A note on (I, J)-e-continuous and (I, J)-e^*-continuous functions

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Abstract. In this paper the notions of I-e-open set and I-e^*-open set are introduced and used to define a large number of modifications of the concept of continuous function, such as (I, J)-e-continuous functions, (I, J)-e^*-continuous functions, contra (I, J)-e-continuous functions, contra (I, J)-e^*-continuous functions, almost weakly (I, J)-e-continuous functions, almost weakly (I, J)-e^*-continuous functions, almost (I, J)-

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e-continuous functions, almost (I, J)-e^*-continuous functions, almost contra (I, J)-e-continuous functions and almost contra (I, J)-e^*-continuous functions. Also, several characterizations of these new classes of functions are given and finally relations between them are investigated.

Keywords: topological ideal, (I, J)-e-continuous functions, (I, J)-e^*-continuous functions, contra (I, J)-e-continuous functions, almost contra (I, J)-e^*-continuous functions.

1. Introduction

In 2008, E. Ekici [10] introduced a new class of generalized open sets in a topological space called e-open sets and, in 2009, [11] introduced a new generalization of open sets called e^*-open sets. Also, [5], [6], [7], [8], [9], [17] studied another generalized forms of open sets using e-open sets and e^*-open sets. Currently using the notion of topological ideal, different types of continuous functions have been introduced and studied. The concept of ideal topological spaces has been introduced and studied by Kuratowski [13] and the local function of a subset A of a topological space (X, τ) was introduced by Vaidyanathaswamy [16] as follows: given a topological space (X, τ) with an ideal I on X and $P(X)$ the set of all subsets of X, a set operator $(.)^* : P(X) \to P(X)$, defined for each $A \subseteq X$, as $A^*(\tau, I) = \{x \in X \ / \ U \cap A \notin I \text{ for every } U \in \tau_x\}$, where $\tau_x = \{U \in \tau : x \in U\}$, is called the local function of A with respect to τ and I. A Kuratowski closure operator $cl^*(\cdot)$ for a topology $\tau^*(\tau, I)$ called the $*$-topology, finer than τ, is defined by $cl^*(A) = A \cup A^*(\tau, I)$. We will denote $A^*(\tau, I)$ by A^* and $\tau^*(\tau, I)$ by τ^*. Note that when $I = \{\emptyset\}$, (respectively, $I = \mathcal{P}(X)$) $A^* = cl(A)$ (respectively, $A^* = \emptyset$). In 1990, Jankovic and Hamlett [12] introduced the notion of I-open set in a topological space (X, τ) with an ideal I on X. In 2018, Rosas et al. [15] introduced, studied and investigated the (I, J)-continuous functions and its relations with another functions and, in this same direction, Al-Omeri and Noiri (see [1], [2] [3]) introduced several modifications of continuity that have served as inspiration for other researchers to focus their attention on this topic. Motivated by this, we introduce the notions of I-e-open set and I-e^*-open set in an ideal topological spaces, and using these notions, we define and study a large number of modifications of the concept of continuous function, such as (I, J)-e-continuous functions, (I, J)-e^*-continuous functions, contra (I, J)-e-continuous functions, contra (I, J)-e^*-continuous functions, almost weakly (I, J)-e-continuous functions, almost weakly (I, J)-e^*-continuous functions, almost (I, J)-e-continuous functions, almost (I, J)-e^*-continuous functions, almost contra (I, J)-e-continuous functions and almost contra (I, J)-e^*-continuous functions. Finally, we give several characterizations of these new classes of functions and we investigate some relations between them.

2. Preliminaries

Throughout this paper, (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces in which no separation axioms are assumed, unless explicitly stated. If I is an ideal on X, (X, τ, I) mean an ideal topological space. For a subset A of (X, τ), $\mathrm{Cl}(A)$ and $int(A)$ denote the closure of A with respect to τ and the interior of A with respect to τ, respectively. A subset S of (X, τ, I) is an I-open [12], if $S \subseteq int(S^*)$. The complement of an I-open set is called I-closed set. In the case that $I = \{\emptyset\}$ (respectively, $I = P(X)$,) the I-open sets form the collection of all preopen sets of X (respectively, the only I-open set is $\emptyset$). The I-closure and the I-interior of a subset A of X, denoted by I-$\mathrm{Cl}(A)$ and I-$int(A)$, respectively, can be defined in the same way as $\mathrm{Cl}(A)$ and $int(A)$, respectively. The family of all I-open (resp. I-closed) subsets of a (X, τ, I), denoted by $IO(X)$(resp. $IC(X)$). We set $IO(X, x) = \{A : A \in IO(X) \text{ and } x \in A\}$. It is well known that in an ideal topological space (X, τ, I), the I-$\mathrm{Cl}(A)$ is an I-closed set and I-$int(A)$ is an I-open set and then, the following two results are immediate, using the notions of I-closure and I-interior.

Theorem 2.1. *Let (X, τ, I) be an ideal topological space, $A \subseteq X$ and $x \in X$. $x \in I\text{-}\mathrm{Cl}(A)$ if and only if $U \cap A \neq \emptyset$ for all $U \in IO(X, x)$.*

Theorem 2.2. *Let (X, τ, I) be an ideal topological space and A, B subsets of X. Then, the following conditions hold:*

1. $I\text{-}int(X \setminus A) = X \setminus I\text{-}\mathrm{Cl}(A)$.
2. $I\text{-}\mathrm{Cl}(X \setminus A) = X \setminus I\text{-}int(A)$.
3. $A \subseteq B \Rightarrow I\text{-}\mathrm{Cl}(A) \subseteq I\text{-}\mathrm{Cl}(B)$.
4. $A \subseteq B \Rightarrow I\text{-}int(A) \subseteq I\text{-}int(B)$.
5. $I\text{-}\mathrm{Cl}(A) \cup I\text{-}\mathrm{Cl}(B) \subseteq I\text{-}\mathrm{Cl}(A \cup B)$.
6. $I\text{-}int(A \cap B \subseteq I\text{-}int(A \cap B)$.

Definition 2.3 ([14]). *Let (X, τ, I) be an ideal topological space. $A \subseteq X$ is said to be:*

1. *I-regular open if $A = I\text{-}int(I\text{-}\mathrm{Cl}(A))$;*
2. *I-semiopen if $A \subseteq I\text{-}\mathrm{Cl}(I\text{-}int(A))$;*
3. *I-preopen if $A \subseteq I\text{-}int(I\text{-}\mathrm{Cl}(A))$.*

The class of I-regular open (resp.,I-semiopen, I-preopen) sets in X, is denoted by $IRO(X)$ (resp., $ISO(X)$, $IPO(X)$). The complement of an I-regular open set is called an I-regular closed set and the class of this sets is denoted by $IRC(X)$. The complement of an I-semiopen set is called an I-semiclosed set and the family of this sets is denoted by $ISC(X)$. Similarly, the complement of an I-preopen set is called an I-preclosed set and the class of this sets is denoted by $IPC(X)$.

Definition 2.4 ([18]). *Let (X, τ, I) be an ideal topological space and $A \subseteq X$:*

1. *A subset A of X is said to be I-δ-open if for each $x \in A$, there exists an I-regular open set G such that $x \in G \subseteq A$.*
2. *A point $x \in X$ is called an I-δ-cluster point of A if I-$int(I\,\mathrm{Cl}(U)) \cap A \neq \emptyset$ for every I-open set U of X containing x.*

The set of all I-δ-cluster point of A is called the I-δ-closure of A and is denoted by I-δ-$\mathrm{Cl}(A)$. If I-δ-$\mathrm{Cl}(A) = A$, A is said to be I-δ-closed. The set $\{x \in X : x \in U \subseteq A$ for some I-regular open set $U \subseteq X\}$ is called I-δ-interior of A and is denoted by I-δ-$int(A)$. A is I-δ-open if I-δ-$int(A) = A$.

Theorem 2.5 ([18]). *Let (X, τ, I) be an ideal topological space and A, B subsets of X. Then, the following conditions hold:*

1. *if $A \subseteq B$ then I-δ-$int(A) \subseteq I$-δ-$int(B)$.*
2. *if $A \subseteq B$ then I-δ-$\mathrm{Cl}(A) \subseteq I$-$\delta$-$\mathrm{Cl}(B)$.*
3. *I-δ-$int(X \setminus A) = X \setminus I$-$\delta$-$\mathrm{Cl}(A)$.*
4. *I-δ-$\mathrm{Cl}(X \setminus A) = X \setminus I$-$\delta$-$int(A)$.*
5. *I-δ-$int(A) \subseteq I$-$int(A) \subseteq I$-$\mathrm{Cl}(A) \subseteq I$-$\delta$-$\mathrm{Cl}(A)$.*

3. Contra (I, J)-e-continuous functions and contra (I, J)-e^*-continuous functions

In this section, we introduced and defined a notion of I-e-open sets and I-e^*-open sets in order to define and characterize a new notions of continuous functions.

Definition 3.1. *Let (X, τ, I) be an ideal topological space. $A \subseteq X$ is said to be:*

1. *I-e-open set if $A \subseteq I$-$\mathrm{Cl}(I$-δ-$int(A)) \cup I$-$int(I$-δ-$\mathrm{Cl}(A))$.*
2. *I-e*-open set if $A \subseteq I$-$\mathrm{Cl}(I$-$int(I$-δ-$\mathrm{Cl}(A)))$.*

The complement of an I-e-open (respectively, I-e^*-open) set is called an I-e-closed (respectively, I-e^*-closed) set. The family of all I-e-open (respectively, I-e^*-open) sets is denoted by I-$eO(X)$ (respectively, I-$e^*O(X)$).

Theorem 3.2. *Let (X, τ, I) be an ideal topological space. The following conditions hold:*

1. *the union of any collection of I-e-open sets is an I-e-open set.*
2. *the union of any collection of I-e*-open sets is an I-e*-open set.*

We define the I-e-closure (respectively, I-e-closure) of $A \subseteq X$ denoted by I-e-Cl(A) (respectively, I-e^*-Cl(A)) as the intersection of all I-e-closed (respectively, I-e^*-closed) sets containing A. From the above A is I-e-closed (respectively, I-e^*-closed) if $A = I$-e-Cl(A) (respectively, $A = I$-e^*-Cl(A)).

Theorem 3.3. *Let (X, τ, I) be an ideal topological space. The following conditions hold:*

1. *every $I - \delta$-open is an I-e-open set, but not conversely.*
2. *every I-e-open set is an I-e*-open set, but not conversely.*

Proof. The proof follows directly from the definition. □

Example 3.4. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $I = \{\emptyset\}$. Then, we obtain that:
$IO(X) = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$,
$IRO(X) = \{\emptyset, X, \{c\}, \{a, b\}\}$,
I-δ-open set $= \{\emptyset, X, \{c\}, \{a, b\}, \{a, b, c\}\}$,
I-$eO(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$,
I-$e^*O(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$.

Now, it is easy to see that: $\{a, c\}$ is an I-e-open but is not I-δ-open. In the same form $\{a, d\}$ is an I-e^*-open but is not I-e-open.

In a natural form, we define the (I, J)-e-continuous and (I, J)-e^*-continuous functions.

Definition 3.5. *A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be:*

1. *(I, J)-e-continuous if $f^{-1}(U)$ is I-e-open for every J-open set U in Y;*
2. *(I, J)-e*-continuous if $f^{-1}(U)$ is I-e*-open for every J-open set U in Y.*

The characterization of (I, J)-e-continuous are very similar as the characterization of (I, J)-continuous functions due in [15].

Definition 3.6. *A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be:*

1. *contra (I, J)-e-continuous if $f^{-1}(U)$ is I-e-closed for every J-open set U in Y;*
2. *contra (I, J)-e*-continuous if $f^{-1}(U)$ is I-e*-closed for every J-open set U in Y.*

Example 3.7. Let $X = Y = \{a, b, c, d\}$, $\tau = \sigma = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$ and $I = J = \{\emptyset\}$. Then, we obtain that:

$$IO(Y) = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}.$$

Define $f : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $f(a) = a$, $f(b) = b$, $f(c) = d$ and $f(d) = c$. It is easy to see that f is contra (I, J)-e-continuous.

Example 3.8. Let $X = Y = \{a,b,c,d\}$, $\tau = \sigma = \{\emptyset, X, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$ and $I = J = \{\emptyset\}$. Then, we obtain that:
$IO(Y) = \{\emptyset, X, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$, I-$e^*C(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$.

Define $f : (X,\tau,I) \to (Y,\sigma,J)$ as follows: $f(a) = a$, $f(b) = b$, $f(c) = d$ and $f(d) = c$. It is easy to see that f is contra (I,J)-e^*-continuous.

Definition 3.9. *Let (X,τ,I) be an ideal topological space and $A \subseteq X$. The I-kernel of A, denoted by I-$ker(A)$ is defined as the intersection of all I-open sets that contains A, that is I-$ker(A) = \cap\{U : U \in IO(X), A \subseteq U\}$.*

In a natural form as in topological spaces, we have the following result.

Theorem 3.10. *Let (X,τ,I) be an ideal topological space, $A \subseteq X$ and $x \in X$, then:*

1. *$x \in I$-$ker(A)$ if and only if $A \cap F \neq \emptyset$ for every I-closed set F containing x;*
2. *$A \subseteq I$-$ker(A)$ and $A = I$-$ker(A)$ if A is an I-open set.*

Using the above notion, we obtain the following characterizations of contra (I,J)-e-continuous and contra (I,J)-e^*-continuous functions.

Theorem 3.11. *For a function $f : (X,\tau,I) \to (Y,\sigma,J)$, the following conditions are equivalent:*

1. *f is contra (I,J)-e-continuous;*
2. *for each $x \in X$ and each J-closed set F of Y containing $f(x)$ there exists $U \in I$-$eO(X)$ such that $f(U) \subseteq F$;*
3. *for each J-closed subset F of Y, $f^{-1}(F)$ is an I-e-open set;*
4. *$f(I$-e-$\mathrm{Cl}(A)) \subseteq J$-$ker(f(A))$ for all $A \subseteq X$;*
5. *I-e-$\mathrm{Cl}(f^{-1}(B)) \subseteq f^{-1}(J$-$ker(B))$ for all $B \subseteq Y$.*

Proof. (1) $\Rightarrow$ (2). Let $x \in X$ and F any J-closed set of Y containing $f(x)$. Then, $Y \setminus F$ is a J-open set in Y and by hypothesis $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F)$ is I-e-closed in X, in consequence, $f^{-1}(F)$ is an I-e-open. Taking $U = f^{-1}(F)$, $x \in U$ and $f(U) \subseteq F$.

(2) $\Rightarrow$ (3). Let F be any J-closed subset of Y. Consider $x \in f^{-1}(F)$, then $f(x) \in F$. By hypothesis, there exists $U \in I$-$eO(X)$ such that $f(U) \subseteq F$. In consequence, $U \subseteq f^{-1}(f(U)) \subseteq f^{-1}(F)$. It follows that $f^{-1}(F) = \cup_{x\in f^{-1}(F)} U$ and then by Theorem 3.2, $f^{-1}(F)$ is an I-e-open set.

(3) $\Rightarrow$ (4). Consider that for some subset A of X, $y \in f(I$-e-$\mathrm{Cl}(A))$ but $y \notin J$-$ker(f(A))$. This implies that there exists a J-closed set F such that $y \in F$ and $f(A) \cap F = \emptyset$. It follows that $A \cap f^{-1}(F) = \emptyset$. Since F is a J-closed

set, by hypothesis, $f^{-1}(F)$ is an I-e-open set and then, I-e-$\mathrm{Cl}(A) \cap f^{-1}(F) = \emptyset$. Since $y \in f(I\text{-}e\text{-}\mathrm{Cl}(A)$, then $y = f(x)$ for some $x \in I$-e-$\mathrm{Cl}(A)$, since $f(x) \in F$, then $x \in f^{-1}(F)$ and hence $x \in I$-e-$\mathrm{Cl}(A) \cap f^{-1}(F)$, which is a contradiction.

(4) $\Rightarrow$ (5). Let B be any subset of Y. By hypothesis, $f(I\text{-}e\text{-}\mathrm{Cl}(f^{-1}(B))) \subseteq J\text{-}ker(B)$. Thus I-e-$\mathrm{Cl}(f^{-1}(B)) \subseteq f^{-1}(J\text{-}ker(B))$ for all $B \subseteq Y$.

(5) $\Rightarrow$ (1). Let V any J-open set of Y. By hypothesis, I-e-$\mathrm{Cl}(f^{-1}(V)) \subseteq f^{-1}(J\text{-}ker(V)) = f^{-1}(V)$. Follows that I-e-$\mathrm{Cl}(f^{-1}(V)) = f^{-1}(V)$. Hence $f^{-1}(V)$ is an I-e-closed set in X. □

Theorem 3.12. *For a function $f : (X, \tau, I) \to (Y, \sigma, J)$, the following conditions are equivalent:*

1. *f is contra (I, J)-e*-continuous;*
2. *for each J-closed subset F of Y, $f^{-1}(F)$ is an I-e*-open set;*
3. *for each $x \in X$ and each J-closed set F of Y containing $f(x)$ there exists $U \in I\text{-}e^*O(X)$ such that $f(U) \subseteq F$;*
4. *$f(I\text{-}e^*\text{-}\mathrm{Cl}(A)) \subseteq J\text{-}ker(f(A))$ for all $A \subseteq X$;*
5. *$I\text{-}e^*\text{-}\mathrm{Cl}(f^{-1}(B)) \subseteq f^{-1}(J\text{-}ker(B))$ for all $B \subseteq Y$.*

Proof. The proof is similar to that of Theorem 3.11. □

Definition 3.13. *A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be:*

1. *almost weakly (I, J)-e-continuous at a point $x \in X$, if for each J-open set V of Y containing $f(x)$, there exists an I-e-open set U containing x such that $f(U) \subseteq J\text{-}\mathrm{Cl}(V)$;*
2. *almost weakly (I, J)-e*-continuous at a point $x \in X$, if for each J-open set V of Y containing $f(x)$, there exists an I-e*-open set U containing x such that $f(U) \subseteq J\text{-}\mathrm{Cl}(V)$.*

If $f : (X, \tau, I) \to (Y, \sigma, J)$ is almost weakly (I, J)-e-continuous (respectively, almost weakly (I, J)-e*-continuous) at each point $x \in X$, then f is said to be almost weakly (I, J)-e-continuous (respectively, almost weakly (I, J)-e*-continuous).

Theorem 3.14. *For a function $f : (X, \tau, I) \to (Y, \sigma, J)$, the following conditions are satisfied:*

1. *If f is (I, J)-e-continuous function then f is almost weakly (I, J)-e-continuous;*
2. *If f is (I, J)-e*-continuous function then f is almost weakly (I, J)-e*-continuous.*

Proof. The proof is a consequence of the Definition 3.13 and the notion of J-closure of a set. □

The following examples shows that the converse of Theorem 3.14 are not necessarily true.

Example 3.15. As in Example 3.7. Then, we obtain that:
$IO(X) = JO(Y) = \{\emptyset, X, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$,
I-$eO(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$.

Define $f : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $f(a) = d$, $f(b) = a$, $f(c) = b$ and $f(d) = c$. It is easy to see that f is almost weakly (I, J)-e-continuous but is not (I, J)-e-continuous.

Example 3.16. Let $X = Y = \{a, b, c\}$, $\tau = \sigma = \{\emptyset, X, \{b, c\}\}$, $I = J = \{\emptyset, \{b\}\}$. Then, we obtain that:
$JO(Y) = \{\emptyset, Y, \{c\}, \{a,c\}, \{b,c\}\}$,
I-$e^*O(X) = \{\emptyset, X, \{a\}, \{c\}, \{a,c\}, \{b,c\}\}$.

Define $f : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $f(a) = c$, $f(b) = a$ and $f(c) = b$. It is easy to see that f is almost weakly (I, J)-e^*-continuous but is not (I, J)-e^*-continuous.

Definition 3.17. *A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be:*

1. *almost (I, J)-e-continuous if and only if for each $x \in X$ and each J-regular open set V of Y containing $f(x)$, there exists an I-e-open set U containing x such that $f(U) \subseteq V$.*

2. *almost (I, J)-e*-continuous if and only if for each $x \in X$ and each J-regular open set V of Y containing $f(x)$, there exists an I-e*-open set U containing x such that $f(U) \subseteq V$.*

Example 3.18. As in Example 3.7. Then, we obtain that:
$IO(Y) = \{\emptyset, Y, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$,
$IRO(Y) = \{\emptyset, Y, \{c\}, \{a,b\}\}$,
I-$eO(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$,
I-$e^*O(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$.

Define $f : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $f(a) = a$, $f(b) = b$, $f(c) = d$ and $f(d) = c$. It is easy to see that f is not almost (I, J)-e-continuous.

Example 3.19. As in Example 3.7. Then, we obtain that:
$IO(Y) = \{\emptyset, Y, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$,
$IRO(Y) = \{\emptyset, Y, \{c\}, \{a,b\}\}$,
I-$eO(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$,

I-$e^*O(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$.

Define $f : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $f(a) = d$, $f(b) = c$, $f(c) = b$ and $f(d) = a$. It is easy to see that f is almost (I, J)-e-continuous but is not (I, J)-e-continuous function.

Example 3.20. As in Example 3.15, f is almost weakly (I, J)-e-continuous but is not almost (I, J)-e-continuous.

4. Almost contra (I, J)-e-continuous functions and almost contra (I, J)-e^*-continuous functions

In this section, we introduced and defined the notions of almost contra (I, J)-e-continuous and almost contra (I, J)-e^*-continuous functions in order to characterize it and find its relations with another notions of continuous functions.

Definition 4.1. *A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be:*

1. *almost contra (I, J)-e-continuous if $f^{-1}(V)$ is I-e-closed for every J-regular open set V of Y,*
2. *almost contra (I, J)-e^*-continuous if $f^{-1}(V)$ is I-e^*-closed for every J-open set V in Y.*

Example 4.2. As in Example 3.7. Then, we obtain that:
$IO(X) = JO(Y) = \{\emptyset, X, \{a\}, \{c\}, \{a, b\}, \{a, c\}, \{a, b, c\}, \{a, c, d\}\}$,
$IRO(X) = \{\emptyset, X, \{c\}, \{a, b\}\}$,
I-$eC(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, d\}, \{c, d\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}\}$.

Define $f : (X, \tau, I) \to (Y, \sigma, J)$ as follows: $f(a) = d$, $f(b) = a$, $f(c) = b$ and $f(d) = c$. It is easy to see that f is almost contra (I, J)-e-continuous.

Theorem 4.3. *For a function $f : (X, \tau, I) \to (Y, \sigma, J)$, the following conditions are equivalent:*

1. *f is almost contra (I, J)-e-continuous;*
2. *$f^{-1}(F)$ is I-e-open for every J-regular closed set F of Y;*
3. *for each $x \in X$ and each J-regular open set F of Y containing $f(x)$ there exists $U \in I$-$eO(X, x)$ such that $f(U) \subseteq F$;*
4. *for each $x \in X$ and each J-regular open set V of Y containing $f(x)$ there exists a I-e-closed set K containing x such that $f^{-1}(V) \subseteq K$.*

Proof. (1) $\Rightarrow$ (2). Let F be any J-regular closed set of Y. Then, $Y \setminus F$ is a J-regular open set of Y, and $f^{-1}(Y \setminus F) = X \setminus f^{-1}(F) \in I$-$eC(X)$. Therefore, $f^{-1}(F) \in I$-$eO(X)$.

(2) $\Rightarrow$ (3). Let F be any J-regular closed set of Y and $x \in X$ such that $f(x) \in F$. Then, $f^{-1}(F) \in I\text{-}eO(X,x)$, $x \in f^{-1}(F)$. Therefore, take $U = f^{-1}(F)$, $f(U) \subseteq F$.

(3) $\Rightarrow$ (4) Let V any J-regular open set of Y such that $f(x) \notin V$, then $f(x) \in Y\setminus V$ and $Y\setminus V$ is a J-regular closed set of Y. By hypothesis, there exists $U \in I\text{-}eO(X,x)$ such that $f(U) \subseteq Y\setminus V$, therefore $U \subseteq f^{-1}(Y\setminus V) \subseteq X\setminus f^{-1}(V)$. Follows $f^{-1}(U) \subseteq X \setminus U$, $X \setminus U$ is an I-e-closed set and $x \notin X \setminus U$.

(4) $\Rightarrow$ (1). Straighforward. □

Theorem 4.4. *For a function $f : (X,\tau,I) \to (Y,\sigma,J)$, the following conditions are equivalent:*

1. *f is almost contra (I,J)-e^*-continuous;*
2. *$f^{-1}(F)$ is I-e^*-open for every J-regular closed set F of Y;*
3. *for each $x \in X$ and each J-regular open set F of Y containing $f(x)$ there exists $U \in I\text{-}e^*O(X,x)$ such that $f(U) \subseteq F$;*
4. *for each $x \in X$ and each J-regular open set F of Y containing $f(x)$ there exists a I-e^*-closed set K containing x such that $f^{-1}(V) \subseteq K$.*

Proof. The proof is similar to that of Theorem 4.3. □

Theorem 4.5. *If $f : (X,\tau,I) \to (Y,\sigma,J)$ is almost contra (I,J)-e-continuous then f is almost contra (I,J)-e^*-continuous.*

Proof. The proof follows from the fact that every I-e-open set is an I-e^*-open set. □

The following example, shows that there exists an almost contra (I,J)-e^*-continuous that is not almost contra (I,J)-e-continuous.

Example 4.6. As in Example 4.2. Then, we obtain that:
$IO(X) = JO(Y) = \{\emptyset, X, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$,
$IRC(Y) = \{\emptyset, Y, \{c,d\}, \{a,b,d\}\}$,
$I\text{-}eO(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$,
$I\text{-}e^*O(X) = \{\emptyset, X, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{a,d\}, \{b,c\}, \{b,d\}, \{c,d\}, \{a,b,c\}, \{a,b,d\}, \{a,c,d\}, \{b,c,d\}\}$.

Define $f : (X,\tau,I) \to (Y,\sigma,J)$ as follows: $f(a) = c, f(b) = a, f(c) = b, f(d) = d$. It is easy to see that f is almost contra (I,J)-e^*-continuous but is not almost contra (I,J)-e-continuous.

Definition 4.7. *An ideal topological space (X,τ,I) is said to be I-extremally disconnected if $I\text{-}\mathrm{Cl}(U) \in IO(X)$ for each $U \in IO(X)$.*

Example 4.8. Let $X = \{a, b, c, d\}$, $\tau = \mathcal{P}(X)$, $I = \{\emptyset\}$. Follows that: $IO(X) = \mathcal{P}(X)$, the I-Cl$(U) = U$ for all $U \in IO(X)$. It follows that (X, τ, I) is extremally disconnected.

Theorem 4.9. *Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be a function, where Y is J-extremally disconnected. The following conditions hold:*

1. *f is almost contra (I, J)-e-continuous if and only if f is almost (I, J)-e-continuous,*
2. *f is almost contra (I, J)-e^*-continuous if and only if f is almost (I, J)-e^*-continuous.*

Proof. (1). Suppose that $x \in X$ and V is any J-regular open set of Y containing $f(x)$. Since Y is J-extremally disconnected, J-Cl$(V) \in JO(Y)$ and then V is J-regular closed, follows that V is J-clopen. Using Theorem 4.3, there exists $U \in IO(X, x)$ such that $f(U) \subseteq V$ and then f is almost (I, J)-e-continuous. Conversely, Let f be almost (I, J)-e-continuous function and W be any J-regular closed set of Y. By hypothesis, Y is J-extremally disconnected, then W is J-regular open, therefore $f^{-1}(W)$ is an I-e-open set of X. Take $U = f^{-1}(W)$ and obtain that $f(U) \subseteq W$.

(2). The proof is similar to that of part (1). □

Definition 4.10. *An ideal topological space (X, τ, I) is said to be I-e^*-$T_{1/2}$ if each I-e^*-closed set is I-δ-closed.*

Example 4.11. Let $X = \{a, b, c, d\}$, $\tau = \mathcal{P}(X)$, $I = \{\emptyset\}$. Then $IO(X) = \mathcal{P}(X)$, $IRO(X) = \mathcal{P}(X)$, I-δ-open sets$=\mathcal{P}(X)$, I-$e^*O(X) = \mathcal{P}(X)$.

It follows that (X, τ, I) is an I-e^*-$T_{1/2}$.

Theorem 4.12. *Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be a function and (X, τ, I) an I-e^*-$T_{1/2}$. Then, f is almost (I, J)-e-continuous if and only if f is almost (I, J)-e^*-continuous.*

Proof. By Theorem 3.10, each I-e-open set is I-e^*-open and then each almost (I, J)-e-continuous f is almost (I, J)-e^*-continuous. Conversely, since (X, τ, I) is an I-e^*-$T_{1/2}$-space, each I-e^*-open set is I-δ-open. By Theorem 3.10, each I-δ-open set is I-e-open and, then each almost (I, J)-e^*-continuous f is almost (I, J)-e-continuous. □

Theorem 4.13. *Let $f : (X, \tau, I) \to (Y, \sigma, J)$ be a function and (X, τ, I) an I-e^*-$T_{1/2}$-space. Then, f is almost contra (I, J)-e-continuous if and only if f is almost contra (I, J)-e^*-continuous.*

Proof. The proof is similar to that of Theorem 4.12. □

Definition 4.14. *A function $f : (X, \tau, I) \to (Y, \sigma, J)$ is said to be:*

1. *(I, J)-e-irresolute if $f^{-1}(V)$ is I-e-open for every J-e-open set V of Y;*

2. (I,J)-e^**-irresolute if* $f^{-1}(V)$ *is* I-e^**-open for every* J-e^**-open set* V *of* Y.

Theorem 4.15. 1. *If* $f:(X,\tau,I)\to(Y,\sigma,J)$ *is an* (I,J)*-e-irresolute function and* $g:(Y,\sigma,J)\to(Z,\beta,L)$ *is an* (J,L)*-e-irresolute function, then* $g\circ f:(X,\tau,I)\to(Z,\beta,L)$ *is an* (I,L)*-e-irresolute function;*

2. *If* $f:(X,\tau,I)\to(Y,\sigma,J)$ *is an* (I,J)-e^**-irresolute function and* $g:(Y,\sigma,J)\to(Z,\beta,L)$ *is an* (J,L)-e^**-irresolute function, then* $g\circ f:(X,\tau,I)\to(Z,\beta,L)$ *is an* (I,L)-e^**-irresolute function;*

3. *If* $f:(X,\tau,I)\to(Y,\sigma,J)$ *is an* (I,J)*-e-irresolute function and* $g:(Y,\sigma,J)\to(Z,\beta,L)$ *is an* (J,L)*-e-continuous function, then* $g\circ f:(X,\tau,I)\to(Z,\beta,L)$ *is an* (I,L)*-e- continuous function;*

4. *If* $f:(X,\tau,I)\to(Y,\sigma,J)$ *is an* (I,J)-e^**-irresolute function and* $g:(Y,\sigma,J)\to(Z,\beta,L)$ *is an* (J,L)-e^**-continuous function, then* $g\circ f:(X,\tau,I)\to(Z,\beta,L)$ *is an* (I,L)-e^**- continuous function.*

5. Conclusion

In this work, in the theoretical framework of an ideal topological space, we have introduced the notions of I-e-open set and I-e^*-open set. By using these notions we have defined the (I,J)-e-continuous functions, (I,J)-e^*-continuous functions, contra (I,J)-e-continuous functions, contra (I,J)-e^*-continuous functions, almost weakly (I,J)-e-continuous functions, almost weakly (I,J)-e^*-continuous functions, almost (I,J)-e-continuous functions, almost (I,J)-e^*-continuous functions, almost contra (I,J)-e-continuous functions and almost contra (I,J)-e^*-continuous functions. Also, we gave various characterizations of these new classes of functions and have obtained some relationships between them. For future research, notions similar to those studied in the reference [4] can be investigated, such as defining an I-semi*-open (respectively, I-pre *-open) set A to the one that satisfies the inclusion $A\subseteq I\text{-Cl}(I\text{-}\delta\text{-}int(A))$ (respectively, $A\subseteq I\text{-}int(I\text{-}\delta\text{-Cl}(A)))$.

In consequence, these notions can be applied in the study of new modifications of continuous functions that are similar to those presented in [4].

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