On lacunary statistical convergence of difference sequences

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Abstract. In this paper, the $S_\theta(\Delta)$ and $N_\theta(\Delta)$ summabilities are used along with the notion of weakly unconditionally Cauchy series (in brief wuC series) to characterize a Banach space. We examine these two kinds of summabilities which are regular methods and we recall some features. Furthermore, we investigate the spaces $S_{N_\theta}(\sum_p \Delta w_p)$ and $S_{S_\theta}(\sum_p \Delta w_p)$ which will be thought to characterize the completeness of a space.
Keywords: completeness, unconditionally Cauchy series, lacunary convergence, difference sequence.

1. Introduction and background

The notion of statistical convergence was introduced under the name almost convergence by Zygmund [1]. It was formally presented by Fast [2]. Later the idea was associated with summability theory by Fridy [3] and many others (see [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14]).

By a lacunary sequence we mean an increasing integer sequence $\theta = \{n_r\}$ such that $n_0 = 0$ and $h_r = n_r - n_{r-1} \to \infty$ as $r \to \infty$ and ratio $\frac{n_r}{n_{r-1}}$ will be abbreviated by q_r. Throughout this paper the intervals determined by θ will be denoted by $I_r = (n_{r-1}, n_r]$. Utilizing lacunary sequence, Fridy and Orhan [15] presented the notion of lacunary statistical convergence. Some works in lacunary statistical convergence can be found in [16, 17, 18, 19, 20].

Let us define the forward difference matrix $\Delta^F = (c_{nk})$ and the backward difference matrix $\Delta^B = (d_{nk})$ by

$$c_{nk} = \begin{cases} (-1)^{n-k}, & n \le k \le n+1, \\ 0, & 0 \le k < n \text{ or } k > n+1, \end{cases}$$

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$$d_{nk} = \begin{cases} (-1)^{n-k}, & n-1 \leq k \leq n, \\ 0, & 0 \leq k < n-1 \text{ or } k > n, \end{cases}$$

for all $k, n \in \mathbb{N} = \{0, 1, 2, ...\}$. Then, the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced by Kızmaz [21], can be seen as the domain of forward difference matrix Δ^F in the classical spaces l_∞, c and c_0 of bounded, convergent and null sequences, respectively. Quite recently, the difference space bv_p was introduced as the domain of the backward difference matrix Δ^B in the classical space l_p of absolutely p-summable sequences for $0 < p < 1$ by Altay and Başar [22], and for $1 \leq p \leq \infty$ by Başar and Altay [23].

Later on the notion was generalized by Et and Çolak [24]. Başarır [25] investigated the Δ-statistical convergence of sequences. Also, the generalized difference sequence spaces were worked by various authors [26, 27, 28, 29, 30].

The characterization of a Banach space through various types of convergence has been examined by authors such as Kolk [31], Connor et al. [32].

The purpose of this study originates in the PhD thesis of the second author [33] who identified a relationship between features of a normed space Y and some sequence spaces which are named convergence spaces associated to a wuC series. These sequence spaces associated to a wuC series were examined [33] in terms of the norm topology and the usual weak topology of the space. These types of consequences have been researched in various convergence spaces connected with a wuC series utilizing different types of convergence [34, 35, 36, 37, 38, 40, 39]. The readers can refer to the recent papers [41, 42, 43, 44], and references therein on the wuC series in a normed space and the examples of multiplier convergent series that characterizes the uc and wuC series, and related topics.

Y be a normed space and $\sum w_i$ also be a series in Y. In [33], the authors defined the space of convergence $S(\sum w_i)$ connected with the series $\sum w_i$, which is introduced as the space of sequence (β_i) in l_∞ such that $\sum \beta_i w_i$ converges. They demonstrated that the necessary and sufficient condition for Y to be a complete space is that for every wuC series $\sum w_i$, the space $S(\sum w_i)$ is complete. Diestel [45] showed that $\sum w_i$ is wuC iff $\sum |f(w_i)| < \infty$ for all $f \in Y^*$. In [46, 47], a Banach space is characterized by means of the strong p-Cesàro summability and ideal-convergence.

In this paper, we examine the completeness of a normed space through the lacunary statistical convergence and lacunary strongly convergence of series for difference sequences. We also describe the summability spaces associated with these summabilities with strongly (p, Δ)-Cesàro summability spaces for difference sequences.

2. Main results

We identify the notion of lacunary Δ-statistically convergent sequence for Banach spaces.

Let $A \subset \mathbb{N}$ and $r \in \mathbb{N}$. $d_\theta^r(A)$ is named the rth partial lacunary density of A, if

$$d_\theta^r(A) = \frac{|A \cap I_r|}{h_r},$$

where $I_r = (k_{r-1}, k_r]$.

The number $d_\theta(A)$ is indicated the lacunary density (θ-density) of A if

$$d_\theta(A) = \lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r : k \in A\}|, \text{ (i.e., } d_\theta(A) = \lim_{r\to\infty} d_\theta^r(A))$$

exists. Also, $\Lambda = \{A \subset \mathbb{N} : d_\theta(A) = 0\}$ is called to be zero density set.

It is easy to demonstrate that this density is a finitely additive measure and we can introduce the notion of lacunary statistically convergent difference sequences for Banach spaces.

Definition 2.1. *Let Y be a Banach space and $\theta = \{n_r\}$ a lacunary sequence. A sequence $w = (w_p)$ is a lacunary Δ-statistically convergent or sequence to $\xi \in Y$ if given $\zeta > 0$,*

$$d_\theta\left(\{p \in I_r : \|\Delta w_p - \xi\| \geq \zeta\}\right) = 0,$$

or equivalently,

$$d_\theta\left(\{p \in I_r : \|\Delta w_p - \xi\| < \zeta\}\right) = 1,$$

we say that (w_p) is $S_\theta(\Delta)$-convergent and is written as S_θ-$\lim \Delta w_p = \xi$.

Definition 2.2. *A sequence $w = (w_p)$ in Y is lacunary strongly Δ-convergent or $N_\theta(\Delta)$-summable to $\xi \in Y$ if*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{p \in I_r} \|\Delta w_p - \xi\| = 0,$$

and we write N_θ-$\lim \Delta w_p = \xi$.

Theorem 2.1. *Let Y be a Banach space and (w_p) a sequence in Y. Note that $S_\theta(\Delta)$ and $N_\theta(\Delta)$ are regular methods.*

Proof. First, we prove that $S_\theta(\Delta)$ is a regular method. If $(\Delta w_p) \to \xi$, then N_θ-$\lim \Delta w_p = \xi$. Let $\zeta > 0$, then there is p_0 such that if $p \geq p_0$, then

$$\|\Delta w_p - \xi\| < \zeta.$$

Therefore, there is $r_0 \in \mathbb{N}$ with $r_0 \geq p_0$ such that if $r \geq r_0$ we obtain

$$\frac{1}{h_r} \sum_{p \in I_r} \|\Delta w_p - \xi\| < \frac{1}{h_r} \sum_{p \in I_r} \zeta = \frac{h_r}{h_r} \zeta = \zeta,$$

which gives that $\lim_{r\to\infty} \frac{1}{h_r} \sum_{p \in I_r} \|\Delta w_p - \xi\| = 0$.

Now, we show that $N_\theta(\Delta)$ is a regular method. If $(\Delta w_p) \to \xi$, then $S_\theta - \lim \Delta w_p = \xi$. One can easily observe that $(\Delta w_p) \to \xi$, given $\zeta > 0$ there is p_0 such that for every $p > p_0$ we obtain

$$card\left(\{p \in I_r : \|\Delta w_p - \xi\| \geq \zeta\}\right) = 0,$$

which gives

$$d_\theta\left(\{p \in I_r : \|\Delta w_p - \xi\| \geq \zeta\}\right) = 0$$

for every $p > p_0$. □

The reverse is not true, as was shown in Example 2.1, in which we introduce an unbounded sequence that is $N_\theta(\Delta)$-summable and Example 2.2 where an unbounded $S_\theta(\Delta)$-convergent sequence is given.

Example 2.1. There exist unbounded sequences which are $N_\theta(\Delta)$-summable. Let $\theta = \{n_r\}$ be a lacunary sequence with $n_0 = 0$ and $n_r = 2^r$. Think that

$$h_1 = n_1 - n_0 = 2 \text{ and } h_r = 2^{r-1}, \text{ for every } r \geq 2,$$

$$I_1 = (n_0, n_1] = (0, 2] \text{ and } I_r = \left(2^{r-1}, 2^r\right], \text{ for every } r \geq 2.$$

Think the sequence determined by

$$\Delta w_p = \begin{cases} 0, & \text{if } p \neq 2^j \text{ for all } j, \\ j - 1, & \text{if } p = 2^j \text{ for all } j. \end{cases}$$

Notice that, (w_p) is unbounded and observe that

$$\frac{\sum_{p \in I_r} |\Delta w_p - 0|}{h_r} = \left\{ \begin{array}{cl} 0, & \text{if } r = 1 \\ \frac{r-1}{2^{r-1}}, & \text{if } r \geq 2 \end{array} \right\} \to 0, \text{ as } r \to \infty$$

which gives that N_θ-$\lim \Delta w_p = 0$.

Theorem 2.2. *Let Y be a Banach space and $\theta = \{n_r\}$ be a lacunary sequence. Then, we have the followings:*

(*i*) N_θ-$\lim \Delta w_p = \xi$ *implies* S_θ-$\lim \Delta w_p = \xi$,

(*ii*) (w_p) *is bounded and* S_θ-$\lim \Delta w_p = \xi$ *imply* N_θ-$\lim \Delta w_p = \xi$.

Proof. (*i*) If N_θ-$\lim \Delta w_p = \xi$ then, for every $\zeta > 0$,

$$\sum_{p \in I_r} \|\Delta w_p - \xi\| \geq \sum_{\substack{p \in I_r \\ \|\Delta w_p - \xi\| \geq \zeta}} \|\Delta w_p - \xi\| \geq \zeta \left|\{p \in I_r : \|\Delta w_p - \xi\| \geq \zeta\}\right|,$$

which gives that S_θ-$\lim \Delta w_p = \xi$.

(ii) Let us assume that (w_p) is bounded and S_θ-$\lim \Delta w_p = \xi$. Since (w_p) is bounded, there exists $H > 0$ such that $\|\Delta w_p - \xi\| < H$ for every $p \in \mathbb{N}$. Given $\zeta > 0$,

$$\frac{1}{h_r}\sum_{p\in I_r} \|\Delta w_p - \xi\| = \frac{1}{h_r}\sum_{\substack{p\in I_r \\ \|\Delta w_p - \xi\| \geq \zeta}} \|\Delta w_p - \xi\| + \frac{1}{h_r}\sum_{\substack{p\in I_r \\ \|\Delta w_p - \xi\| < \zeta}} \|\Delta w_p - \xi\|$$

$$\leq \frac{H}{h_r}\left|\{p \in I_r : \|\Delta w_p - \xi\| \geq \zeta\}\right| + \zeta,$$

so, we obtain N_θ-$\lim \Delta w_p = \xi$. □

Next, we give an example to demonstrate that the assumption over the sequence to be bounded is necessary and cannot be removed.

Example 2.2. There exist unbounded $S_\theta(\Delta)$-convergent sequences to ξ which are not $N_\theta(\Delta)$-summable to ξ. Let $\theta = \{n_r\}$ be a lacunary sequence with $n_0 = 0$ and $n_r = 2^r$. Consider that

$$h_1 = n_1 - n_0 = 2 \text{ and } h_r = 2^{r-1} \text{ for every } r \geq 2,$$

$$I_1 = (n_0, n_1] = (0, 2] \text{ and } I_r = \left(2^{r-1}, 2^r\right] \text{ for every } r \geq 2.$$

Think the sequence determined by

$$\Delta w_p = \begin{cases} 0, & \text{if } p \neq 2^j \text{ for all } j, \\ 2^j, & \text{if } p = 2^j \text{ for all } j. \end{cases}$$

Given $\zeta > 0$, it is simply to denote that

$$\frac{|\{p \in I_r : \|\Delta w_p - 0\| \geq \zeta\}|}{h_r} \to 0 \text{ as } r \to \infty,$$

which gives that S_θ-$\lim \Delta w_p = 0$. Also, note that (w_p) is an unbounded sequence. However,

$$\frac{\sum_{p\in I_r} |\Delta w_p - 0|}{h_r} = \left\{ \begin{array}{ll} \frac{2}{2} = 1, & \text{if } r = 1, \\ \frac{2^r}{2^{r-1}} = 2, & \text{if } r \geq 2 \end{array} \right\} \to 2, \text{ as } r \to \infty$$

which gives that N_θ-$\lim \Delta w_p \neq 0$.

Definition 2.3. *Take Y as a Banach space. A sequence $w = (w_p)$ is named to be lacunary Δ-statistically Cauchy sequence if there exists a subsequence $(w_{p'(r)})$ of (w_p) sucht that $p'_r \in I_r$, for every $r \in \mathbb{N}$, $\lim_{r\to\infty} \Delta w_{p'(r)} = \xi$, for some $\xi \in Y$ and for each $\zeta > 0$,*

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{p \in I_r : \left\|\Delta w_p - \Delta w_{p'(r)}\right\| \geq \zeta\right\}\right| = 0,$$

or equivalently,

$$\lim_{r\to\infty} \frac{1}{h_r}\left|\left\{p \in I_r : \left\|\Delta w_p - \Delta w_{p'(r)}\right\| < \zeta\right\}\right| = 1.$$

In this case, we say that (w_p) is $S_\theta(\Delta)$-Cauchy.

The following consequence is acquired for sequences in Banach spaces, and we involve the proof for the sake of completeness.

Theorem 2.3. *Take Y as a Banach space. A sequence $w=(w_p)$ is $S_\theta(\Delta)$-convergent iff it is $S_\theta(\Delta)$-Cauchy.*

Proof. Let $w=(w_p)$ be an $S_\theta(\Delta)$-convergent sequence in Y and for each $p\in\mathbb{N}$, we determine

$$K_q=\left\{p\in\mathbb{N}:\|\Delta w_p-\xi\|<\frac{1}{q}\right\}.$$

Observe that $K_q\supseteq K_{q+1}$ and $\frac{\operatorname{card}(K_q\cap I_r)}{h_r}\to 1$ as $r\to\infty$.

Establish m_1 such that $r\leq m_1$ then $\operatorname{card}(K_1\cap I_r)/h_r>0$, i.e., $K_1\cap I_r\neq\emptyset$. Next, select $m_2>m_1$ such that if $r\geq m_2$, then $K_2\cap I_r\neq\emptyset$. Now, for each $m_1\leq r\leq m_2$, we select $p'_r\in I_r$ such that $p'_r\in I_r\cap K_1$, i.e., $\left\|\Delta w_{p'(r)}-\xi\right\|<1$. Technically, we select $m_{k+1}>m_k$, such that if $r>m_{k+1}$, then $I_r\cap K_{k+1}\neq\emptyset$. So, for all r such that $m_k\leq r<m_{k+1}$, we select $p'_r\in I_r\cap K_k$, and we obtain $\left\|\Delta w_{p'(r)}-\xi\right\|<\frac{1}{k}$.

Therefore, we get a sequence (p'_r) such that $p'_r\in I_r$ for every $r\in\mathbb{N}$ and $\lim_{r\to\infty}\Delta w_{p'(r)}=\xi$. As a result, we acquire

$$\frac{1}{h_r}\left|\left\{p\in I_r:\left\|\Delta w_p-\Delta w_{p'(r)}\right\|\geq\zeta\right\}\right|\leq\frac{1}{h_r}\left|\left\{p\in I_r:\|\Delta w_p-\xi\|\geq\frac{\zeta}{2}\right\}\right|+\frac{1}{h_r}\left|\left\{p\in I_r:\left\|\Delta w_{p'(r)}-\xi\right\|\geq\frac{\zeta}{2}\right\}\right|.$$

Since S_θ-$\lim\Delta w_p=\xi$ and $\lim_{r\to\infty}\Delta w_{p'(r)}=\xi$ we conclude that (w_p) is $S_\theta(\Delta)$-Cauchy.

Conversely, if (w_p) is $S_\theta(\Delta)$-Cauchy sequence, for every $\zeta>0$,

$$\left|\left\{p\in I_r:\|\Delta w_p-\xi\|\geq\zeta\right\}\right|\leq\left|\left\{p\in I_r:\left\|\Delta w_p-\Delta w_{p'(r)}\right\|\geq\tfrac{\zeta}{2}\right\}\right|+\left|\left\{p\in I_r:\left\|\Delta w_{p'(r)}-\xi\right\|\geq\tfrac{\zeta}{2}\right\}\right|.$$

Since (w_p) is $S_\theta(\Delta)$-Cauchy and $\lim_{r\to\infty}\Delta w_{p'(r)}=\xi$, we conclude that S_θ-$\lim\Delta w_p=\xi$. □

Now, we examine some features of the statistical lacunary summability spaces for Banach spaces.

Let us think Y a real Banach space, $\sum_j\Delta w_j$ a series in Y and $\theta=(n_r)$ a lacunary sequence. We identify

$$S_{S_\theta}\Big(\sum_j\Delta w_j\Big)=\left\{(a_j)_j\in l_\infty:\sum_j a_j\Delta w_j\text{ is }S_\theta\text{-summable}\right\}$$

endowed with the supremum norm. The space will be called as the space of $S_\theta(\Delta)$-summability connected with $\sum_j\Delta w_j$. We will describe the completeness of the space $S_{S_\theta}(\sum_j\Delta w_j)$ in Theorem 2.4, but first we have to give the following Lemma.

Lemma 2.1. *Let Y be a Banach space and presume that the series $\sum_j \Delta w_j$ is not wuC. Then, there is $f \in Y^*$ and a null sequence $(a_j)_j \in c_0$ such that*

$$\sum_j a_j f(\Delta w_j) = +\infty$$

and

$$a_j f(\Delta w_j) \geq 0.$$

Proof. Since $\sum_{j=1}^{\infty} |f(\Delta w_j)| = +\infty$, there exists t_1 such that $\sum_{j=1}^{t_1} |f(\Delta w_j)| > 2 \cdot 2$. We itendify $a_j = \frac{1}{2}$ if $f(\Delta w_j) \geq 0$ and $a_j = -\frac{1}{2}$ if $f(\Delta w_j) < 0$ for $j = \{1, 2, ..., t_1\}$. This gives that $\sum_{j=1}^{t_1} a_j f(\Delta w_j) > 2$ and $a_j f(\Delta w_j) \geq 0$ if $j = \{1, 2, ..., t_1\}$. Let $t_2 > t_1$ be such that $\sum_{j=t_1+1}^{t_2} |f(\Delta w_j)| > 2^2 \cdot 2^2$. We determine $a_j = \frac{1}{2^2}$ if $f(\Delta w_j) \geq 0$ and $a_j = -\frac{1}{2^2}$ if $f(\Delta w_j) < 0$ for $j = \{t_1 + 1, ..., t_2\}$. Hence, $\sum_{j=t_1+1}^{t_2} a_j f(\Delta w_j) > 2^2$ and $a_j f(\Delta w_j) \geq 0$ if $j = \{t_1 + 1, ..., t_2\}$. So, we have acquired a sequence $(a_j)_j \in c_0$ with the above features. □

Theorem 2.4. *Let Y be a Banach space and $\theta = \{n_r\}$ a lacunary sequence. The subsequent are equivalent:*

(i) *The series $\sum_j \Delta w_j$ is wuC.*

(ii) *The space $S_{S_\theta}(\sum_j \Delta w_j)$ is complete.*

(iii) *The space c_0 of all null sequences is included in $S_{S_\theta}(\sum_j \Delta w_j)$.*

Proof. $(i) \Rightarrow (ii)$: Since $\sum_j \Delta w_j$ is wuC, the subsequent supremum is finite:

$$Q = \sup \left\{ \left\| \sum_{j=1}^{n} \beta_j \Delta w_j \right\| : |\beta_j| \leq 1,\ 1 \leq j \leq n,\ n \in \mathbb{N} \right\} < +\infty.$$

Let $(\beta^s)_s \subset S_{S_\theta}(\sum_j \Delta w_j)$ such that $\lim_s \left\| \beta^s - \beta^0 \right\|_\infty = 0$, with $\beta^0 \in l_\infty$. We will denote that $\beta^0 \in S_{S_\theta}(\sum_j \Delta w_j)$. Let us assume without any loss of generality that $\left\| \beta^0 \right\|_\infty \leq 1$. Then, the partial sums $S_p^0 = \sum_{j=1}^{p} \beta_j^0 \Delta w_j$ satisfy $\left\| S_p^0 \right\| \leq Q$ for every $p \in \mathbb{N}$, i.e., the sequence $\left(S_p^0\right)$ is bounded. Then, $\beta^0 \in S_{S_\theta}(\sum_j \Delta w_j)$ iff $\left(S_p^0\right)$ is $S_\theta(\Delta)$-summable to some $\xi \in Y$. In accordance with Theorem 2.3, $\left(S_p^0\right)$ is lacunary Δ-statistically convergent to $\xi \in Y$ iff $\left(S_p^0\right)$ is lacunary Δ-statistically Cauchy sequence.

Let $\zeta > 0$ and $n \in \mathbb{N}$. Then, we acquire statement (ii) if we indicate that there is a subsequence $\left(S_{p'(r)}\right)$ such that $p'_r \in I_r$ for every $r \in \mathbb{N}$, $\lim_{r \to \infty} S_{p'(r)} = \xi$ and

$$d_\theta \left(\left\{ p \in I_r : \left\| S_p^0 - S_{p'(r)}^0 \right\| < \zeta \right\} \right) = 1.$$

Since $\beta^s \to \beta^0$ in l_∞, there is $s_0 > n$ such that $\left\|\beta^s - \beta^0\right\|_\infty < \frac{\zeta}{4Q}$ for all $s > s_0$, and since $S_p^{s_0}$ is $S_\theta(\Delta)$-Cauchy, there is $p'_r \in I_r$ such that $\lim_{r\to\infty} S_{p'(r)}^{s_0} = \xi$ for some ξ and

$$d_\theta\left(\left\{p \in I_r : \left\|S_p^{s_0} - S_{p'(r)}^{s_0}\right\| < \frac{\zeta}{2}\right\}\right) = 1.$$

Think $r \in \mathbb{N}$ and fix $p \in I_r$ such that

$$\left\|S_p^{s_0} - S_{p'(r)}^{s_0}\right\| < \frac{\zeta}{2}. \tag{1}$$

We will signify that $\left\|S_p^0 - S_{p'(r)}^0\right\| < \zeta$, and this will evidence that

$$\left\{p \in I_r : \left\|S_p^{s_0} - S_{p'(r)}^{s_0}\right\| < \frac{\zeta}{2}\right\} \subset \left\{p \in I_r : \left\|S_p^0 - S_{p'(r)}^0\right\| < \zeta\right\}.$$

Since the first set has density 1, the second will also have density 1 and we will be done.

Let us observe first that for each $i \in \mathbb{N}$,

$$\left\|\sum_{j=1}^{i} \frac{4Q}{\zeta}\left(\beta_j^s - \beta_j^{s_0}\right)\Delta w_j\right\| \le Q,$$

for every $s > s_0$, therefore

$$\left\|S_i^0 - S_i^{s_0}\right\| = \left\|\sum_{j=1}^{i}\left(\beta_j^0 - \beta_j^{s_0}\right)\Delta w_j\right\| \le \frac{\zeta}{4}. \tag{2}$$

Then, by using the triangular inequality,

$$\begin{aligned}\left\|S_p^0 - S_{p'(r)}^0\right\| &\le \left\|S_p^0 - S_p^{s_0}\right\| + \left\|S_p^{s_0} - S_{p'(r)}^{s_0}\right\| + \left\|S_{p'(r)}^{s_0} - S_{p'(r)}^0\right\| \\ &< \tfrac{\zeta}{4} + \tfrac{\zeta}{2} + \tfrac{\zeta}{4} = \zeta.\end{aligned}$$

Therefore, by applying (1) and (2), the last inequality yields the desired result.

(ii) $\Rightarrow$ *(iii)*: Let us observe that if $S_{S_\theta}(\sum_j \Delta w_j)$ is complete, then it includes the space of ultimately zero sequences c_{00} and therefore the thesis comes, since the supremum norm completion of c_{00} is c_0.

(iii) $\Rightarrow$ *(i)*: By utilizing the contradiction, presume that the series $\sum \Delta w_j$ is not wuC. So, there is $f \in Y^*$ such that $\sum_{j=1}^{\infty} |f(\Delta w_j)| = +\infty$. By Lemma 2.1, we can create technically a sequence $(\beta_j)_j \in c_0$ such that

$$\sum_j \beta_j f(\Delta w_j) = +\infty$$

and

$$\beta_j f(\Delta w_j) \ge 0.$$

Now, we will examine that the sequence $(S_p) = (\sum_{j=1}^{p} \beta_j f(\Delta w_j))$ is not $S_\theta(\Delta)$-summable to any $\xi \in \mathbb{R}$. By utilizing the contradiction, assume that it is $S_\theta(\Delta)$-summable to $\xi \in \mathbb{R}$, then we obtain

$$\frac{1}{h_r} |\{p \in I_r : |S_p - \xi| \geq \zeta\}| = \frac{1}{h_r} \sum_{\substack{p=n_r-1 \\ |S_p-\xi|\geq\zeta}}^{n_r} 1 \to 0 \text{ as } r \to \infty.$$

Since S_p is an inreasing sequence and $S_p \to \infty$, there is n_0 such that $|S_p - \xi| \geq \zeta$ for every $p \geq n_0$. Let us presume that $n_r > n_0$ for every r. Consequently,

$$\frac{1}{h_r} \sum_{\substack{n=n_r-1 \\ |S_p-\xi|\geq\zeta}}^{n_r} 1 = \frac{h_r}{h_r} = 1 \nrightarrow 0 \text{ as } r \to \infty,$$

a contradiction. This gives that (S_p) is not $S_\theta(\Delta)$-convergent and is a contradiction with (iii). □

Now, we examine some features of the lacunary strongly Δ-summability space for Banach spaces.

Let Y a real Banach space, $\sum_j \Delta w_j$ a series in Y and $\theta = (n_r)$ a lacunary sequence. We itendify

$$S_{N_\theta}(\sum_j \Delta w_j) = \left\{ (a_j)_j \in l_\infty : \sum_j a_j \Delta w_j \text{ is } N_\theta\text{-summable} \right\}$$

endowed with the supremum norm. This will be characterised as the space of $N_\theta(\Delta)$-summability connected with the series $\sum_j \Delta w_j$. We can now give a theorem very same as that of Theorem 2.4 but for the case of $N_\theta(\Delta)$-summability. Actually Theorem 2.5 describes the completeness of the space $S_{N_\theta}(\sum_j \Delta w_j)$.

Theorem 2.5. *Let Y be a real Banach space and $\theta = (n_r)$ a lacunary sequence. The subsequent are equivalent:*

- (i) *The series $\sum_j \Delta w_j$ is wuC.*
- (ii) *The space $S_{N_\theta}(\sum_j \Delta w_j)$ is complete.*
- (iii) *The space c_0 of all null sequences is included in $S_{N_\theta}(\sum_j \Delta w_j)$.*

Proof. $(i) \Rightarrow (ii)$: Since $\sum_j \Delta w_j$ is wuC, the subsequent supremum is finite:

$$Q = \sup \left\{ \left\| \sum_{j=1}^{k} \beta_j \Delta w_j \right\| : |\beta_j| \leq 1, 1 \leq j \leq k, k \in \mathbb{N} \right\} < \infty.$$

Let $(\beta^s)_s \subset S_{N_\theta}(\sum_j \Delta w_j)$ such that $\lim_s \left\|\beta^s - \beta^0\right\|_\infty = 0$, with $\beta^0 \in l_\infty$. We will denote that $\beta^0 \in S_{N_\theta}(\sum_j \Delta w_j)$. With no loss of generality, we can presume that $\left\|\beta^0\right\|_\infty \leq 1$. So, the partial sums $S_p^0 = \sum_{j=1}^p \beta_j^0 \Delta w_j$ satisfy $\left\|S_p^0\right\| \leq Q$ for every $p \in \mathbb{N}$, i.e., the sequence $\left(S_p^0\right)$ is bounded. Then, $\beta^0 \in S_{N_\theta}(\sum_j \Delta w_j)$ iff $\left(S_p^0\right)$ is $N_\theta(\Delta)$-summable to some $\xi \in Y$. Since $\left(S_p^0\right)$ is bounded, it is sufficient to show that (S_p) is $S_\theta(\Delta)$-convergent, as a consequence of Theorem 2.1 due to Fridy and Orhan [15]. The results follows similarly as in Theorem 2.4.

$(ii) \Rightarrow (iii)$: It is adequate to observe that $S_{N_\theta}(\sum_j \Delta w_j)$ is a complete space and it includes the space of ultimately zero sequences c_{00}, so it involves the completion of c_{00} with regards to the supremum norm, hence it includes c_0.

$(iii) \Rightarrow (i)$: By utilizing the contradiction, presume that the series $\sum \Delta w_j$ is not wuC. So, there is $f \in Y^*$ such that $\sum_{j=1}^\infty |f(\Delta w_j)| = +\infty$. By Lemma 2.1, we can create technically a sequence $(\beta_j)_j \in c_0$ such that $\sum_j \beta_j f(\Delta w_j) = +\infty$ and $\beta_j f(\Delta w_j) \geq 0$.

The sequence $S_p = \sum_{j=1}^p \beta_j f(\Delta w_j)$ is not $N_\theta(\Delta)$-summable to any $\xi \in \mathbb{R}$. Since $S_p \to \infty$, for every $H > 0$, there is p_0 such that $|S_p| > H$ if $p > p_0$. Then, we acquire

$$\frac{1}{h_r} \sum_{p \in I_r} |S_p| > \frac{h_r Q}{h_r} = Q.$$

Hence S_p is not $N_\theta(\Delta)$-summable to any $\xi \in \mathbb{R}$, on the other hand

$$\infty \leftarrow \frac{1}{h_r} \sum_{p \in I_r} |S_p| \leq |\xi| + \sum_{p \in I_r} |S_p - \xi| \to |\xi|$$

We can deduce that S_p is not $N_\theta(\Delta)$-convergent, a contradiction with (iii). □

A Banach space Y can be characterized by the completeness of the space $S_{N_\theta}(\sum_p \Delta w_p)$ for every wuC series $\sum_p \Delta w_p$, as we will show, nextly.

Theorem 2.6. *Take Y as a normed real vector space. Then, Y is complete iff $S_{N_\theta}(\sum_p \Delta w_p)$ is a complete space for every wuC series $\sum_p \Delta w_p$.*

Proof. The necessary condition is obvious from Theorem 2.4. Now, suppose that Y is not complete, hence there is a series $\sum_p \Delta w_p$ in Y such that $\|\Delta w_p\| \leq \frac{1}{p 2^p}$ and $\sum \Delta w_p = w^{**} \in Y^{**} \diagdown Y$. We will provide a wuC series $\sum_p \Delta y_p$ such that $S_{N_\theta}(\sum_p \Delta y_p)$ is not complete, a contradiction. Set $S_M = \sum_{p=1}^M \Delta w_p$. As Y^{**} is a Banach space endowed with the dual topology, $\sup_{\|y^*\| \leq 1} |y^*(S_M) - w^{**}(y^*)|$ tends to 0 as $M \to \infty$, i.e.,

$$\lim_{M \to +\infty} y^*(S_M) = \lim_{M \to +\infty} \sum_{p=1}^M y^*(\Delta w_p) = w^{**}(y^*), \tag{3}$$

for every $\|y^*\| \leq 1$. Put $\Delta y_p = p \Delta w_p$ and let us observe that $\|\Delta y_p\| < \frac{1}{2^p}$. Therefore, $\sum_p \Delta y_p$ is absolutely convergent, so it is unconditionally convergent and weakly unconditionally Cauchy. We claim that the series $\sum_p \frac{1}{p}\Delta y_p$ is not N_θ-summable in Y. Using contradiction assume that $S_M = \sum_{p=1}^{M} \frac{1}{p}\Delta y_p$ is N_θ-summable in Y, i.e., there exists ξ in Y such that $\lim_{r\to\infty} \frac{1}{h_r}\sum_{p\in I_r} \|S_p - \xi\| = 0$. This gives that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{p\in I_r} y^*(S_p) = y^*(\xi), \tag{4}$$

for every $\|y^*\| \le 1$. By the relations (3) and (4), the uniqueness of the limit and since $N_\theta(\Delta)$ is a regular method, we get $w^{**}(y^*) = y^*(\xi)$ for every $\|y^*\| \le 1$, so we acquire $w^{**} = \xi \in Y$, a contradiction. Hence, $S_M = \sum_{p=1}^{M} \frac{1}{p}\Delta y_p$ is not N_θ-summable to any $\xi \in Y$.

Finally, let us observe that since $\sum_p \Delta y_p$ is a weakly unconditionally Cauchy series and $S_M = \sum_{p=1}^{M} \frac{1}{p}\Delta y_p$ is not N_θ-summable, we get $\left(\frac{1}{p}\right) \notin S_{N_\theta}(\sum_p \Delta y_p)$ and this means that $c_0 \nsubseteq S_{N_\theta}(\sum_p \Delta y_p)$ which contradicts Part (iii) of Theorem 2.5. This completes the proof. □

Definition 2.4. *Let $0 < p < \infty$, the sequence $w = (w_n)$ is named to be strongly (p, Δ)-Cesàro or $|\sigma_p|(\Delta)$-summable if there is $\xi \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} \frac{1}{t} \sum_{i=1}^{t} \|\Delta w_i - \xi\|^p = 0$$

and is writen as $(\Delta w_n) \underset{|\sigma_p|}{\to} \xi$ or $|\sigma_p| - \lim_{n\to\infty} \Delta w_n = \xi$.

Let $\sum \Delta w_i$ be a series in a real Banach space Y,

$$S_{|\sigma_p|}(\sum_i \Delta w_i) = \left\{ (a_i)_i \in l_\infty : \sum_i a_i \Delta w_i \text{ is } |\sigma_p|\text{-summable} \right\}$$

endowed with the supremum norm.

Corollary 2.1. *Take Y as a normed real vector space and $p \ge 1$. The subsequent are equivalent:*

(i) Y is complete.

(ii) $S_{N_\theta}(\sum_p \Delta w_p)$ is complete for every wuC series $\sum_p \Delta w_p$.

(iii) $S_{S_\theta}(\sum_p \Delta w_p$ is complete for every wuC series $\sum_p \Delta w_p$.

(iv) $S_{|\sigma_p|}(\sum_p \Delta w_p)$ is complete, for every wuC series $\sum_p \Delta w_p$.

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