On finite groups with s-weakly normal subgroups

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Abstract. A subgroup H of a group G is weakly normal in G if $H^g \leq N_G(H)$ implies that $g \in N_G(H)$ for any element $g \in G$. A subgroup H of a group G is s-weakly normal in G if there exists a normal subgroup T such that $G = HT$ and $H \cap T$ is weakly normal in G. Clearly a weakly normal subgroup of G is an s-weakly normal subgroup of G. In this paper, we investigate the influence of s-weakly normal subgroups on the structure of a finite group, especially some criteria for supersolvability, nilpotency, formation and hypercenter of a finite group are proved. Based on our results, some recent results can be generalized easily.

Keywords: finite group, weakly normal subgroup, s-weakly normal subgroup, supersolvable group, nilpotent group.

1. Introduction

The groups which appear throughout this paper are assumed to be finite groups and G always denotes a finite group. Let's first introduce some frequently used notations and terminologies, any unexplained terms can be found in [9, 11, 12, 20].

Let $|G|$ be the order of G and $\pi(G)$ be the set of all prime divisors of $|G|$. For a p-group P, where p is a prime, we write $\Omega_1(P) = \langle x \in P | x^p = 1 \rangle$ and $\Omega_2(P) = \langle x \in P | x^{p^2} = 1 \rangle$; we say a p'-group H means a group H satisfying $p \nmid |H|$.

Let $\mathfrak{F}$ be a class of groups. Recall that $\mathfrak{F}$ is said to be a *formation* if $\mathfrak{F}$ is closed under taking homomorphic image and finite subdirect product, that is, for each group G and a normal subgroup N of G, $G \in \mathfrak{F}$ implies that $G/N \in \mathfrak{F}$, moreover, if $M \trianglelefteq G$, then $G/N \in \mathfrak{F}$ and $G/M \in \mathfrak{F}$ imply $G/(N \cap M) \in \mathfrak{F}$. A

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formation $\mathfrak{F}$ is said to be *saturated* if $G \in \mathfrak{F}$ whenever $G/\Phi(G) \in \mathfrak{F}$, where $\Phi(G)$ is the intersection of all maximal subgroups of G.

We denote by $\mathfrak{U}$ the class of all supersolvable groups and by $\mathfrak{N}$ the class of all nilpotent groups. It is known that $\mathfrak{U}$ and $\mathfrak{N}$ are both saturated formations (see [12]). We denote by $Z_{\mathfrak{F}}(G)$ the product of all $\mathfrak{F}$-hypercentral subgroups of G. In particular, $Z_{\mathfrak{U}}(G)$ denotes the product of all normal subgroups N of G such that each chief factor of G below N has prime order. It is known that for the formation $\mathfrak{N}$, $Z_{\mathfrak{N}}(G) = Z_{\infty}(G)$ is the hypercenter of G.

It is known that a subgroup H of a group G is *pronormal* in G if the subgroups H and H^g are conjugate in $\langle H, H^g \rangle$ for each element g of G. This concept was introduced by P. Hall [13] and the first general results about pronormality appeared in a paper by J. S. Rose [21]. A subgroup H of G is *c-normal* in G if there is a normal subgroup N of G such that $G = HN$ and $H \cap N \leq H_G = Core_G(H)$, see for example [22]. In [7], the authors introduced the concept of $\mathcal{H}$-subgroup of a group and proved a number of interesting results about such subgroups. A subgroup H of a group G is called an *$\mathcal{H}$-subgroup* provided that $H^g \cap N_G(H) \leq H$ for all $g \in G$. It is easy to see that the Sylow p-subgroups, normal subgroups and self-normalizing subgroups of an arbitrary group are $\mathcal{H}$-subgroups. Following Müller [18], a subgroup H of a group G is *weakly normal* in G if $H^g \leq N_G(H)$ implies that $g \in N_G(H)$ for any element $g \in G$. It is known that every pronormal subgroup and $\mathcal{H}$-subgroup of G are weakly normal in G, but the converse is not true (see [1, p.28] and [6]) for more details. In [17], the authors investigated the behaviour of weakly normal subgroups, and obtained some characterizations about the supersolvability and nilpotency of G by assuming that some subgroups of prime power order of G are weakly normal in G. Recently, the authors in [24] gave some results about formation under the condition that some subgroups of prime square order are weakly normal in G.

It is known that there is no obvious general relationship between the concepts of c-normal subgroup and $\mathcal{H}$-subgroup. For a generalization of both $\mathcal{H}$-subgroup and c-normal subgroup, Assad, et al.[2] introduced the concept of weakly $\mathcal{H}$-subgroup, which describes subgroup embedding properties of a finite group. A subgroup H of a group G is called a weakly $\mathcal{H}$-subgroup in G if there exists a normal subgroup K of G such that $G = HK$ and $H \cap K$ is a $\mathcal{H}$-subgroup of G. The authors in [2] determined the structure of a finite group G when all maximal subgroups of every Sylow subgroup of certain subgroups of G are weakly $\mathcal{H}$-subgroups in G.

Inspired by the above works, we consider the following question:

How is the structure of a finite group G determined by its subgroup H with the property that there exists a normal subgroup T of G such that $HT = G$ and $H \cap T$ is weakly normal in G?

We first introduce a new notion of s-weakly normal subgroup which is a generalization of c-normal subgroup, $\mathcal{H}$-subgroup, and weakly normal subgroup.

Definition 1.1. *A subgroup H of a group G is an s-weakly normal subgroup of G if there exists a normal subgroup T of G such that $G = HT$ and $H \cap T$ is weakly normal in G.*

Clearly a c-normal subgroup, $\mathcal{H}$-subgroup or weakly normal subgroup of G is an s-weakly normal subgroup of G, but the converse is not true. The following two examples will show this.

Example 1.1. Let $G = S_4$ be the symmetric group of degree 4. Suppose that $H = \langle(12)\rangle$ and A_4 is the alternating group of degree 4. Since A_4 is a normal subgroup of G such that $G = HA_4$ and $H \cap A_4 = \{(1)\}$ is weakly normal in G, H is s-weakly normal in G. It is easy to see that $N_G(H) = \{(1), (12), (34), (12)(34)\}$. Let $g = (13)(24) \in G$. Since $H^g = \{(1), (34)\}$ and $H^g \cap N_G(H) = \{(1), (34)\} \nleq H$, H is not an $\mathcal{H}$-subgroup of G. Also note that $H^g \leq N_G(H)$, but $g = (13)(24) \notin N_G(H)$. It follows that H is not weakly normal in G.

Example 1.2. Let $G = S_4$, $H = \{(1), (12), (13), (23), (123), (132)\}$. It is easy to check that $N_G(H) = H$, this means that H is an $\mathcal{H}$-subgroup of G and so it is weakly normal in G. Hence H is s-weakly normal in G. But H is not c-normal in G. In fact, since $H_G = \{(1)\}$, there is no such a normal subgroup T of G such that $G = HT$ and $H \cap T \leq H_G$.

The aim of this paper is to obtain some new characterizations of the nilpotency and supersolvability of finite groups by studying the s-weakly normality properties of some certain primary subgroups. In Section 2, some necessary lemmas are given. In Section 3, some criteria for supersolvability, nilpotency, formation and hypercenter of a finite group are proved, based on these criteria, some recent results can be improved and extended easily. These results show that the concept of s-weakly normal subgroup provides us a useful tool to investigate the structure of finite groups.

2. Preliminaries

In this section we list some basic facts which are needed in this paper.

Lemma 2.1 ([17], Lemma 2.1, Lemma 2.2)**.** *Let N, H and K be subgroups of a finite group G. Then:*

(1) *If H is weakly normal in G, and $H \leq K \leq G$, then H is weakly normal in K.*

(2) *Let $N \trianglelefteq G$ and $N \leq H$. Then H is weakly normal in G if and only if H/N is weakly normal in G/N.*

(3) *If H is weakly normal in G and $H \trianglelefteq \trianglelefteq K \leq G$, then $H \trianglelefteq K$.*

(4) *If $N \trianglelefteq G$, P is a weakly normal p-subgroup of G such that $(|N|, p) = 1$, then PN is weakly normal in G and PN/N is weakly normal in G/N.*

Lemma 2.2. *Let G be a finite group and N, H, K be subgroups of group G.*

(1) *If H is s-weakly normal in G, and $H \leq K \leq G$, then H is s-weakly normal in K.*

(2) *Let $N \leq H$ and $N \trianglelefteq G$. Then H is s-weakly normal in G if and only if H/N is s-weakly normal in G/N.*

(3) *Let N be a normal subgroup of G. If H is a p-subgroup of G such that $(|N|, |H|) = 1$ and H is s-weakly normal in G, then HN/N is s-weakly normal in G/N.*

Proof. (1) Since H is an s-weakly normal subgroup of G, there exists a normal subgroup T of G such that $G = HT$ and $H \cap T$ is weakly normal in G. It is easy to see that $K = K \cap HT = H(K \cap T)$ and $H \cap (K \cap T) = H \cap T$ is weakly normal in G. It follows by Lemma 2.1(1) that $H \cap (K \cap T) = H \cap T$ is weakly normal in K. Hence H is s-weakly normal in K.

(2) Assume that H is s-weakly normal in G, that is, there exists a normal subgroup T of G such that $G = HT$ and $H \cap T$ is weakly normal in G. Since $N \trianglelefteq G$, it is clear that $G/N = (H/N)(TN/N)$ and $TN/N \trianglelefteq G/N$. Then it follows from Lemma 2.1(2) that $(H/N) \cap (TN/N) = N(H \cap T)/N$ is weakly normal in G/N. Hence H/N is s-weakly normal in G/N.

Conversely, suppose that H/N is s-weakly normal in G/N. Then there exists a normal subgroup T/N of G/N such that $G/N = (H/N)(T/N)$ and $(H/N) \cap (T/N) = (H \cap T)/N$ is weakly normal in G/N. It is easy to see that $G = HT$ and $H \cap T$ is weakly normal in G by Lemma 2.1(2), that is, H is s-weakly normal in G.

(3) Suppose that H is s-weakly normal in G. Then there exists a normal subgroup T of G such that $G = HT$ and $H \cap T$ is weakly normal in G. Note that $(|N|, |H|) = 1$, we have $N \leq T$, and hence $HN \cap T = N(H \cap T)$. It is easy to see that $HN \cap T = N(H \cap T)$ is weakly normal in G. By Lemma 2.1(4), $N(H \cap T)/N$ is weakly normal in G/N. Note that $G/N = (HN/N)(T/N)$ and $(HN/N) \cap (T/N) = (HN \cap T)/N = N(H \cap T)/N$ is weakly normal in G/N. Hence HN/N is s-weakly normal in G/N. □

Lemma 2.3 ([14], Satz 5.4, p.434)**.** *Let G be a group and $p \in \pi(G)$, If G is a minimal non-p-nilpotent group, that is, G is not nilpotent but all of its proper subgroups are p-nilpotent, then*

(*i*) *$G = PQ$, where P is a normal Sylow p-subgroup of G and Q is a non-normal cyclic Sylow q-subgroup of G.*

(*ii*) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*

(*iii*) *If $p > 2$, then $exp(P)$ is p, and when $p = 2$, $exp(P)$ is at most 4, where $exp(P)$ is the exponent of group P.*

Lemma 2.4 ([23], Theorem 6.3, p.221 and Corollary 7.8, p.33)**.** *Let P be a normal p-subgroup of a group G such that $|G/C_G(P)|$ is a power of prime p. Then $P \leq Z_{\mathfrak{U}}(G)$.*

Lemma 2.5 ([9], Theorem 6.10, p.390). *If a class of groups $\mathfrak{F}$ is a saturated formation, then $[G^{\mathfrak{F}}, Z_{\mathfrak{F}}(G)] = 1$.*

Lemma 2.6. *Let H be an s-weakly normal subgroup of G and K be a subgroup of G such that $H \leq K$. If $K/\Phi(K)$ is a chief factor of G, then H is weakly normal in G.*

Proof. Since H is s-weakly normal in G, there is a normal subgroup T of G such that $G = HT$ and $H \cap T$ is weakly normal in G. Note that $K/\Phi(K)$ is a chief factor of G. Thus either $(K\cap T)\Phi(K)/\Phi(K) = 1$ or $(K\cap T)\Phi(K)/\Phi(K) = K/\Phi(K)$. In the former case, since $K \cap T \leq \Phi(K)$, we have $K = K \cap HT = H(K \cap T) = H$. This implies that $H \trianglelefteq G$, and clearly, H is weakly normal in G. In the latter case, we have $(K \cap T)\Phi(K) = K$, and hence $K \cap T = K$ and $G = T$. This also implies that H is weakly normal in G. □

Let G be a finite group. It is known that the Fitting subgroup $F(G)$ of G is the unique maximal normal nilpotent subgroup of G, and the generalized Fitting subgroup $F^*(G)$ of G is the unique maximal normal quasinilpotent subgroup of G. The following results about $F^*(G)$ and $F(G)$are useful in our paper.

Lemma 2.7 ([15], Chapter X 13). *Let G be a group.*
(1) *Suppose that $F^*(G)$ is solvable. Then $F^*(G) = F(G)$;*
(2) *$C(F^*(G)) \leq F(G)$;*
(3) *If N is a normal subgroup of G, then $F^*(N) = N \cap F^*(G)$.*

Lemma 2.8 ([6], Lemma 2). *Let H be a p-subgroup of G. Then the following properties are equivalent:*
(1) *H is a pronormal subgroup of G;*
(2) *H is a weakly normal subgroup of G.*

Lemma 2.9 ([4], Theorem 4.1). *Let p be the smallest prime of $\pi(G)$ and P a Sylow p-subgroup of G. If every subgroup of P of order p or 4 (when $p = 2$) is pronormal in G, then G is p-nilpotent.*

From Lemma 2.8 and Lemma 2.9, we immediately get the following result.

Lemma 2.10. *Let G be a group and p be the smallest prime of $\pi(G)$. If P is a Sylow p-subgroup of G and every subgroup of $P$of order p or 4 (when $p = 2$) is weakly normal in G, then G is p-nilpotent.*

Lemma 2.11 ([16], Lemma 2.8). *Let P be a normal p-subgroup of G contained in $Z_\infty(G)$. Then $O^p(G) \leq C_G(P)$.*

Lemma 2.12 ([10], Lemma 2.4). *Let P be a p-group. If α is a p'-automorphism of P which centralizes $\Omega_1(P)$, then $\alpha = 1$ unless P is a non-abelian 2-group. If $[\alpha, \Omega_2(P)] = 1$, then $\alpha = 1$ without restriction.*

3. Main results

In the sequel, we discuss the influence of s-weakly normal subgroups on the structure of a group.

Theorem 3.1. *Let G be a group with a Sylow p-subgroup P, where p is the smallest prime in $\pi(G)$. Suppose that every subgroup of P of order p or 4 (when $p = 2$) is s-weakly normal in G. Then G is p-nilpotent.*

Proof. Suppose that the required result is not true and let G be a counterexample of minimal order.

Firstly, suppose that p is an odd prime. If every subgroup of P of order p is weakly normal in G, then by Lemma 2.10, G is p-nilpotent, a contradiction. Hence there exists a subgroup P_1 of P such that $|P_1| = p$ and P_1 is not weakly normal in G. By the hypotheses of the theorem, P_1 is s-weakly normal in G, i.e. there is a normal subgroup T of G such that $G = P_1T$ and $P_1 \cap T$ is weakly normal in G. If $P_1 \cap T \neq 1$, then $P_1 \cap T = P_1$ is weakly normal in G, a contradiction. Thus $P_1 \cap T = 1$, and therefore T is a proper subgroup of G. By Lemma 2.2(1), T satisfies the hypotheses of the theorem. Then T is p-nilpotent by the minimal choice of G. Let $T_{p'}$ be a normal p-complement of G. Clearly, $T_{p'}$ *char* G. Note that T is normal in G, and therefore $T_{p'} \trianglelefteq G$. This means that G is p-nilpotent, which is a contradiction.

Now suppose that $p = 2$. Since G is not 2-nilpotent, it follows that G contains a minimal non-2-nilpotent subgroup K. Then K is a minimal non-nilpotent subgroup of G and $K = K_2 \rtimes K_q$, where K_2 is a normal Sylow 2-subgroup of K and K_q is a non-normal Sylow q-subgroup of K, where $q > 2$, and $exp(K_2)$ is at most 4. By using Lemma 2.1(1), we can easily see that the hypothesis is inherited by K. Then by Lemma 2.10, K_2 contains a subgroup L of order 2 or 4 such that L is not weakly normal in K. By the hypotheses of the theorem, L is s-weakly normal in G and thereby L is s-weakly normal in K by Lemma 2.2(1), that is, there is a normal subgroup T of K such that $K = LT$ and $L \cap T$ is weakly normal in K. If $T = K$, then $L \cap T = L$ is weakly normal in K, a contradiction. Thus T is a proper subgroup of K. If $|L| = 2$, then $L \cap T = 1$. Since T is a normal nilpotent subgroup of K, K_q *char* $T \trianglelefteq K$, and hence $K_q \trianglelefteq K$, a contradiction. If $|L| = 4$, then we can always conclude that $K_q \trianglelefteq K$ when $|L \cap T| = 1$ or $|L \cap T| = 2$ since T is a normal nilpotent subgroup of K, a contradiction. This completes the proof of the theorem. □

Theorem 3.2. *Let G be a group with a normal p-subgroup P, where $p \in \pi(G)$. Suppose that every subgroup of P of order p or of order 4 (when $p = 2$) is s-weakly normal in G. Then we have $P \leq Z_{\mathfrak{U}}(G)$.*

Proof. We proceed the proof of the theorem by induction on $|G| + |P|$ and distinguish the following two cases.

Case (1): p is an odd prime.

If every subgroup of P of order p is normal in G, then it is easy to see from [5, Theorem 1.1] that $P \leq Z_{\mathfrak{U}}(G)$. Now we assume that there exists a subgroup H of order p in P such that H is not normal in G. By the hypothesis of the theorem, H is s-weakly normal in G, that is, there exists a normal subgroup T of G such that $G = HT$ and $H \cap T$ is weakly normal in G. Assume that $H \cap T \neq 1$. Then $H \cap T = H$ is weakly normal in G. It is easy to know that H is subnormal in G, then we have that $H \trianglelefteq G$ by Lemma 2.1(3), which is a contradiction. Therefore $H \cap T = 1$. Note that $P \cap T \trianglelefteq G$. By hypothesis of the theorem, every subgroup of $P \cap T$ of order p is s-weakly normal in G. Hence, by induction on $|G| + |P|$, we have that $P \cap T \leq Z_{\mathfrak{U}}(G)$. Since $P = H(P \cap T)$, we have $P/(P \cap T) = H(P \cap T)/(P \cap T)$ is a normal subgroup of $G/(P \cap T)$ of order p. Consequently, $P/(P \cap T) \leq Z_{\mathfrak{U}}(G/(P \cap T))$. Note that $P \cap T \leq Z_{\mathfrak{U}}(G)$, and then by [23, Theorem 7.7, p.32], we have

$$Z_{\mathfrak{U}}(G/(P \cap T)) = Z_{\mathfrak{U}}(G)/(P \cap T).$$

Therefore, $P \leq Z_{\mathfrak{U}}(G)$.

Case (2): $p = 2$.

Let Q be any Sylow q-subgroup of G, where $q \neq 2$. Then it is clear that PQ is a subgroup of G. By Lemma 2.2(1) and Theorem 3.1, PQ is 2-nilpotent, this implies that $PQ = P \times Q$. And then $|G/C_G(P)|$ is a power of 2. By Lemma 2.4, we have $P \leq Z_{\mathfrak{U}}(G)$. □

Theorem 3.3. *Let G be a group and $\mathfrak{F}$ be a saturated formation containing the class of supersolvable groups $\mathfrak{U}$. Then G lies in $\mathfrak{F}$ if and only if there is a normal subgroup H of G such that $G/H \in \mathfrak{F}$, and every subgroup of H of prime order or of order 4 is s-weakly normal in G.*

Proof. The necessity is obvious, and we only need to prove the sufficiency part. We use induction on the order of group G. By Lemma 2.2(1) and using repeated applications of Theorem 3.1, H has a Sylow tower of supersolvable type. Without loss of generality, let p be the largest prime of $\pi(H)$ and P be the Sylow p-subgroup of H. Clearly P is a characteristic subgroup of H, and note $H \trianglelefteq G$, we have $P \trianglelefteq G$. This implies that $H/P \trianglelefteq G/P$ and $(G/P)/(H/P) \cong G/H \in \mathfrak{F}$. It follows from Lemma 2.2(3) that every subgroup of H/P of prime order or of order 4 is s-weakly normal in G/P. By induction on $|G|$, we have $G/P \in \mathfrak{F}$. It is easy to see from Theorem 3.2 that $P \leq Z_{\mathfrak{U}}(G)$. And by Lemma[9, Propositin 3.11, p.362], $Z_{\mathfrak{U}}(G) \leq Z_{\mathfrak{F}}(G)$, consequently, we have $P \leq Z_{\mathfrak{F}}(G)$. Therefore, $G \in \mathfrak{F}$. □

As an immediate consequence of Theorem 3.3, we have the following two corollaries.

Corollary 3.1. *Let G be a group with a normal subgroup E. If G/E is supersolvable and every subgroup of E of prime order or of order 4 is s-weakly normal in G, then G is supersolvable.*

Corollary 3.2. *Suppose that every subgroup of prime order or of order* 4 *is* s*-weakly normal in a group* G*. Then* G *is supersolvable.*

Theorem 3.4. *Let* G *be a group and* $\mathfrak{F}$ *be a saturated formation containing the class of supersolvable groups* $\mathfrak{U}$*. Then* G *lies in* $\mathfrak{F}$ *if and only if there is a normal subgroup* E *of* G *such that* $G/E \in \mathfrak{F}$*, and every subgroup of* $F^*(E)$ *of prime order or of order* 4 *is* s*-weakly normal in* G*.*

Proof. We only need to prove the sufficiency part. We use induction on the order of group G. By Lemma 2.2(1), every subgroup of $F^*(E)$ of prime order or of order 4 is s-weakly normal in $F^*(E)$. It follows from Corollary 3.2 that $F^*(E)$ is supersolvable. By Lemma 2.7(1), we have $F^*(E) = F(E)$, and then $F(E) \leq Z_{\mathfrak{U}}(G)$ by Theorem 3.2. Since $Z_{\mathfrak{U}}(G) \leq Z_{\mathfrak{F}}(G)$, we have $F(E) \leq Z_{\mathfrak{F}}(G)$. Hence by Lemma 2.5, we have $G/C_G(F(E)) \in \mathfrak{F}$. This implies that $G/(E \cap C_G(F(E))) = G/C_E(F(E)) \in \mathfrak{F}$. Since $C_E(F(E)) \leq F(E)$ by Lemma 2.7(2) and $F^*(E) = F(E)$, we have

$$G/F(E) \cong (G/C_E(F(E)))/(F(E)/C_E(F(E))) \in \mathfrak{F}.$$

And then it is easy to see that $G \in \mathfrak{F}$ by Theorem 3.3. □

Corollary 3.3. *Let* G *be a group and* $\mathfrak{F}$ *be a saturated formation containing the class of supersolvable groups* $\mathfrak{U}$*. Then* G *lies in* $\mathfrak{F}$ *if and only if there is a solvable normal subgroup* E *such that* $G/E \in \mathfrak{F}$ *and every subgroup of* $F(E)$ *of prime order or of order* 4 *is* s*-weakly normal in* G*.*

In the following part, we characterize the nilpotency of finite groups by the s-weakly normality of some subgroups of prime power order in G.

Theorem 3.5. *Let* G *be a group with a normal subgroup* E *such that* G/E *is nilpotent. If every minimal subgroup of* E *is contained in* $Z_\infty(G)$*, and every cyclic subgroup of* E *of order* 4 *is* s*-weakly normal in* G*, then* G *is nilpotent.*

Proof. Assume that the result is false and let (G, E) be a counterexample such that $|G| + |E|$ is minimal. Then we prove the theorem via the following steps.

(1) G is a minimal non-nilpotent group, that is, $G = P \rtimes Q$, where P is a normal Sylow p-subgroup of G and Q is a non-normal cyclic Sylow q-subgroup of G for some prime $q \neq p$; $P/\Phi(P)$ is a chief factor of G; $exp(P) = p$ when $p > 2$ and $exp(P)$ is at most 4 when $p = 2$.

Let K be any proper subgroup of G. Then $K/(E \cap K) \cong EK/E \leq G/E$ is nilpotent, and every minimal subgroup of $E \cap K$ is contained in $Z_\infty(G) \cap K \leq Z_\infty(K)$. By hypothesis, every cyclic subgroup of $E \cap K$ of order 4 is s-weakly normal in G. Thus by Lemma 2.2(1), every cyclic subgroup of $E \cap K$ of order 4 is s-weakly normal in K. Hence $(K, E \cap K)$ satisfies the hypothesis of the theorem. Then the choice of (G, E) implies that K is nilpotent. Hence G is a minimal non-nilpotent group, and so (1) holds by [14, Chapter III, Satz 5.2].

(2) $P \leq E$.

If $P \nleq E$, then clearly $P \cap E < P$, and so $(P \cap E)Q < G$. By (1), $(P \cap E)Q$ is nilpotent. This implies that $Q \trianglelefteq (P \cap E)Q$. Since $G/(P \cap E) \lesssim G/P \times G/E$ is nilpotent, $(P \cap E)Q \trianglelefteq G$, and thus $Q \trianglelefteq G$, which contradicts to (1).

(3) Final contradiction.

If $exp(P) = p$, then $P \leq Z_\infty(G)$, and so G is nilpotent, which is impossible. Hence we may assume that $p = 2$ and $exp(P) = 4$. Then by Lemma 2.6, every cyclic subgroup of P of order 4 is weakly normal in G, and so every cyclic subgroup of P of order 4 is normal in G by Lemma 2.1(3). Take an element $x \in P \backslash \Phi(P)$. Since $P/\Phi(P)$ is a chief factor of G, $P = \langle x \rangle^G \Phi(P) = \langle x \rangle^G$. If x is of order 2, then $P = \langle x \rangle^G \leq Z_\infty(G)$, also we have G is nilpotent. a contradiction. Now assume that x is of order 4. Then $\langle x \rangle \trianglelefteq G$, and so $P = \langle x \rangle$ is cyclic. By [20, (10.1.9)], G is 2-nilpotent, and so $Q \trianglelefteq G$, a contradiction. This completes the proof of the theorem. □

Theorem 3.6. *Let G be a group with a normal subgroup E such that G/E is nilpotent. If every minimal subgroup of $F^*(E)$ is contained in $Z_\infty(G)$ and every cyclic subgroup of $F^*(E)$ of order 4 is s-weakly normal in G, then G is nilpotent.*

Proof. Assume that the result is false and let (G, E) be a counterexample such that $|G| + |E|$ is minimal. Then we prove the theorem via the following steps.

(1) Every proper normal subgroup of G is nilpotent.

Let K be any proper normal subgroup of G. Then $K/(E \cap K) \cong EK/E \leq G/E$ is nilpotent. By Lemma 2.7(3), $F^*(E \cap K) = F^*(E) \cap K$. Hence by Lemma 2.2(1), $(K, E \cap K)$ satisfies the hypothesis of the theorem. The choice of (G, E) implies that K is nilpotent.

(2) $E = G = \gamma_\infty(G)$ and $F^*(G) = F(G) < G$, where $\gamma_\infty(G)$ is the nilpotent residual of G.

If E is a proper subgroup of G, then E is nilpotent by (1), and so $F^*(E) = F(E) = E$. By Theorem 3.5, G is nilpotent, a contradiction. Thus $E = G$. Now suppose that $F^*(G) = G$. Then by Theorem 3.5 again, G is nilpotent, which is impossible. Hence $F^*(G) < G$, and $F^*(G) = F(G)$ by (1). If $\gamma_\infty(G) < G$, then by (1), $\gamma_\infty(G) \leq F(G)$, and so $G/F(G)$ is nilpotent. It follows that G is nilpotent, a contradiction. Thus $\gamma_\infty(G) = G$.

(3) Every cyclic subgroup of $F(G)$ of order 4 is contained in $Z(G)$.

By hypothesis and (2), every cyclic subgroup H of $F(G)$ of order 4 is s-weakly normal in G. Then there exists a normal subgroup T of G such that $G = HT$ and $H \cap T$ is weakly normal in G. If $T < G$, then $T \leq F(G)$ by (1), and thereby $F(G) = G$, a contradiction. Hence $T = G$, and so H is weakly normal in G. By Lemma 2.1(3), $H \trianglelefteq G$. This implies that $G/C_G(H)$ is abelian. Then by (2), $C_G(H) = \gamma_\infty(G) = G$, and so $H \leq Z(G)$. Thus (3) holds.

(4) Final contradiction.

Let p be any prime divisor of $|F(G)|$ and P be the Sylow p-subgroup of $F(G)$. Then $P \trianglelefteq G$. If p is odd, then by hypothesis, $\Omega_1(P) \leq Z_\infty(G)$. It follows from Lemma 2.11 that $O^p(G) \leq C_G(\Omega_1(P))$, and so $O^p(G) \leq C_G(P)$ by Lemma 2.12. Then by (2), $C_G(P) = \gamma_\infty(G) = G$. Now consider that $p = 2$. Then by hypothesis and (3), $\Omega_2(P) \leq Z_\infty(G)$. A similar discussion as above also shows that $C_G(P) = G$. Therefore, we have $C_G(F(G)) = G$, which contradicts the fact that $C_G(F(G)) \leq F(G)$ by (2) and Lemma 2.7(2). This completes the proof of the theorem. □

Remark 3.1. Note that a c-normal subgroup, $\mathcal{H}$-subgroup and weakly normal subgroup of G is an s-weakly normal subgroup of G, thus some recent results can be generalized and improved by applications of the results given in this paper. For example,[22, Theorem 4.2] and [3, Theorem 3.6] are immediate results of Theorem 3.3; It is easy to obtain [8, Theorem 11]) and [17, Theorem 3.1] by Corollary 3.1; [17, Theorem 3.2] and [17, Corollary 3.4] are immediate result of Theorem 3.4; [17, Theorem 3.5] and [17, Theorem 3.6] are immediate results of Theorem 3.5 and Theorem 3.6, respectively.

Acknowledgement

The authors would like to thank the anonymous referee for a very careful reading of the manuscript and the valuable comments and suggestions. This work is partially supported by the following grants: National Natural Science Foundation of China (Grant No. 11947071); Natural Science Foundation of Chongqing, China (Grant No. cstc2021jcyj-msxmX0575, CSTB2022NSCQ-MSX0831).

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Accepted: January 23, 2022