On finite groups with $s$-weakly normal subgroups

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Abstract. A subgroup $H$ of a group $G$ is weakly normal in $G$ if $H^g \leq N_G(H)$ implies that $g \in N_G(H)$ for any element $g \in G$. A subgroup $H$ of a group $G$ is $s$-weakly normal in $G$ if there exists a normal subgroup $T$ such that $G = HT$ and $H \cap T$ is weakly normal in $G$. Clearly a weakly normal subgroup of $G$ is an $s$-weakly normal subgroup of $G$. In this paper, we investigate the influence of $s$-weakly normal subgroups on the structure of a finite group, especially some criteria for supersolvability, nilpotency, formation and hypercenter of a finite group are proved. Based on our results, some recent results can be generalized easily.

Keywords: finite group, weakly normal subgroup, $s$-weakly normal subgroup, supersolvable group, nilpotent group.

1. Introduction

The groups which appear throughout this paper are assumed to be finite groups and $G$ always denotes a finite group. Let’s first introduce some frequently used notations and terminologies, any unexplained terms can be found in [9, 11, 12, 20].

Let $|G|$ be the order of $G$ and $\pi(G)$ be the set of all prime divisors of $|G|$. For a $p$-group $P$, where $p$ is a prime, we write $\Omega_1(P) = \langle x \in P | x^p = 1 \rangle$ and $\Omega_2(P) = \langle x \in P | x^{p^2} = 1 \rangle$; we say a $p'$-group $H$ means a group $H$ satisfying $p \nmid |H|$. Let $\mathfrak{F}$ be a class of groups. Recall that $\mathfrak{F}$ is said to be a formation if $\mathfrak{F}$ is closed under taking homomorphic image and finite subdirect product, that is, for each group $G$ and a normal subgroup $N$ of $G$, $G \in \mathfrak{F}$ implies that $G/N \in \mathfrak{F}$, moreover, if $M \unlhd G$, then $G/N \in \mathfrak{F}$ and $G/M \in \mathfrak{F}$ imply $G/(N \cap M) \in \mathfrak{F}$. A

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formation $\mathcal{F}$ is said to be saturated if $G \in \mathcal{F}$ whenever $G/\Phi(G) \in \mathcal{F}$, where $\Phi(G)$ is the intersection of all maximal subgroups of $G$.

We denote by $\mathcal{U}$ the class of all supersolvable groups and by $\mathcal{N}$ the class of all nilpotent groups. It is known that $\mathcal{U}$ and $\mathcal{N}$ are both saturated formations (see [12]). We denote by $Z_F(G)$ the product of all $F$-hypercentral subgroups of $G$. In particular, $Z_U(G) = Z_\infty(G)$ is the hypercenter of $G$.

It is known that a subgroup $H$ of a group $G$ is pronormal in $G$ if the subgroups $H$ and $H^g$ are conjugate in $\langle H, H^g \rangle$ for each element $g$ of $G$. This concept was introduced by P. Hall [13] and the first general results about pronormality appeared in a paper by J. S. Rose [21]. A subgroup $H$ of $G$ is c-normal in $G$ if there is a normal subgroup $N$ of $G$ such that $G = HN$ and $H \cap N \leq H_G = \text{Core}_G(H)$, see for example [22]. In [7], the authors introduced the concept of $\mathcal{H}$-subgroup of a group and proved a number of interesting results about such subgroups. A subgroup $H$ of a group $G$ is called an $\mathcal{H}$-subgroup provided that $H^g \cap N_G(H) \leq H$ for all $g \in G$. It is easy to see that the Sylow $p$-subgroups, normal subgroups and self-normalizing subgroups of an arbitrary group are $\mathcal{H}$-subgroups. Following M"{u}ller [18], a subgroup $H$ of a group $G$ is weakly normal in $G$ if $H^g \leq N_G(H)$ implies that $g \in N_G(H)$ for any element $g \in G$. It is known that every pronormal subgroup and $\mathcal{H}$-subgroup of $G$ are weakly normal in $G$, but the converse is not true (see [1, p.28] and [6]) for more details. In [17], the authors investigated the behaviour of weakly normal subgroups, and obtained some characterizations about the supersolvability and nilpotency of $G$ by assuming that some subgroups of prime power order of $G$ are weakly normal in $G$. Recently, the authors in [24] gave some results about formation under the condition that some subgroups of prime square order are weakly normal in $G$.

It is known that there is no obvious general relationship between the concepts of c-normal subgroup and $\mathcal{H}$-subgroup. For a generalization of both $\mathcal{H}$-subgroup and c-normal subgroup, Assad, et al.[2] introduced the concept of weakly $\mathcal{H}$-subgroup, which describes subgroup embedding properties of a finite group. A subgroup $H$ of a group $G$ is called a weakly $\mathcal{H}$-subgroup in $G$ if there exists a normal subgroup $K$ of $G$ such that $G = HK$ and $H \cap K$ is a $\mathcal{H}$-subgroup of $G$. The authors in [2] determined the structure of a finite group $G$ when all maximal subgroups of every Sylow subgroup of certain subgroups of $G$ are weakly $\mathcal{H}$-subgroups in $G$.

Inspired by the above works, we consider the following question:

How is the structure of a finite group $G$ determined by its subgroup $H$ with the property that there exists a normal subgroup $T$ of $G$ such that $HT = G$ and $H \cap T$ is weakly normal in $G$?

We first introduce a new notion of s-weakly normal subgroup which is a generalization of c-normal subgroup, $\mathcal{H}$-subgroup, and weakly normal subgroup.
Definition 1.1. A subgroup \( H \) of a group \( G \) is an \( s \)-weakly normal subgroup of \( G \) if there exists a normal subgroup \( T \) of \( G \) such that \( G = HT \) and \( H \cap T \) is weakly normal in \( G \).

Clearly a \( c \)-normal subgroup, \( H \)-subgroup or weakly normal subgroup of \( G \) is an \( s \)-weakly normal subgroup of \( G \), but the converse is not true. The following two examples will show this.

Example 1.1. Let \( G = S_4 \) be the symmetric group of degree 4. Suppose that \( H = \langle (12) \rangle \) and \( A_4 \) is the alternating group of degree 4. Since \( A_4 \) is a normal subgroup of \( G \) such that \( G = HA_4 \) and \( H \cap A_4 = \langle (1) \rangle \) is weakly normal in \( G \), \( H \) is \( s \)-weakly normal in \( G \). It is easy to see that \( N_G(H) = \{(1), (12), (34), (12)(34)\} \) and \( H^g \cap N_G(H) = \{(1), (34)\} \not\leq H \), \( H \) is not an \( H \)-subgroup of \( G \). Also note that \( H^g \leq N_G(H) \), but \( g = (13)(24) \not\in N_G(H) \). It follows that \( H \) is not weakly normal in \( G \).

Example 1.2. Let \( G = S_4 \), \( H = \{(1), (12), (13), (23), (123), (132)\} \). It is easy to check that \( N_G(H) = H \), this means that \( H \) is an \( H \)-subgroup of \( G \) and so it is weakly normal in \( G \). Hence \( H \) is \( s \)-weakly normal in \( G \). But \( H \) is not \( c \)-normal in \( G \). In fact, since \( H_G = \{(1)\} \), there is no such a normal subgroup \( T \) of \( G \) such that \( G = HT \) and \( H \cap T \leq H_G \).

The aim of this paper is to obtain some new characterizations of the nilpotency and supersolvability of finite groups by studying the \( s \)-weakly normality properties of some certain primary subgroups. In Section 2, some necessary lemmas are given. In Section 3, some criteria for supersolvability, nilpotency, formation and hypercenter of a finite group are proved, based on these criteria, some recent results can be improved and extended easily. These results show that the concept of \( s \)-weakly normal subgroup provides us a useful tool to investigate the structure of finite groups.

2. Preliminaries

In this section we list some basic facts which are needed in this paper.

Lemma 2.1 ([17], Lemma 2.1, Lemma 2.2). Let \( N, H \) and \( K \) be subgroups of a finite group \( G \). Then:

1. If \( H \) is weakly normal in \( G \), and \( H \leq K \leq G \), then \( H \) is weakly normal in \( K \).

2. Let \( N \trianglelefteq G \) and \( N \leq H \). Then \( H \) is weakly normal in \( G \) if and only if \( H/N \) is weakly normal in \( G/N \).

3. If \( H \) is weakly normal in \( G \) and \( H \leq K \leq G \), then \( H \leq K \).

4. If \( N \leq G \), \( P \) is a weakly normal \( p \)-subgroup of \( G \) such that \( ([N], p) = 1 \), then \( PN \) is weakly normal in \( G \) and \( PN/N \) is weakly normal in \( G/N \).
Lemma 2.2. Let $G$ be a finite group and $N, H, K$ be subgroups of group $G$.

(1) If $H$ is s-weakly normal in $G$, and $H \leq K \leq G$, then $H$ is s-weakly normal in $K$.

(2) Let $N \leq H$ and $N \trianglelefteq G$. Then $H$ is s-weakly normal in $G$ if and only if $H/N$ is s-weakly normal in $G/N$.

(3) Let $N$ be a normal subgroup of $G$. If $H$ is a p-subgroup of $G$ such that $([N], |H|) = 1$ and $H$ is s-weakly normal in $G$, then $HN/N$ is s-weakly normal in $G/N$.

Proof. (1) Since $H$ is an s-weakly normal subgroup of $G$, there exists a normal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T$ is weakly normal in $G$. It is easy to see that $K = K \cap HT = H(K \cap T)$ and $H \cap (K \cap T) = H \cap T$ is weakly normal in $G$. It follows by Lemma 2.1(1) that $H \cap (K \cap T) = H \cap T$ is weakly normal in $K$. Hence $H$ is s-weakly normal in $K$.

(2) Assume that $H$ is s-weakly normal in $G$, that is, there exists a normal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T$ is weakly normal in $G$. Since $N \trianglelefteq G$, it is clear that $G/N = (H/N)(TN/N)$ and $TN/N \trianglelefteq G/N$. Then it follows from Lemma 2.1(2) that $(H/N) \cap (TN/N) = N(H \cap T)/N$ is weakly normal in $G/N$. Hence $H/N$ is s-weakly normal in $G/N$.

Conversely, suppose that $H/N$ is s-weakly normal in $G/N$. Then there exists a normal subgroup $T/N$ of $G/N$ such that $G/N = (H/N)(T/N)$ and $(H/N) \cap (T/N) = (H \cap T)/N$ is weakly normal in $G/N$. It is easy to see that $G = HT$ and $H \cap T$ is weakly normal in $G$ by Lemma 2.1(2), that is, $H$ is s-weakly normal in $G$.

(3) Suppose that $H$ is s-weakly normal in $G$. Then there exists a normal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T$ is weakly normal in $G$. Note that $([N], |H|) = 1$, we have $N \leq T$, and hence $HN \cap T = N(H \cap T)$. It is easy to see that $HN \cap T = N(H \cap T)$ is weakly normal in $G$. By Lemma 2.1(4), $N(H \cap T)/N$ is weakly normal in $G/N$. Note that $G/N = (HN/N)(T/N)$ and $(HN/N) \cap (T/N) = (HN \cap T)/N = N(H \cap T)/N$ is weakly normal in $G/N$. Hence $HN/N$ is s-weakly normal in $G/N$.

Lemma 2.3 ([14], Satz 5.4, p.434). Let $G$ be a group and $p \in \pi(G)$, If $G$ is a minimal non-$p$-nilpotent group, that is, $G$ is not nilpotent but all of its proper subgroups are $p$-nilpotent, then

(i) $G = PQ$, where $P$ is a normal Sylow $p$-subgroup of $G$ and $Q$ is a non-normal cyclic Sylow $q$-subgroup of $G$.

(ii) $P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.

(iii) If $p > 2$, then $\exp(P)$ is $p$, and when $p = 2$, $\exp(P)$ is at most 4, where $\exp(P)$ is the exponent of group $P$.

Lemma 2.4 ([23], Theorem 6.3, p.221 and Corollary 7.8, p.33). Let $P$ be a normal $p$-subgroup of a group $G$ such that $|G/C_G(P)|$ is a power of prime $p$. Then $P \leq Z_d(G)$.
Lemma 2.5 ([9], Theorem 6.10, p.390). If a class of groups $\mathfrak{F}$ is a saturated formation, then $[G^{\mathfrak{F}}, Z_{\mathfrak{F}}(G)] = 1$.

Lemma 2.6. Let $H$ be an $s$-weakly normal subgroup of $G$ and $K$ be a subgroup of $G$ such that $H \leq K$. If $K/\Phi(K)$ is a chief factor of $G$, then $H$ is weakly normal in $G$.

Proof. Since $H$ is $s$-weakly normal in $G$, there is a normal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T$ is weakly normal in $G$. Note that $K/\Phi(K)$ is a chief factor of $G$. Thus either $(K \cap T)\Phi(K)/\Phi(K) = 1$ or $(K \cap T)\Phi(K)/\Phi(K) = K/\Phi(K)$. In the former case, since $K \cap T \leq \Phi(K)$, we have $K = K \cap HT = H(K \cap T) = H$. This implies that $H \leq G$, and clearly, $H$ is weakly normal in $G$. In the latter case, we have $(K \cap T)\Phi(K) = K$, and hence $K \cap T = K$ and $G = T$. This also implies that $H$ is weakly normal in $G$. \qed

Let $G$ be a finite group. It is known that the Fitting subgroup $F(G)$ of $G$ is the unique maximal normal nilpotent subgroup of $G$, and the generalized Fitting subgroup $F^*(G)$ of $G$ is the unique maximal normal quasinilpotent subgroup of $G$. The following results about $F^*(G)$ and $F(G)$ are useful in our paper.

Lemma 2.7 ([15], Chapter X 13). Let $G$ be a group.

1. Suppose that $F^*(G)$ is solvable. Then $F^*(G) = F(G)$;
2. $C(F^*(G)) \leq F(G)$;
3. If $N$ is a normal subgroup of $G$, then $F^*(N) = N \cap F^*(G)$.

Lemma 2.8 ([6], Lemma 2). Let $H$ be a $p$-subgroup of $G$. Then the following properties are equivalent:

1. $H$ is a pronormal subgroup of $G$;
2. $H$ is a weakly normal subgroup of $G$.

Lemma 2.9 ([4], Theorem 4.1). Let $p$ be the smallest prime of $\pi(G)$ and $P$ a Sylow $p$-subgroup of $G$. If every subgroup of $P$ of order $p$ or $4$ (when $p = 2$) is pronormal in $G$, then $G$ is $p$-nilpotent.

From Lemma 2.8 and Lemma 2.9, we immediately get the following result.

Lemma 2.10. Let $G$ be a group and $p$ be the smallest prime of $\pi(G)$. If $P$ is a Sylow $p$-subgroup of $G$ and every subgroup of $P$ of order $p$ or $4$ (when $p = 2$) is weakly normal in $G$, then $G$ is $p$-nilpotent.

Lemma 2.11 ([16], Lemma 2.8). Let $P$ be a normal $p$-subgroup of $G$ contained in $Z_\infty(G)$. Then $O^p(G) \leq C_G(P)$.

Lemma 2.12 ([10], Lemma 2.4). Let $P$ be a $p$-group. If $\alpha$ is a $p'$-automorphism of $P$ which centralizes $\Omega_1(P)$, then $\alpha = 1$ unless $P$ is a non-abelian $2$-group. If $[\alpha, \Omega_2(P)] = 1$, then $\alpha = 1$ without restriction.
3. Main results

In the sequel, we discuss the influence of s-weakly normal subgroups on the structure of a group.

**Theorem 3.1.** Let $G$ be a group with a Sylow $p$-subgroup $P$, where $p$ is the smallest prime in $\pi(G)$. Suppose that every subgroup of $P$ of order $p$ or 4 (when $p = 2$) is s-weakly normal in $G$. Then $G$ is $p$-nilpotent.

**Proof.** Suppose that the required result is not true and let $G$ be a counterexample of minimal order.

Firstly, suppose that $p$ is an odd prime. If every subgroup of $P$ of order $p$ is weakly normal in $G$, then by Lemma 2.10, $G$ is $p$-nilpotent, a contradiction. Hence there exists a subgroup $P_1$ of $P$ such that $|P_1| = p$ and $P_1$ is not weakly normal in $G$. By the hypotheses of the theorem, $P_1$ is $s$-weakly normal in $G$, i.e. there is a normal subgroup $T$ of $G$ such that $G = P_1T$ and $P_1 \cap T$ is weakly normal in $G$. If $P_1 \cap T \neq 1$, then $P_1 \cap T = P_1$ is weakly normal in $G$, a contradiction. Thus $P_1 \cap T = 1$, and therefore $T$ is a proper subgroup of $G$. By Lemma 2.2(1), $T$ satisfies the hypotheses of the theorem. Then $T$ is $p$-nilpotent by the minimal choice of $G$. Let $T_p'$ be a normal $p$-complement of $G$. Clearly, $T_p' \lhd G$. Note that $T$ is normal in $G$, and therefore $T_p' \lhd G$. This means that $G$ is $p$-nilpotent, which is a contradiction.

Now suppose that $p = 2$. Since $G$ is not 2-nilpotent, it follows that $G$ contains a minimal non-2-nilpotent subgroup $K$. Then $K$ is a minimal non-nilpotent subgroup of $G$ and $K = K_2 \rtimes K_q$, where $K_2$ is a normal Sylow 2-subgroup of $K$ and $K_q$ is a non-normal Sylow $q$-subgroup of $K$, where $q > 2$, and $exp(K_2)$ is at most 4. By using Lemma 2.1(1), we can easily see that the hypothesis is inherited by $K$. Then by Lemma 2.10, $K_2$ contains a subgroup $L$ of order 2 or 4 such that $L$ is not weakly normal in $K$. By the hypotheses of the theorem, $L$ is $s$-weakly normal in $G$ and thereby $L$ is $s$-weakly normal in $K$ by Lemma 2.2(1), that is, there is a normal subgroup $T$ of $K$ such that $K = LT$ and $L \cap T$ is weakly normal in $K$. If $T = K$, then $L \cap T = L$ is weakly normal in $K$, a contradiction. Thus $T$ is a proper subgroup of $K$. If $|L| = 2$, then $L \cap T = 1$. Since $T$ is a normal nilpotent subgroup of $K$, $K_q \lhd T \lhd K$, and hence $K_q \lhd K$, a contradiction. If $|L| = 4$, then we can always conclude that $K_q \lhd K$ when $|L \cap T| = 1$ or $|L \cap T| = 2$ since $T$ is a normal nilpotent subgroup of $K$, a contradiction. This completes the proof of the theorem.

**Theorem 3.2.** Let $G$ be a group with a normal $p$-subgroup $P$, where $p \in \pi(G)$. Suppose that every subgroup of $P$ of order $p$ or of order 4 (when $p = 2$) is $s$-weakly normal in $G$. Then we have $P \leq Z_2(G)$.

**Proof.** We proceed the proof of the theorem by induction on $|G| + |P|$ and distinguish the following two cases.

Case (1): $p$ is an odd prime.
If every subgroup of $P$ of order $p$ is normal in $G$, then it is easy to see from [5, Theorem 1.1] that $P \leq Z_4(G)$. Now we assume that there exists a subgroup $H$ of order $p$ in $P$ such that $H$ is not normal in $G$. By the hypothesis of the theorem, $H$ is $s$-weakly normal in $G$, that is, there exists a normal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T$ is weakly normal in $G$. Assume that $H \cap T \neq 1$. Then $H \cap T = H$ is weakly normal in $G$. It is easy to know that $H$ is subnormal in $G$, then we have that $H \leq G$ by Lemma 2.1(3), which is a contradiction. Therefore $H \cap T = 1$. Note that $P \cap T \leq H$. By hypothesis of the theorem, every subgroup of $P \cap T$ of order $p$ is $s$-weakly normal in $G$. Hence, by induction on $|G| + |P|$, we have that $P \cap T \leq Z_4(G)$. Since $P = H(P \cap T)$, we have $P/(P \cap T) = H(P \cap T)/(P \cap T)$ is a normal subgroup of $G/(P \cap T)$ of order $p$. Consequently, $P/(P \cap T) \leq Z_4(G/(P \cap T))$. Note that $P \cap T \leq Z_4(G)$, and then by [23, Theorem 7.7, p.32], we have

$$Z_4(G/(P \cap T)) = Z_4(G)/(P \cap T).$$

Therefore, $P \leq Z_4(G)$.


Let $Q$ be any Sylow $q$-subgroup of $G$, where $q \neq 2$. Then it is clear that $PQ$ is a subgroup of $G$. By Lemma 2.2(1) and Theorem 3.1, $PQ$ is 2-nilpotent, this implies that $PQ = P \times Q$. And then $|G/C_G(P)|$ is a power of 2. By Lemma 2.4, we have $P \leq Z_4(G)$.

**Theorem 3.3.** Let $G$ be a group and $\mathfrak{F}$ be a saturated formation containing the class of supersolvable groups $\Delta$. Then $G$ lies in $\mathfrak{F}$ if and only if there is a normal subgroup $H$ of $G$ such that $G/H \in \mathfrak{F}$, and every subgroup of $H$ of prime order or of order 4 is $s$-weakly normal in $G$.

**Proof.** The necessity is obvious, and we only need to prove the sufficiency part. We use induction on the order of group $G$. By Lemma 2.2(1) and using repeated applications of Theorem 3.1, $H$ has a Sylow tower of supersolvable type. Without loss of generality, let $p$ be the largest prime of $\pi(H)$ and $P$ be the Sylow $p$-subgroup of $H$. Clearly $P$ is a characteristic subgroup of $H$, and note $H \leq G$, we have $P \leq G$. This implies that $H/P \leq G/P$ and $(G/P)/(H/P) \cong G/H \in \mathfrak{F}$. It follows from Lemma 2.2(3) that every subgroup of $H/P$ of prime order or of order 4 is $s$-weakly normal in $G/P$. By induction on $|G|$, we have $G/P \in \mathfrak{F}$. It is easy to see from Theorem 3.2 that $P \leq Z_4(G)$. And by Lemma [9, Proposition 3.11, p.362], $Z_4(G) \leq Z_4(G)$, consequently, we have $P \leq Z_4(G)$. Therefore, $G \in \mathfrak{F}$.

As an immediate consequence of Theorem 3.3, we have the following two corollaries.

**Corollary 3.1.** Let $G$ be a group with a normal subgroup $E$. If $G/E$ is supersolvable and every subgroup of $E$ of prime order or of order 4 is $s$-weakly normal in $G$, then $G$ is supersolvable.
Corollary 3.2. Suppose that every subgroup of prime order or of order 4 is $s$-weakly normal in a group $G$. Then $G$ is supersolvable.

Theorem 3.4. Let $G$ be a group and $\mathcal{F}$ be a saturated formation containing the class of supersolvable groups $\mathcal{U}$. Then $G$ lies in $\mathcal{F}$ if and only if there is a normal subgroup $E$ of $G$ such that $G/E \in \mathcal{F}$, and every subgroup of $F^*(E)$ of prime order or of order 4 is $s$-weakly normal in $G$.

Proof. We only need to prove the sufficiency part. We use induction on the order of group $G$. By Lemma 2.2(1), every subgroup of $F^*(E)$ of prime order or of order 4 is $s$-weakly normal in $F^*(E)$. It follows from Corollary 3.2 that $F^*(E)$ is supersolvable. By Lemma 2.7(1), we have $F^*(E) = F(E)$, and then $F(E) \leq Z_\mathcal{U}(G)$ by Theorem 3.2. Since $Z_\mathcal{U}(G) \leq Z_\mathcal{F}(G)$, we have $F(E) \leq Z_\mathcal{F}(G)$. Hence by Lemma 2.5, we have $G/C_G(F(E)) \in \mathcal{F}$. This implies that $G/(E \cap C_G(F(E))) = G/C_E(F(E)) \in \mathcal{F}$. Since $C_E(F(E)) \leq F(E)$ by Lemma 2.7(2) and $F^*(E) = F(E)$, we have

$$G/F(E) \cong (G/C_E(F(E)))/(F(E)/C_E(F(E))) \in \mathcal{F}.$$ 

And then it is easy to see that $G \in \mathcal{F}$ by Theorem 3.3.

Corollary 3.3. Let $G$ be a group and $\mathcal{F}$ be a saturated formation containing the class of supersolvable groups $\mathcal{U}$. Then $G$ lies in $\mathcal{F}$ if and only if there is a solvable normal subgroup $E$ such that $G/E \in \mathcal{F}$ and every subgroup of $F(E)$ of prime order or of order 4 is $s$-weakly normal in $G$.

In the following part, we characterize the nilpotency of finite groups by the $s$-weakly normality of some subgroups of prime power order in $G$.

Theorem 3.5. Let $G$ be a group with a normal subgroup $E$ such that $G/E$ is nilpotent. If every minimal subgroup of $E$ is contained in $Z_\infty(G)$, and every cyclic subgroup of $E$ of order 4 is $s$-weakly normal in $G$, then $G$ is nilpotent.

Proof. Assume that the result is false and let $(G, E)$ be a counterexample such that $|G| + |E|$ is minimal. Then we prove the theorem via the following steps.

1. $G$ is a minimal non-nilpotent group, that is, $G = P \times Q$, where $P$ is a normal Sylow $p$-subgroup of $G$ and $Q$ is a non-normal cyclic Sylow $q$-subgroup of $G$ for some prime $q \neq p$; $P/\Phi(P)$ is a chief factor of $G$; $\exp(P) = p$ when $p > 2$ and $\exp(P)$ is at most 4 when $p = 2$.

Let $K$ be any proper subgroup of $G$. Then $K/(E \cap K) \cong EK/E \leq G/E$ is nilpotent, and every minimal subgroup of $E \cap K$ is contained in $Z_\infty(G) \cap K \leq Z_\infty(K)$. By hypothesis, every cyclic subgroup of $E \cap K$ of order 4 is $s$-weakly normal in $G$. Thus by Lemma 2.2(1), every cyclic subgroup of $E \cap K$ of order 4 is $s$-weakly normal in $K$. Hence $(K, E \cap K)$ satisfies the hypothesis of the theorem. Then the choice of $(G, E)$ implies that $K$ is nilpotent. Hence $G$ is a minimal non-nilpotent group, and so (1) holds by [14, Chapter III, Satz 5.2].

2. $P \leq E$. 

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If $P \not\subseteq E$, then clearly $P \cap E \leq P$, and so $(P \cap E)Q < G$. By (1), $(P \cap E)Q$ is nilpotent. This implies that $Q \leq (P \cap E)Q$. Since $G/(P \cap E) \leq G/P \times G/E$ is nilpotent, $(P \cap E)Q \leq G$, and thus $Q \leq G$, which contradicts to (1).

(3) Final contradiction.

If $\exp(P) = p$, then $P \leq Z_{\infty}(G)$, and so $G$ is nilpotent, which is impossible. Hence we may assume that $p = 2$ and $\exp(P) = 4$. Then by Lemma 2.6, every cyclic subgroup of $P$ of order 4 is weakly normal in $G$, and so every cyclic subgroup of $P$ of order 4 is normal in $G$ by Lemma 2.1(3). Take an element $x \in P \setminus \Phi(P)$. Since $P/\Phi(P)$ is a chief factor of $G$, $P = \langle x \rangle^G\Phi(P) = \langle x \rangle^G$. If $x$ is of order 2, then $P = \langle x \rangle^G \leq Z_{\infty}(G)$, also we have $G$ is nilpotent. a contradiction. Now assume that $x$ is of order 4. Then $\langle x \rangle \leq G$, and so $P = \langle x \rangle$ is cyclic. By [20, (10.1.9)], $G$ is 2-nilpotent, and so $Q \leq G$, a contradiction. This completes the proof of the theorem. 

\[ \Box \]

**Theorem 3.6.** Let $G$ be a group with a normal subgroup $E$ such that $G/E$ is nilpotent. If every minimal subgroup of $F^*(E)$ is contained in $Z_{\infty}(G)$ and every cyclic subgroup of $F^*(E)$ of order 4 is $s$-weakly normal in $G$, then $G$ is nilpotent.

**Proof.** Assume that the result is false and let $(G, E)$ be a counterexample such that $|G| + |E|$ is minimal. Then we prove the theorem via the following steps.

(1) Every proper normal subgroup of $G$ is nilpotent.

Let $K$ be any proper normal subgroup of $G$. Then $K/(E \cap K) \cong EK/E \leq G/E$ is nilpotent. By Lemma 2.7(3), $F^*(E \cap K) = F^*(E) \cap K$. Hence by Lemma 2.2(1), $(K, E \cap K)$ satisfies the hypothesis of the theorem. The choice of $(G, E)$ implies that $K$ is nilpotent.

(2) $E = G = \gamma_{\infty}(G)$ and $F^*(G) = F(G) < G$, where $\gamma_{\infty}(G)$ is the nilpotent residual of $G$.

If $E$ is a proper subgroup of $G$, then $E$ is nilpotent by (1), and so $F^*(E) = F(E) = E$. By Theorem 3.5, $G$ is nilpotent, a contradiction. Thus $E = G$. Now suppose that $F^*(G) = G$. Then by Theorem 3.5 again, $G$ is nilpotent, which is impossible. Hence $F^*(G) < G$, and $F^*(G) = F(G)$ by (1). If $\gamma_{\infty}(G) < G$, then by (1), $\gamma_{\infty}(G) \leq F(G)$, and so $G/F(G)$ is nilpotent. It follows that $G$ is nilpotent, a contradiction. Thus $\gamma_{\infty}(G) = G$.

(3) Every cyclic subgroup of $F(G)$ of order 4 is contained in $Z(G)$.

By hypothesis and (2), every cyclic subgroup $H$ of $F(G)$ of order 4 is $s$-weakly normal in $G$. Then there exists a normal subgroup $T$ of $G$ such that $G = HT$ and $H \cap T$ is weakly normal in $G$. If $T < G$, then $T \leq F(G)$ by (1), and thereby $F(G) = G$, a contradiction. Hence $T = G$, and so $H$ is weakly normal in $G$. By Lemma 2.1(3), $H \leq G$. This implies that $G/C_G(H)$ is abelian. Then by (2), $C_G(H) = \gamma_{\infty}(G) = G$, and so $H \leq Z(G)$. Thus (3) holds.

(4) Final contradiction.

Let $p$ be any prime divisor of $|F(G)|$ and $P$ be the Sylow $p$-subgroup of $F(G)$. Then $P \leq G$. If $p$ is odd, then by hypothesis, $\Omega_1(P) \leq Z_{\infty}(G)$. It follows from Lemma 2.11 that $O^p(G) \leq C_G(\Omega_1(P))$, and so $O^p(G) \leq C_G(P)$ by Lemma 2.12. Then by (2), $C_G(P) = \gamma_{\infty}(G) = G$. Now consider that $p = 2$. Then by
hypothesis and (3), $\Omega_2(P) \leq Z_\infty(G)$. A similar discussion as above also shows that $C_G(P) = G$. Therefore, we have $C_G(F(G)) = G$, which contradicts the fact that $C_G(F(G)) \leq F(G)$ by (2) and Lemma 2.7(2). This completes the proof of the theorem. □

**Remark 3.1.** Note that a $c$-normal subgroup, $H$-subgroup and weakly normal subgroup of $G$ is an $s$-weakly normal subgroup of $G$, thus some recent results can be generalized and improved by applications of the results given in this paper. For example, [22, Theorem 4.2] and [3, Theorem 3.6] are immediate results of Theorem 3.3; It is easy to obtain [8, Theorem 11] and [17, Theorem 3.1] by Corollary 3.1; [17, Theorem 3.2] and [17, Corollary 3.4] are immediate result of Theorem 3.4; [17, Theorem 3.5] and [17, Theorem 3.6] are immediate results of Theorem 3.5 and Theorem 3.6, respectively.

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**References**


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