hold.

Table 6: Cayley table for the binary operation "$*$"

$*$	0	a	b	c	d	e	f
0	a	a	a	a	a	a	a
a	0	a	b	c	d	e	f
b	0	0	d	f	d	0	f
c	0	0	b	d	d	0	d
d	0	a	b	c	d	e	f
e	0	a	0	0	0	a	0
f	0	e	b	d	d	e	d

Definition 3.11. *A (weak) eGE-algebra $(X, *, E)$ is said to be transitive if it satisfies:*

$$(\forall x, y, z \in X)(x * y \leq_e (z * x) * (z * y)). \tag{10}$$

It is clear that every transitive eGE-algebra is a transitive weak eGE-algebra.

Example 3.12. Let $X = \{0, a, b, c, d\}$ be a set with the Cayley table which is given in Table 7.

Table 7: Cayley table for the binary operation "$*$"

$*$	0	a	b	c	d
0	0	a	b	c	d
a	0	0	b	c	0
b	0	a	b	c	d
c	0	a	0	0	a
d	b	b	b	c	b

Then $(X, *, E)$ with $E = \{0, b\}$ is a transitive (weak) eGE-algebra.

Lemma 3.1. *Every transitive weak eGE-algebra $(X, *, E)$ satisfies:*

$$(\forall x, y, z \in X)(y \leq_e z \;\Rightarrow\; x * y \leq_e x * z,\; z * x \leq_e y * x). \tag{11}$$

$$(\forall x, y, z \in X)(x \leq_e y,\; y \leq_e z \;\Rightarrow\; x \leq_e z). \tag{12}$$

Proof. Let $x, y, z \in X$ be such that $y \leq_e z$. Then $y * z \in E$, which implies from (eGE3) and (10) that

$$(x * y) * (x * z) = (y * z) * ((x * y) * (x * z)) \in E,$$

that is, $x * y \leq_e x * z$. The combination of (GE3), (eGE3) and (10) induces

$$\begin{aligned}(z * x) * (y * x) &= (y * z) * ((z * x) * (y * x)) \\ &= (y * z) * ((z * x) * ((y * z) * (y * x))) \\ &= (z * x) * ((y * z) * (y * x)) \in E,\end{aligned}$$

and so $z * x \leq_e y * x$. Hence, (11) is valid. Let $x, y, z \in X$ be such that $x \leq_e y$ and $y \leq_e z$. Then $x * y \in E$ and $y * z \in E$. Using (eGE3) and (10), we have

$$x * z = (y * z) * ((x * y) * (x * z)) \in E,$$

and thus $x \leq_e z$. □

Corollary 3.1. *Every transitive eGE-algebra* $(X, *, E)$ *satisfies* (11) *and* (12).

The following example shows that any weak eGE-algebra $(X, *, E)$ does not satisfy the following assertion.

$$(13) \qquad (\forall x, y, z \in X)(y * z \in E \ \Rightarrow x * (y * z) \in E).$$

Example 3.13. Let $X = \{0, a, b, c, d\}$ be a set with the Cayley table which is given in Table 8.

Table 8: Cayley table for the binary operation "$*$"

$*$	0	a	b	c	d
0	0	a	b	c	d
a	b	b	b	c	c
b	0	a	b	c	d
c	a	a	b	b	a
d	0	0	b	b	0

Then, $(X, *, E)$ with $E = \{0, b\}$ is a weak eGE-algebra. But it doesn't satisfy (13). In fact, $d * 0 = 0 \in E$ but $c * (d * 0) = c * 0 = a \notin E$.

Proposition 3.3. *For any weak eGE-algebra* $(X, *, E)$ *satisfying the condition* (13), *we have*

$$(14) \qquad (\forall x, y, z \in X)\left(x \leq_e y * z \ \Rightarrow \begin{cases} y * (x * z) \in E \\ y * (x * (y * z)) \in E \end{cases} \right).$$

Proof. Let $(X, *, E)$ be a weak eGE-algebra that satisfies the condition (13). Let $x, y, z \in X$ be such that $x \leq_e y * z$. Then $x * (y * z) \in E$ and hence

$$y * (x * z) = y * (x * (y * z)) \in E$$

by (GE3) and (13). □

Since every eGE-algebra $(X, *, E)$ satisfies the condition (13) (see [3]), we have the following corollary.

Corollary 3.2. *Every eGE-algebra* $(X, *, E)$ *satisfies the condition* (14).

Theorem 3.4. *Let $(X, *, E)$ be a weak eGE-algebra satisfying the condition* (13) *where E contains the constant* 1*, and let $Y := \{1\} \cup (X \setminus E)$. Define a binary operation "$\circledast$" on Y as follows:*

$$\circledast : Y \times Y \to Y,\ (x, y) \mapsto \begin{cases} x * y, & \text{if } x \neq 1 \neq y,\ x * y \notin E, \\ 1, & \text{if } x \neq 1 \neq y,\ x * y \in E, \\ y, & \text{if } x = 1, \\ 1, & \text{if } y = 1. \end{cases} \tag{15}$$

Then $(Y, \circledast, 1)$ is a GE-algebra.

Proof. (GE1) and (GE2) are directly identified by the definition of $\circledast$. Let $x, y, z \in X$. It is clear that if $x = 1$, $y = 1$ or $z = 1$, then

$$x \circledast (y \circledast z) = x \circledast (y \circledast (x \circledast z)).$$

Assume that $x \neq 1$, $y \neq 1$ and $z \neq 1$. If $y * z \in E$, then $y \circledast z = 1$ and so $x \circledast (y \circledast z) = x \circledast 1 = 1$.

On the other hand, if $x * z \in E$, then $x \circledast z = 1$. Hence,

$$x \circledast (y \circledast (x \circledast z)) = x \circledast (y \circledast 1) = x \circledast 1 = 1.$$

If $x * z \notin E$, then $x \circledast z = x * z$. Since $y * z \in E$, we have $x * (y * z) \in E$, that is, $x \leq_e y * z$ by (13), and thus $y * (x * z) \in E$ by Proposition 3.3. Hence, $y \circledast (x \circledast z) = y \circledast (x * z) = 1$, and so $x \circledast (y \circledast (x \circledast z)) = x \circledast 1 = 1$. This shows that $x \circledast (y \circledast z) = x \circledast (y \circledast (x \circledast z))$ when $y * z \in E$. If $y * z \notin E$, then $y \circledast z = y * z$, and either $x * (y * z) \in E$ or $x * (y * z) \notin E$. For the case $x * (y * z) \in E$, we get $x \circledast (y \circledast z) = x \circledast (y * z) = 1$, and $y * (x * z) \in E$ by Proposition 3.3. Thus $y \circledast (x \circledast z) = y \circledast (x * z) = 1$ when $x * z \notin E$. If $x * z \in E$, then $x \circledast z = 1$ and so $y \circledast (x \circledast z) = y \circledast 1 = 1$. Hence,

$$x \circledast (y \circledast (x \circledast z)) = x \circledast 1 = 1 = x \circledast (y \circledast z).$$

For the case $x * (y * z) \notin E$, we get $y * z \notin E$ by (13), and $y * (x * z) \notin E$ by Proposition 3.3. Then $x * z \notin E$ by (13). Since $1 \in E$ and

$$x * (y * (x * z)) = x * (y * z) \notin E$$

by (GE3), it follows that

$$\begin{aligned} x \circledast (y \circledast z) &= x \circledast (y * z) = x * (y * z) \\ &= x * (y * (x * z)) \\ &= x \circledast (y * (x * z)) \\ &= x \circledast (y \circledast (x \circledast z)). \end{aligned}$$

Therefore, $(Y, \circledast, 1)$ is a GE-algebra. $\square$

Note that, every eGE-algebra $(X, *, E)$ satisfies the condition (13) and it is a weak eGE-algebra. Hence, we have the next corollary.

Corollary 3.3 ([3]). *Let $(X, *, E)$ be an eGE-algebra where E contains the constant 1 and consider $Y := \{1\} \cup (X \setminus E)$. If we give a binary operation "$\circledast$" on Y by (15), then $(Y, \circledast, 1)$ is a GE-algebra.*

The following example illustrates Theorem 3.4.

Example 3.14. Let $X = \{0, 1, 2, 3, 4\}$ be a set with the Cayley table which is given in Table 9.

Table 9: Cayley table for the binary operation "$*$"

$*$	0	1	2	3	4
0	0	1	2	3	4
1	0	1	2	3	4
2	0	0	0	0	4
3	0	0	0	0	4
4	1	1	1	3	1

Then $(X, *, E)$ with $E = \{0, 1\}$ is a weak eGE-algebra satisfying the condition (13), and $Y = \{1\} \cup (X \setminus E) = \{1, 2, 3, 4\}$. The operation $\circledast$ on Y is given by Table 10, and $(Y, \circledast, 1)$ is a GE-algebra.

Table 10: Cayley table for the binary operation "$\circledast$"

$\circledast$	1	2	3	4
1	1	2	3	4
2	1	1	1	4
3	1	1	1	4
4	1	1	3	1

4. Extended GE-filters

Given a superset F of E in a weak eGE-algebra $(X, *, E)$, we consider the next arguments:

$$(\forall a \in E)(\forall x, y \in X)(x * (a * y) \in F \;\Rightarrow\; a * y \in F), \tag{16}$$

$$(\forall a \in E^c)(\forall x, y \in X)(x * (a * y) \in F \;\Rightarrow\; a * y \in F). \tag{17}$$

The following example shows that there exists a weak eGE-algebra $(X, *, E)$ in which any supserset F of E does not satisfy the assertion (16) or (17).

Table 11: Cayley table for the binary operation "$*$"

$*$	0	a	b	c	d
0	0	a	b	c	d
a	0	a	b	c	d
b	c	a	a	c	c
c	0	0	0	0	0
d	0	0	0	0	0

Example 4.1. 1. Let $X = \{0, a, b, c, d\}$ be a set with the Cayley table which is given in Table 11.

Then, $(X, *, E)$ with $E = \{0, a\}$ is a weak eGE-algebra. If we take a superset $F = \{0, a, b\}$ of E, then $d * (a * c) = d * c = 0 \in F$ and $a \in E$ but $a * c = c \notin F$. Hence, F does not satisfy the assertion (16).

2. Let $X = \{0, a, b, c, d\}$ be a set with the Cayley table which is given in Table 12.

Table 12: Cayley table for the binary operation "$*$"

$*$	0	a	b	c	d
0	b	a	b	a	a
a	0	c	0	c	c
b	0	a	b	c	d
c	0	a	b	c	d
d	0	a	0	c	c

Then, $(X, *, E)$ with $E = \{b, c\}$ is a weak eGE-algebra, and the set $F = \{b, c, d\}$ does not satisfy (17) since $0 * (a * 0) = 0 * 0 = b \in F$ but $a * 0 = 0 \notin F$.

We provide a condition for a superset of E in a weak eGE-algebra $(X, *, E)$ to be an eGE-filter.

Theorem 4.1. *Let F be a superset of E in a weak eGE-algebra $(X, *, E)$. If F satisfies* (16)*, then F is an eGE-filter of $(X, *, E)$.*

Proof. Let $x, y \in X$ be such that $x \in F$ and $x * y \in F$. Then $x * (E * y) = x * \{y\} \subseteq F$, and so $x * (a * y) \in F$, for all $a \in E$. It follows from (16) that $a * y \in F$, for all $a \in E$. Hence, $\{y\} = E * y \subseteq F$, and thus $y \in F$. Therefore, F is an eGE-filter of $(X, *, E)$. $\square$

Question 4.2. *If a superset F of E in a weak eGE-algebra $(X, *, E)$ satisfies* (17)*, then is F an eGE-filter of $(X, *, E)$?*

The answer to the Question 4.2 is negative as seen in the next example.

Table 13: Cayley table for the binary operation "$*$"

$*$	0	a	b	c	d
0	b	b	b	d	d
a	b	b	b	b	b
b	0	a	b	c	d
c	0	a	b	c	d
d	b	b	b	b	b

Example 4.2. Let $X = \{0, a, b, c, d\}$ be a set with the Cayley table which is given in Table 13. Then $(X, *, E)$ with $E = \{b, c\}$ is a weak eGE-algebra, and the set $F = \{b, c, d\}$ satisfies (17). But F is not eGE-filter of $(X, *, E)$ since $d * 0 = b \in F$ and $d \in F$ but $0 \notin F$.

Theorem 4.3. *Let F be a superset of E in a weak eGE-algebra $(X, *, E)$. Then F is an eGE-filter of $(X, *, E)$ if and only if it satisfies:*

$$(\forall x \in E)(\forall y, z \in X)(x * (y * z) \in F,\ x * y \in F \ \Rightarrow\ x * z \in F). \tag{18}$$

Proof. Assume that F is an eGE-filter of $(X, *, E)$. Let $x, y, z \in X$ be such that $x \in E$, $x * (y * z) \in F$ and $x * y \in F$. Then $y * z = x * (y * z) \in F$ and $y = x * y \in F$ by (eGE3). It follows from (eGE3) and (5) that $x * z = z \in F$.

Conversely, suppose that F satisfies (18). Assume that $x \in F$ and $x * y \in F$, for all $x, y \in X$. Then $E * x = \{x\} \subseteq F$ and $E * (x * y) = \{x * y\} \subseteq F$. It follows that $a * x \in F$ and $a * (x * y) \in F$, for all $a \in E$. Hence, $a * y \in F$, for all $a \in E$ by (18), and so $\{y\} = E * y \subseteq F$, that is $y \in F$. Therefore, F is an eGE-filter of $(X, *, E)$. $\square$

Given an eGE-algebra $(X, *, E)$ and any element $a, b \in X$, consider the following set.

$$E_a := \{x \in X \mid a * x \in E\}, \tag{19}$$

$$E(a, b) := \{x \in X \mid a * (b * x) \in E\}. \tag{20}$$

The set E_a(resp. $E(a, b)$) is called an upper set of a(resp. of a and b).

Proposition 4.1. *Let $(X, *, E)$ be an eGE-algebra and $a, b \in X$. Then*

(i) $a \in E_a$ *and* $a, b \in E(a, b)$.

(ii) $E_a \subseteq E(a, x)$, *for all* $x \in X$.

(iii) $E(a, b) = E(b, a)$.

(iv) $a \leq_e b \ \Rightarrow\ b \in E(a, c)$, *for all* $c \in X$.

(v) $b \in E \ \Rightarrow\ E(a, b) \subseteq E_a$.

(vi) $E_a = \bigcap_{x \in X} E(a, x)$.

Proof. (i) It is straightforward.

(ii) If $z \in E_a$ and $x \in X$, then $a * z \in E$ and so $x * (a * z) \in x * E \subseteq E$ by (eGE2). It follows from (14) that $a * (x * z) \in E$. Hence, $z \in E(a, x)$, and thus $E_a \subseteq E(a, x)$, for all $x \in X$.

(iii) it is straightforward by (14).

(iv) Assume $a \leq_e b$ and let $c \in X$. Then $a*b \in E$, and so $c*(a*b) \in c*E \subseteq E$. Hence, $b \in E(c, a) = E(a, c)$.

(v) Let $b \in E$. Then $b * x \in E * x = \{x\}$ by (eGE3), and so $b * x = x$, for all $x \in X$. If $y \in E(a, b)$, then $a * y = a * (b * y) \in E$, i.e., $y \in E_a$. Hence, $E(a, b) \subseteq E_a$.

(vi) We have $E_a \subseteq \bigcap_{x \in X} E(a, x)$ by (ii). If $y \in \bigcap_{x \in X} E(a, x)$, then $y \in E(a, x)$, i.e., $a * (x * y) \in E$, for all $x \in X$ and so $a * (b * y) \in E$ for $b \in E$. It follows from (eGE3) that $a * y = a * (b * y) \in E$, that is, $y \in E_a$. Hence, $\bigcap_{x \in X} E(a, x) \subseteq E_a$, and therefore (vi) is valid. $\square$

The following example shows that the set E_a may not be an eGE-filter of a weak eGE-algebra $(X, *, E)$.

Example 4.3. Let $X = \{0, a, b, c, d\}$ be a set with the Cayley table which is given in Table 14. Then $(X, *, E)$ with $E = \{a, b\}$ is a weak eGE-algebra which

Table 14: Cayley table for the binary operation "$*$"

$*$	0	a	b	c	d
0	a	a	a	a	a
a	0	a	b	c	d
b	0	a	b	c	d
c	0	a	0	a	d
d	b	a	b	c	b

is not eGE-algebra. We can observe that $E_d = \{0, a, b, d\} \subseteq E$. But E_d is not an eGE-filter of $(X, *, E)$. In fact, $0 * c = a \in E_d$ and $0 \in E_d$ but $c \notin E_d$.

We provide conditions for the set E_a to be an eGE-filter.

Theorem 4.4. *If a weak eGE-algebra $(X, *, E)$ satisfies:*

$$(\forall x, y, z \in X)(x * (y * z) = (x * y) * (x * z)), \tag{21}$$

then E_a is an eGE-filter of $(X, *, E)$, for all $a \in X$.

Proof. It is clear that E_a is a superset of E. Let $x, y \in X$ be such that $x \in E_a$ and $x * y \in E_a$. Then $a * x \in E$ and $(a * x) * (a * y) = a * (x * y) \in E$ by (21). Since E is an eGE-filter of X, it follows from (5) that $a * y \in E$, that is, $y \in E_a$. Therefore, E_a is an eGE-filter of X. $\square$

Corollary 4.1. *If an eGE-algebra* $(X, *, E)$ *satisfies* (21)*, then* E_a *is an eGE-filter of* $(X, *, E)$*, for all* $a \in X$*.*

The following example shows that there exist $a, b \in X$ such that the set $E(a, b)$ may not be an eGE-filter of a weak eGE-algebra $(X, *, E)$.

Example 4.4. Let $X = \{0, a, b, c, d, e\}$ be a set with the Cayley table which is given in Table 15. Then $(X, *, E)$ with $E = \{a, b\}$ is a weak eGE-algebra

Table 15: Cayley table for the binary operation "$*$"

$*$	0	a	b	c	d	e
0	a	a	a	a	a	a
a	0	a	b	c	d	e
b	0	a	b	c	d	e
c	0	a	b	b	a	e
d	0	b	b	c	b	b
e	b	d	b	c	d	b

which is not an eGE-algebra. Let $c, d \in X$. Then we can observe that $E(d, c) = \{a, b, d, e\}$ and $E \subseteq E(d, c)$. But $E(d, c)$ is not an eGE-filter of $(X, *, E)$ since $e * 0 = b \in E(d, c)$ and $e \in E(d, c)$ but $0 \notin E(d, c)$.

We provide a condition for the set $E(a, b)$ to be an eGE-filter, for all $a, b \in X$.

Theorem 4.5. *If a weak eGE-algebra* $(X, *, E)$ *satisfies* (21)*, then* $E(a, b)$ *is an eGE-filter of* $(X, *, E)$*, for all* $a, b \in X$*.*

Proof. Let $a, b \in X$. It is clear that $E(a, b)$ is a superset of E. Let $x, y \in X$ be such that $x \in E(a, b)$ and $x * y \in E(a, b)$. Then $a * (b * x) \in E$ and $a * (b * (x * y)) \in E$. Using (21), we have

$$(a * (b * x)) * (a * (b * y)) = a * ((b * x) * (b * y)) = a * (b * (x * y)) \in E.$$

Since E is an eGE-filter of $(X, *, E)$, it follows from (5) that $a * (b * y) \in E$, i.e., $y \in E(a, b)$. Therefore, $E(a, b)$ is an eGE-filter of $(X, *, E)$, for all $a, b \in X$. $\square$

Corollary 4.2. *If an eGE-algebra* $(X, *, E)$ *satisfies* (21)*, then* $E(a, b)$ *is an eGE-filter of* $(X, *, E)$*, for all* $a, b \in X$*.*

Theorem 4.6. *Let* F *be a nonempty subset of* X *in a weak eGE-algebra* $(X, *, E)$*. Then* F *is an eGE-filter of* $(X, *, E)$ *if and only if it satisfies:*

(22) $$(\forall a, b \in F)(E(a, b) \subseteq F).$$

Proof. Assume that F is an eGE-filter of $(X, *, E)$ and let $x \in E(a, b)$, for all $a, b \in F$. Then $a * (b * x) \in E \subseteq F$, and so $x \in F$ by (5). Hence, $E(a, b) \subseteq F$, for all $a, b \in F$.

Conversely, suppose F satisfies (22). Then F is a superset of E since $E \subseteq E(a,b) \subseteq F$, for all $a,b \in F$. Let $x,y \in X$ be such that $x \in F$ and $x * y \in F$. Since $(x*y)*(x*y) \in E$ by (eGE1), we have $y \in E(x*y,x) \subseteq F$. Consequently, F is an eGE-filter of $(X,*,E)$. $\square$

Corollary 4.3. *Let F be a nonempty subset of X in an eGE-algebra $(X,*,E)$. Then F is an eGE-filter of $(X,*,E)$ if and only if it satisfies* (22).

Proposition 4.2. *If F is an eGE-filter of a weak eGE-algebra $(X,*,E)$, then $F = \bigcup_{a,b\in F} E(a,b)$.*

Proof. Let $x \in F$. The combination of (eGE1) and (eGE3) induces $x*(y*x) \in E$, for all $y \in E$. Hence, $x \in E(x,y)$, and so

$$F \subseteq \bigcup_{x\in F, y\in E} E(x,y) \bigcup_{a,b\in F} E(a,b).$$

If $x \in \bigcup_{a,b\in F} E(a,b)$, then $x \in E(y,z)$ for some $y,z \in F$ and thus $x \in F$ by Theorem 4.6. This shows that $\bigcup_{a,b\in F} E(a,b) \subseteq F$, and we conclude that $F = \bigcup_{a,b\in F} E(a,b)$. $\square$

Corollary 4.4. *If F is an eGE-filter of an eGE-algebra $(X,*,E)$, then $F = \bigcup_{a,b\in F} E(a,b)$.*

5. Conclusion

We have introduced a broader concept than eGE algebra, called weak eGE algebra and its properties are investigated. We have also considered the concept of transitive and tightly (weak) eGE algebra and some properties are discussed. We have provided a way to create a GE-algebra using a weak eGE-algebra with additional conditions. We have introduced the notions of eGE-filter and upper set and associated properties are investigated. Conditions for a superset of E in a weak eGE-algebra $(X,*,E)$ to be an eGE-filter are provided. We have established the characterization of the eGE-filter.

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