A study on Riesz I-convergence in intuitionistic fuzzy normed spaces

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Abstract. The primary objective of this study is to introduce the concept of ideal convergent sequences as a domain of regular Riesz triangular matrix in the settings of intuitionistic fuzzy normed spaces(IFNS). Some properties of this notion with respect to the intuitionistic fuzzy norm are also presented in this study. We demonstrate the Riesz ideal Cauchy criterion in intuitionistic fuzzy normed spaces and later on, we show that in an arbitrary IFNS, a sequence is Riesz ideal convergent if and only if it satisfies Riesz ideal Cauchy criterion. We also present major counterexamples for the converse part of some results. Lastly, we define the notion of Riesz I^*–convergence in intuitionistic fuzzy normed spaces and establish the relationship with Riesz I–convergence in IFNS with a certain counterexample.

Keywords: intuitionistic fuzzy normed space, Riesz matrix, I–convergence, I–Cauchy.

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1. Introduction and preliminaries

The field of fuzzy topology plays a pivotal role to obtain significant applications in the theory of quantum particle physics [40]. Zadeh [47] gave a major breakthrough in this field by introducing the idea of fuzzy set, afterward many authors came forward to establish fuzzy analogs of classical theories. Atanassov ([2], [3], [4], [5]) defined intuitionistic fuzzy sets (IFS) and the characteristics of these IFS are given by Deschrijver and Kerre [13]. Inspired by these notions, Coker ([9], [10]) introduced intuitionistic fuzzy topological spaces. Saadati and Park [42, 43] studied these spaces and their generalization which helped them to obtain the concept of intuitionistic fuzzy normed space(IFNS). Mursaleen [36] presented the notion of statistical convergence with respect to the intuitionistic fuzzy norm and proved some fundamental results.

Definition 1.1 ([42]). *The five-tuple $(X, f_1, f_2, *, \diamond)$ is said to be an intuitionistic fuzzy normed space if X is a linear space over a field F, $*$ is a continuous t-norm, $\diamond$ is a continuous t-co-norm and f_1 & f_2 are the fuzzy sets on $X\times(0,\infty)$ satisfy the following conditions $\forall\ y, z \in X$ and $s, t\ > 0$:*

(a) $f_1(y,t)\ +\ f_2(y,t) \leq 1$;

(b) $f_1(y,t)\ > 0$;

(c) $f_1(y,t)\ =\ 1$ *iff* $y = 0$;

(d) $f_1(cy,t)\ =\ f_1(y, \frac{t}{|c|})$, $\forall c \neq 0,\ c \in F$;

(e) $f_1(y,t) * f_1(z,s) \leq f_1(y+z, t+s)$;

(f) $f_1(y,t) : (0,\infty) \to [0,1]$ *is continuous in* t;

(g) $\lim_{t\to\infty} f_1(y,t)\ =\ 1$ *and* $\lim_{t\to 0} f_1(y,t)\ =\ 0$;

(i) $f_2(y,t)\ > 0$;

(j) $f_2(y,t)\ =\ 0$ *iff* $y = 0$;

(l) $f_2(cy,t)\ =\ f_2(y, \frac{t}{|c|}$, $\forall c \neq 0,\ c \in F$;

(m) $f_2(y,t) \diamond f_2(z,s) \geq f_2(y+z, t+s)$;

(n) $f_2(y,t) : (0,\infty) \to [0,1]$ *is continuous in* t;

(o) $\lim_{t\to\infty} f_2(y,t)\ =\ 0$ *and* $\lim_{t\to 0} f_2(y,t)\ =\ 1$.

In this case, (f_1, f_2) is called intuitionistc fuzzy norm.

Remark 1.1. Hosseini et al. [22] and many others have defined intuitionistic fuzzy normed space by a more complete definition. This study can be studied as a more extended case in this novel setting. The current definition 1.1, however, simplifies the computational aspects.

Example 1.1. If $(X, ||\bullet||)$ forms a normed linear space, let for all $a, b \in [0,1]$, t-norm is defined as $a * b = ab$ and t-co-norm is defined as $a \diamond b = \min\{a+b, 1\}$, then for any $y \in X$ and $\forall\ t > 0$, consider

$$\phi(y,t) = \frac{t}{t + ||y||} \quad \text{and} \quad \psi(y,t) = \frac{||y||}{t + ||y||}$$

Then, $(X, \phi, \psi, *, \diamond)$ forms an intuitionistic fuzzy normed space.

Lemma 1.1 ([42]). *If $m_i \in (0,1),\ i = 1$ to 7. $*$ and $\diamond$ are continuous $t-$norm and continuous $t-$conorm, respectively. Then:*

(1) *If $m_1 > m_2$, $\exists\ m_3, m_4 \in (0,1)$ s.t. $m_1 * m_3 \geq m_2$ and $m_1 \geq m_2 \diamond m_4$.*

(2) *If $m_5 \in (0,1)$, $\exists\ m_6, m_7 \in (0,1)$ s.t. $m_6 * m_6 \geq m_5$ and $m_5 \geq m_7 \diamond m_7$.*

Definition 1.2 ([42]). *In an IFNS $(X, f_1, f_2, *, \diamond)$, a sequence (y_k) is said to be convergent to ζ if for a given $\epsilon > 0$ and $t > 0$, $\exists\ k_0 \in \mathbb{N}$ such that*

$$f_1(y_k - \zeta, t) > 1 - \epsilon \quad \textit{and} \quad f_2(y_k - \zeta, t) < \epsilon,\ \forall\ k \geq k_0.$$

Definition 1.3 ([42]). *In an IFNS $(X, f_1, f_2, *, \diamond)$, a sequence (y_k) is said to be Cauchy if for a given $\epsilon > 0$ and $t > 0$, $\exists\ k_0 \in \mathbb{N}$ such that*

$$f_1(y_k - y_j, t) > 1 - \epsilon \quad \textit{and} \quad f_2(y_k - y_j, t) < \epsilon,\ \forall\ j, k \geq k_0.$$

The notion of convergence of sequence has become a useful notion in the fundamental theory of functional analysis and plays a key role, especially in sequence space.

In 1951, Fast [14] gave the concept of statistical convergence which is a very important extension of usual convergence and is being used widely in different areas of science and technology. Due to the versatility of this concept, various forms of statistical convergence are introduced and these forms are further defined in different settings, e.g. [35, 36]. In the race of inventing the most extended form of statistical convergence, Kostyrko et al. [33] introduced the idea of I-convergence using the notion of ideal defined on $\mathbb{N}$. Nowadays it has become a more important form than many other forms of convergence (see [33]). If $I \subset P(X)$ of any set X with a) $\phi \in I$, b) $A \cup B \in I$ for all $A, B \in I$ and c) $\forall\ A \in I$ and $B \subset A$ then $B \in I$. Then I is called an ideal. If $I \neq 2^X$ then I is called non trivial ideal. If $\{\{x\} : x \in X\} \subset I$ then I is called admissible ideal. If $\mathcal{F} \subset P(X)$ of a set X then $\mathcal{F}$ is called filter if a) $\phi \notin \mathcal{F}$, b) $A \cap B \in \mathcal{F}$ for all $A, B \in \mathcal{F}$, c) $\forall\ A \in \mathcal{F}$ and $A \subset B$ then $B \in \mathcal{F}$. Ŝalát et al. [44, 45] studied the characterization of ideal convergence and also defined the ideal convergence field. Later, many others (e.g. [15, 24, 25, 39]) further investigated the notion of I-convergence from the sequence space point of view and linked it with the summability theory. In addition, numerous researchers are working on the various extended versions of ideal convergence of sequence and further introduced them in several important spaces as Hazarika [18, 19] introduced the ideal convergence for sequence and double sequence in fuzzy normed space and gave the salient features of this notion in fuzzy norm-setting. Mursaleen and Mohiuddine [37, 38] further studied this concept for multiple sequences with respect to the intuitionistic fuzzy norm as well as probabilistic norm. Moreover, Hazarika [17, 20, 21] also extended several forms of ideal convergence in different spaces like random $2-$normed space, probabilistic normed space, intuitionistic fuzzy normed space. Debnath [11, 12] defined ideal convergence via lacunary and lacunary difference mean in intuitionistic fuzzy normed space and established key results with respect to IFN. Recently, Khan et al. [26, 27] also studied the notion of ideal convergence as a domain of the Nörlund matrix and generalized difference matrix, respectively in intuitionistic fuzzy normed space. Khan et al. also defined their respective intuitionistic fuzzy ideal convergent sequence spaces and proved their topological properties. Other important notions with respect to the intuitionistic fuzzy norm, one can refer to [28, 29, 30, 31, 32].

Proposition 1.1 ([33])**.** *Class* $\mathcal{F}(I) = \{A \subset X : A = X\backslash B,$ *for some* $B \in I\}$ *is a filter on* X*, where* $I \subset P(\mathbb{N})$ *is a non trivial ideal.*

$\mathcal{F}(I)$ *is known as the filter associated with the ideal* I*.*

Definition 1.4 ([33])**.** *Let* $I \subset P(\mathbb{N})$ *is a non-trivial ideal, a sequence* $(y_k) \in \omega$ *is called to be* $I-$*convergent to* $\zeta \in \mathbb{R}$ *if* $\forall\ \epsilon > 0$,

$$\{k \in \mathbb{N} : \mid y_k - \zeta \mid \geq \epsilon\} \in I.$$

We denote it as $I - \lim(y_k) = \zeta$.

Definition 1.5 ([33])**.** *Let* $I \subset P(\mathbb{N})$ *is a non-trivial ideal, a sequence* $(y_k) \in \omega$ *is called to be* $I-$*Cauchy if* $\forall\ \epsilon > 0$, *there exists a* $K = K(\epsilon)$ *such that*

$$\{k \in \mathbb{N} : \mid y_k - y_K \mid \geq \epsilon\} \in I.$$

Definition 1.6 ([33])**.** *Let* $I \subset P(\mathbb{N})$ *is a non-trivial ideal,* I^*-*convergence of a sequence* $(y_k) \in \omega$ *to number* $\zeta \in \mathbb{R}$*(i.e.* $I^* - \lim y = \zeta$*) is defined as if there exists a set* $M \in I$*, s.t. for* $A = \mathbb{N}\backslash M = \{k_i \in \mathbb{N} : k_i < k_{i+1},$ *for all* $i \in \mathbb{N}\}$ *we have,* $\lim_{k\to\infty} y_{i_k} = \zeta$.

Definition 1.7 ([33])**.** *An admissible ideal* $I \subset P(\mathbb{N})$ *is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets* $\{A_1, A_2, ...\}$ *belonging to I, there exists a countable family* $\{B_1, B_2, ...\}$ *in* I *such that* $A_i \Delta B_i$ *is a finite set for each* $i \in \mathbb{N}$ *and* $B = \cup_{i=1}^{\infty} B_i \in I$*; where* Δ *is the symmetric difference.*

As a bounded linear operator on the space of all p-summable sequence l_p, several authors used the riesz matrix $\mathcal{R}_n^b$ in the different disciplines of sequence space. Recall in [1, 8, 7, 16, 23, 34, 46] for a sequence of positive numbers (b_k), the entries of infinite riesz matrix $\mathcal{R}_j^b = (r_{jk}^b)$ is defined as

$$r_{jk}^b = \begin{cases} \dfrac{b_k}{\mathcal{B}_j}, & 0 \leq k \leq j \\ 0, & k \geq j. \end{cases} \tag{1}$$

where $\mathcal{B}_j = \sum_{k=0}^{j} b_k$. By (1) clearly, Riesz matrix is lower triangular and is regular if $\mathcal{B}_j \to \infty$ as $j \to \infty$, (see [1, 6, 23, 41]). Recently by using the notion of $I-$convergence and domain of Riesz matrix $\mathcal{R}_j^b$, Khan et al. [24] introduced Riesz $I-$convergent sequence space

$$c^I(\mathcal{R}_j^b) := \{y = (y_k) \in \omega : \{j \in \mathbb{N} : |\mathcal{R}_j^b(y) - L| \geq \varepsilon \text{ for some } L \in \mathbb{R}\} \in I\}, \tag{2}$$

$$c_0^I(\mathcal{R}_j^b) := \left\{ y = (y_k) \in \omega : \{j \in \mathbb{N} : |\mathcal{R}_j^b(y)| \geq \varepsilon\} \in I \right\}, \tag{3}$$

where

$$\mathcal{R}_j^b(y) := \frac{1}{\mathcal{B}_j} \sum_{k=0}^{j} b_k y_k \quad \text{for all } j \in \mathbb{N}. \tag{4}$$

Clearly, Riesz I–convergent sequence is a more generalized form of ideal convergent sequence and each I–convergent sequence is Riesz I–convergent sequence but converse is not true.

Example 1.2. Let I is an ideal defined on the set of natural numbers such that it contains the subsets of natural numbers whose natural density is zero. If we take sequence (b_k) as $b_k = k$ for all k and sequence (y_k) as

$$\mathcal{R}_j^b(y) = \begin{cases} 1, & \text{if } j = m^2, \ (m \in \mathbb{N}) \\ 0, & \text{otherwise.} \end{cases}$$

Then, it is obvious that sequence (y_k) is Riesz I–convergent to 0 but sequence (y_k) is not I–convergent.

Since Riesz ideal convergence is a novel and a more extended variant of ideal convergence and IFNS is a unified and generalized space of various important spaces so these facts motivate us to define Riesz ideal convergence in intuitionistic fuzzy norm-setting.

2. Main results

In this particular section, we define Riesz I-convergence in intuitionistic fuzzy normed space and try to give some theorems about it. Throughout the paper, we assume that sequence $y = (y_k)$ and $\mathcal{R}_j^b(y)$ are related as (4) and I is an admissible ideal of subset of $\mathbb{N}$.

Definition 2.1. *In an IFNS $(X, f_1, f_2, *, \diamond)$, a sequence (y_k) is said to be Riesz I-convergent to ζ if for a given $\epsilon > 0$ and $s > 0$, the following set*

$$\{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y) - \zeta, s) \leq 1 - \epsilon \text{ or } f_2(\mathcal{R}_j^b(y) - \zeta, s) \geq \epsilon\} \in I.$$

In this case, we say Riesz $I_{(f_1,f_2)}-$ limit of sequence (y_k) is ζ and denote it as $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim y = \zeta$.

Proposition 2.1. *Let $(X, f_1, f_2, *, \diamond)$ is an IFNS and $y = (y_k)$ is a sequence in X. Then for every $\epsilon > 0$ and $s > 0$, these following are equivalent:*

(a) $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim y = \zeta$.

(b) $\{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y) - \zeta, s) \leq 1 - \epsilon\} \in I$ *and* $\{j \in \mathbb{N} : f_2(\mathcal{R}_j^b(y) - \zeta, s) \geq \epsilon\} \in I$.

(c) $\{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y) - \zeta, s) > 1 - \epsilon \text{ and } f_2(\mathcal{R}_j^b(y) - \zeta, s) < \epsilon\} \in \mathcal{F}(I)$.

(d) $\{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y) - \zeta, s) > 1 - \epsilon\} \in \mathcal{F}(I)$ *and* $\{j \in \mathbb{N} : f_2(\mathcal{R}_j^b(y) - \zeta, s) < \epsilon\} \in \mathcal{F}(I)$.

(e) $I - \lim_{j \to \infty} f_1(\mathcal{R}_j^b(y) - \zeta, s) = 1$ *and* $I - \lim_{j \to \infty} f_2(\mathcal{R}_j^b(y) - \zeta, s) = 0$.

Theorem 2.1. *Let $y = (y_k)$ be a sequence in IFNS $(X, f_1, f_2, *, \diamond)$. If sequence (y_k) is Riesz $I-$convergent in X, then Riesz $I_{(f_1,f_2)}-$ limit of (y_k) is unique.*

Proof of Theorem 2.1. Let on contrary that ζ_1 and ζ_2 are two different elements such that $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim y = \zeta_1$ and $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim y = \zeta_2$. For a given $\epsilon > 0$, choose $r > 0$ such that $(1 - r) * (1 - r) > 1 - \epsilon$ and $r \diamond r < \epsilon$. For $s > 0$, we define

$$\begin{aligned} A_1 &= \{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y) - \zeta_1, s) \leq 1 - r\}, \\ A_2 &= \{j \in \mathbb{N} : f_2(\mathcal{R}_j^b(y) - \zeta_1, s) \geq r\}, \\ A_3 &= \{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y) - \zeta_2, s) \leq 1 - r\}, \\ A_4 &= \{j \in \mathbb{N} : f_2(\mathcal{R}_j^b(y) - \zeta_2, s) \geq r\} \end{aligned}$$

and $A = (A_1 \cup A_3) \cap (A_2 \cup A_4)$.

Sets A_1, A_2, A_3, A_4 and A must belong to I as (y_k) has two different Riesz $I-$limit with respect to intuitionistic fuzzy norm (f_1, f_2) i.e. ζ_1 and ζ_2. Hence, $A^c \in \mathcal{F}(I)$ then A^c is non empty. Let us say some $j_0 \in A^c$ then either $j_0 \in {A_1}^c \cap {A_3}^c$ or $j_0 \in {A_2}^c \cap {A_4}^c$.

If $j_0 \in {A_1}^c \cap {A_3}^c$ which implies that

$$f_1\left(\mathcal{R}_{j_0}^b(y) - \zeta_1, \frac{s}{2}\right) > 1 - r \text{ and } f_1\left(\mathcal{R}_{j_0}^b(y) - \zeta_2, \frac{s}{2}\right) > 1 - r.$$

Hence,

$$\begin{aligned} f_1(\zeta_1-\zeta_2,s) &\geq f_1\left(\mathcal{R}^b_{j_0}(y)-\zeta_1,\frac{s}{2}\right)*f_1\left(\mathcal{R}^b_{j_0}(y)-\zeta_2,\frac{s}{2}\right)\\ &> (1-r)*(1-r)>1-\epsilon. \end{aligned}$$

As $\epsilon>0$ was arbitrary hence $f_1(\zeta_1-\zeta_2,s)=1$ for all $s>0$. So, we have $\zeta_1=\zeta_2$, which is a contradiction.

If $j_0\in {A_2}^c\cap {A_4}^c$ which implies that

$$f_2\left(\mathcal{R}^b_{j_0}(y)-\zeta_1,\frac{s}{2}\right)<r \text{ and } f_2\left(\mathcal{R}^b_{j_0}(y)-\zeta_2,\frac{s}{2}\right)<r.$$

Hence,

$$\begin{aligned} f_2(\zeta_1-\zeta_2,s) &\leq f_2\left(\mathcal{R}^b_{j_0}(y)-\zeta_1,\frac{s}{2}\right)\diamond f_2\left(\mathcal{R}^b_{j_0}(y)-\zeta_2,\frac{s}{2}\right)\\ &< r\diamond r<\epsilon. \end{aligned}$$

As $\epsilon>0$ was arbitrary hence $f_2(\zeta_1-\zeta_2,s)=0$ for all $s>0$. So, we have $\zeta_1=\zeta_2$, which is a contradiction. Hence, $y=(y_k)$ has unique Riesz $I_{(f_1,f_2)}-$limit.

Theorem 2.2. *Let $y=(y_k)$ be any sequence in IFNS $(X,f_1,f_2,*,\diamond)$ such that ordinary Riesz limit w.r.t. IFN (f_1,f_2) of (y_k) is ζ, then $\mathcal{R}^b_{I_{(f_1,f_2)}}-\lim y=\zeta$.*

Proof of Theorem 2.2. As we are given that $\mathcal{R}^b_{(f_1,f_2)}-\lim y=\zeta$, hence for any $\epsilon>0$, and $s>0$, we can find a natural number $j_0\in\mathbb{N}$ in such a way that

$$f_1(\mathcal{R}^b_j(y)-\zeta,s)>1-\epsilon \text{ and } f_2(\mathcal{R}^b_j(y)-\zeta,s)<\epsilon,$$

for all $j\geq j_0$.

Now, let

$$K=\{j\in\mathbb{N}: f_1(\mathcal{R}^b_j(y)-\zeta,s)\leq 1-\epsilon \text{ or } f_2(\mathcal{R}^b_j(y)-\zeta,s)\geq\epsilon\}.$$

As $K\subset\{1,2,....,j_0-1\}$ and I is an admissible ideal so $K\in I$. Hence, $\mathcal{R}^b_{I_{(f_1,f_2)}}-\lim y=\zeta$.

Remark 2.1. *Converse of Theorem 2.2 may not be hold in general.*

Example 2.1. In the Example 1.1, let X=$\mathbb{R}$ with usual norm, then $(\mathbb{R},f_1,f_2,*,\diamond)$ forms an IFNS.

Suppose $I=\{A\subset\mathbb{N}:\delta(A)=0\}$, where $\delta(A)$ is the natural density of the set A in $\mathbb{N}$ which is defined as $\delta(A)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\chi_A(i)$, where χ_A is the characteristic function on A, hence I is a non-trivial admissible ideal.

Now, for a positive sequence of real numbers $b=(b_k)$, let us define a sequnece $y=(y_k)$ such that

$$\mathcal{R}^b_j(y)=\begin{cases}1, & \text{if } j=m^2,\ (m\in\mathbb{N})\\ 0, & \text{otherwise.}\end{cases}$$

Now, for any $\epsilon > 0$ and $s > 0$, we define

$$\begin{aligned} K(\epsilon, s) &= \{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y), s) \leq 1-\epsilon \text{ or } f_2(\mathcal{R}_j^b(y), s) \geq \epsilon\} \\ &\text{then,} \\ K(\epsilon, s) &= \left\{ j \in \mathbb{N} : \frac{s}{s + ||\mathcal{R}_j^b(y)||} \leq 1-\epsilon \text{ or } \frac{||\mathcal{R}_j^b(y)||}{s + ||\mathcal{R}_j^b(y)||} \geq \epsilon \right\} \\ &= \left\{ j \in \mathbb{N} : |\mathcal{R}_j^b(y)| \geq \frac{\epsilon s}{1-\epsilon} > 0 \right\} \\ &\subseteq \{j \in \mathbb{N} : j = m^2, (m \in \mathbb{N})\}. \end{aligned}$$

Hence, $\delta(K(\epsilon, s)) = 0$, which implies that $K(\epsilon, s) \in I$. Hence, $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim y = 0$. On the other hand, $\mathcal{R}_j^b(y)$ is not convergent with respect to the intuitionistic fuzzy norm (f_1, f_2) as $\mathcal{R}_j^b(y)$ is not convergent in $(\mathbb{R}, ||.||)$.

Theorem 2.3. *Let $y = (y_k)$ and $z = (z_k)$ be any two sequences in IFNS $(X, f_1, f_2, *, \diamond)$ such that $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(y) = \zeta_1$ and $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(z) = \zeta_2$ then:*

(1) $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(y+z) = \zeta_1 + \zeta_2$.

(2) *For any real number α, $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(\alpha y) = \alpha\zeta_1$.*

Proof of Theorem 2.3. (1) For any $\epsilon > 0$, we may find $r > 0$ such that $(1-r) * (1-r) > 1-\epsilon$ and $r \diamond r < \epsilon$.

For $s > 0$, we define

$$\begin{aligned} A_1 &= \{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y) - \zeta_1, s) \leq 1-r\}, \\ A_2 &= \{j \in \mathbb{N} : f_2(\mathcal{R}_j^b(y) - \zeta_1, s) \geq r\}, \\ A_3 &= \{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(z) - \zeta_2, s) \leq 1-r\}, \\ A_4 &= \{j \in \mathbb{N} : f_2(\mathcal{R}_j^b(z) - \zeta_2, s) \geq r\} \\ \text{and } A &= (A_1 \cup A_3) \cap (A_2 \cup A_4). \end{aligned}$$

Sets A_1, A_2, A_3, A_4 and A must belong to I as $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(y) = \zeta_1$ and $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(z) = \zeta_2$. Hence, $A^c \in \mathcal{F}(I)$ then A^c is non empty. Now, we show that

$$A^c \subset \{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y+z) - (\zeta_1+\zeta_2), s) > 1-\epsilon \text{ and } f_2(\mathcal{R}_j^b(y+z) - (\zeta_1+\zeta_2), s) < \epsilon\}.$$

To show this we let $j_0 \in A^c$. So, we have

$$\begin{aligned} f_1\left(\mathcal{R}_{j_0}^b(y) - \zeta_1, \frac{s}{2}\right) &> 1-r, \; f_1\left(\mathcal{R}_{j_0}^b(z) - \zeta_2, \frac{s}{2}\right) > 1-r, \\ f_2\left(\mathcal{R}_{j_0}^b(y) - \zeta_1, \frac{s}{2}\right) &< r \text{ and } f_2\left(\mathcal{R}_{j_0}^b(z) - \zeta_2, \frac{s}{2}\right) < r. \end{aligned}$$

Hence, we have

$$\begin{aligned} f_1(\mathcal{R}^b_{j_0}(y+z)-(\zeta_1+\zeta_2),s) &\geq f_1\left(\mathcal{R}^b_{j_0}(y)-\zeta_1,\frac{s}{2}\right) * f_1\left(\mathcal{R}^b_{j_0}(z)-\zeta_2,\frac{s}{2}\right) \\ &> (1-r)*(1-r) \\ &> 1-\epsilon \end{aligned}$$

and

$$\begin{aligned} f_2(\mathcal{R}^b_{j_0}(y+z)-(\zeta_1+\zeta_2),s) &\leq f_2\left(\mathcal{R}^b_{j_0}(y)-\zeta_1,\frac{s}{2}\right) \diamond f_2\left(\mathcal{R}^b_{j_0}(z)-\zeta_2,\frac{s}{2}\right) \\ &< r\diamond r \\ &< \epsilon \end{aligned}$$

which implies that

$$A^c\subset\{j\in\mathbb{N}: f_1(\mathcal{R}^b_j(y+z)-(\zeta_1+\zeta_2),s)>1-\epsilon \text{ and } f_2(\mathcal{R}^b_j(y+z)-(\zeta_1+\zeta_2),s)<\epsilon\}.$$

As $A^c\in\mathcal{F}(I)$, hence the later set belongs to $\mathcal{F}(I)$, which implies that $\mathcal{R}^b_{I_{(f_1,f_2)}}-\lim(y+z)=\zeta_1+\zeta_2$.

(2) If $\alpha=0$ then for any $\epsilon>0$ and $s>0$,

$$\begin{aligned} f_1(\mathcal{R}^b_j(0y)-(0\zeta_1),s) &= f_1(0,s)=1>1-\epsilon \\ \text{and } f_2(\mathcal{R}^b_j(0y)-(0\zeta_1),s) &= f_2(0,s)=0<\epsilon \end{aligned}$$

which implies that $\mathcal{R}^b_{(f_1,f_2)}-\lim(0y)=\theta$. Hence, by Theorem 2.3, $\mathcal{R}^b_{I_{(f_1,f_2)}}-\lim(0y)=\theta$.

If $\alpha(\neq 0)\in\mathbb{R}$. To prove the result, we will show that for any $\epsilon>0$ and $s>0$, the set

$$\{j\in\mathbb{N}: f_1(\mathcal{R}^b_j(\alpha y)-(\alpha\zeta_1),s)>1-\epsilon \text{ and } f_2(\mathcal{R}^b_j(\alpha y)-(\alpha\zeta_1),s)<\epsilon\}\in\mathcal{F}(I),$$

for any $\alpha(\neq 0)\in\mathbb{R}$.

As we have given that $\mathcal{R}^b_{I_{(f_1,f_2)}}-\lim(y)=\zeta_1$ so we have for any $\epsilon>0$ and $s>0$, the set

$$K=\{j\in\mathbb{N}: f_1(\mathcal{R}^b_j(y)-\zeta_1,s)>1-\epsilon \text{ and } f_2(\mathcal{R}^b_j(y)-\zeta_1,s)<\epsilon\}\in\mathcal{F}(I).$$

Choose any $j_0\in K$, hence we have $f_1(\mathcal{R}^b_{j_0}(y)-\zeta_1,s)>1-\epsilon$ and $f_2(\mathcal{R}^b_{j_0}(y)-\zeta_1,s)<\epsilon$. Now,

$$\begin{aligned} f_1(\mathcal{R}^b_{j_0}(\alpha y)-(\alpha\zeta_1),s) &= f_1\left(\mathcal{R}^b_{j_0}(y)-\zeta_1,\frac{s}{|\alpha|}\right) \\ &\geq f_1(\mathcal{R}^b_{j_0}(y)-\zeta_1,s) * f_1\left(0,\frac{s}{|\alpha|}-s\right) \\ &= f_1(\mathcal{R}^b_{j_0}(y)-\zeta_1,s)*1 \\ &= f_1(\mathcal{R}^b_{j_0}(y)-\zeta_1,s)>1-\epsilon \end{aligned}$$

and

$$\begin{aligned} f_2(\mathcal{R}^b_{j_0}(\alpha y) - (\alpha\zeta_1), s) &= f_2\bigg(\mathcal{R}^b_{j_0}(y) - \zeta_1, \frac{s}{|\alpha|}\bigg) \\ &\le f_2\bigg(\mathcal{R}^b_{j_0}(y) - \zeta_1, s\bigg) \diamond f_2\bigg(0, \frac{s}{|\alpha|} - s\bigg) \\ &= f_2\bigg(\mathcal{R}^b_{j_0}(y) - \zeta_1, s\bigg) \diamond 0 \\ &= f_2\bigg(\mathcal{R}^b_{j_0}(y) - \zeta_1, s\bigg) < \epsilon \end{aligned}$$

which implies that

$j_0 \in \{j \in \mathbb{N} : f_1(\mathcal{R}^b_j(\alpha y) - (\alpha\zeta_1), s) > 1 - \epsilon$ and $f_2(\mathcal{R}^b_j(\alpha y) - (\alpha\zeta_1), s) < \epsilon\}$.

Hence,

$K \subset \{j \in \mathbb{N} : f_1(\mathcal{R}^b_j(\alpha y) - (\alpha\zeta_1), s) > 1 - \epsilon$ and $f_2(\mathcal{R}^b_j(\alpha y) - (\alpha\zeta_1), s) < \epsilon\}$.

Since $K \in \mathcal{F}(I)$, hence the set

$\{j \in \mathbb{N} : f_1(\mathcal{R}^b_j(\alpha y) - (\alpha\zeta_1), s) > 1 - \epsilon$ and $f_2(\mathcal{R}^b_j(\alpha y) - (\alpha\zeta_1), s) < \epsilon\} \in \mathcal{F}(I)$

which implies that $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(\alpha y) = \alpha\zeta_1$.

Theorem 2.4. *Let $y = (y_k)$ be any sequence in IFNS $(X, f_1, f_2, *, \diamond)$ and let I be a non-trivial ideal in $\mathbb{N}$. If $z = (z_k)$ is Riesz I-convergent sequence in X w.r.t. IFN (f_1, f_2) such that the set $\{j \in \mathbb{N} : \mathcal{R}^b_j(y) \ne \mathcal{R}^b_j(z)\} \in I$, then the sequence $y = (y_k)$ is Riesz I-convergent sequence in X w.r.t. IFN (f_1, f_2).*

Proof of Theorem 2.4. Let

$$\{j \in \mathbb{N} : \mathcal{R}^b_j(y) \ne \mathcal{R}^b_j(z)\} \in I$$

and let $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(z) = \zeta$. Then, for any given $\epsilon > 0$ and $s > 0$, we have

$$A = \{j \in \mathbb{N} : f_1(\mathcal{R}^b_j(z) - \zeta, s) \le 1 - \epsilon \text{ or } f_2(\mathcal{R}^b_j(z) - \zeta, s) \ge \epsilon\} \in I.$$

Thus, for any $\epsilon > 0$,

$$\begin{aligned} &\{j \in \mathbb{N} : f_1(\mathcal{R}^b_j(y) - \zeta, s) \le 1 - \epsilon \text{ or } f_2(\mathcal{R}^b_j(y) - \zeta, s) \ge \epsilon\} \\ &\qquad \subseteq \{j \in \mathbb{N} : \mathcal{R}^b_j(y) \ne \mathcal{R}^b_j(z)\} \cup A \end{aligned}$$

which implies that

$$\{j \in \mathbb{N} : f_1(\mathcal{R}^b_j(y) - \zeta, s) \le 1 - \epsilon \text{ or } f_2(\mathcal{R}^b_j(y) - \zeta, s) \ge \epsilon\} \in I.$$

Hence, $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(y) = \zeta$ that is $y = (y_k)$ is Riesz I-convergent sequence in X w.r.t. IFN (f_1, f_2).

Now, we define Riesz I-Cauchy sequence in IFNS and establish results about relathionship with Riesz I-convergence in IFNS.

Definition 2.2. *In an IFNS $(X, f_1, f_2, *, \diamond)$, a sequence (y_k) is said to be Riesz Cauchy w.r.t. IFN (f_1, f_2) if for all $\epsilon > 0$ and $s > 0$, $\exists K \in \mathbb{N}$ such that*

$$f_1(\mathcal{R}^b_j(y) - \mathcal{R}^b_K(y), s) > 1 - \epsilon \text{ and } f_2(\mathcal{R}^b_j(y) - \mathcal{R}^b_K(y), s) < \epsilon \text{ for all } j \geq K.$$

Definition 2.3. *In an IFNS $(X, f_1, f_2, *, \diamond)$, a sequence (y_k) is said to be Riesz I-Cauchy w.r.t. IFN (f_1, f_2) if for a given $\epsilon > 0$ and $s > 0$, $\exists K \in \mathbb{N}$ such that the following set*

$$\{j \in \mathbb{N} : f_1(\mathcal{R}^b_j(y) - \mathcal{R}^b_K(y), s) \geq 1 - \epsilon \text{ or } f_2(\mathcal{R}^b_j(y) - \mathcal{R}^b_K(y), s) \leq \epsilon\} \in I.$$

Theorem 2.5. *Let $y = (y_k)$ be any sequence in IFNS $(X, f_1, f_2, *, \diamond)$ such that (y_k) is Riesz $I-$convergent with respect to intuitiontic fuzzy norm (f_1, f_2) if and only if (y_k) is Riesz $I-$Cauchy with respect to intuitiontic fuzzy norm (f_1, f_2).*

Proof of Theorem 2.5. In X, let $y = (y_k)$ is such that $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(y) = \zeta$, then for any given $\epsilon > 0$, we can choose $0 < r < 1$ in such a way that $(1 - r) * (1 - r) > 1 - \epsilon$ and $r \diamond r < \epsilon$. Then, for any $s > 0$, we define

$$A_1 = \{j \in \mathbb{N} : f_1(\mathcal{R}^b_j(y) - \zeta, s) \leq 1 - r\}, \; A_2 = \{j \in \mathbb{N} : f_2(\mathcal{R}^b_j(y) - \zeta, s) \geq r\}$$

$$\text{and } A = (A_1 \cup A_2).$$

Sets A_1, A_2 and A must belong to I as $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(y) = \zeta$. Hence, $A^c \in \mathcal{F}(I)$ then A^c is non empty. Let if $m \in A^c$, choose a fixed $l \in A^c$. So, we have,

$$\begin{aligned} f_1(\mathcal{R}^b_m(y) - \zeta, \frac{s}{2}) &> 1 - r, \; f_1(\mathcal{R}^b_l(y) - \zeta, \frac{s}{2}) > 1 - r, \\ f_2(\mathcal{R}^b_m(y) - \zeta, \frac{s}{2}) &< r \text{ and } f_2(\mathcal{R}^b_l(y) - \zeta, \frac{s}{2}) < r. \end{aligned}$$

Hence, we have

$$\begin{aligned} f_1(\mathcal{R}^b_m(y) - \mathcal{R}^b_l(y), s) &\geq f_1\left(\mathcal{R}^b_m(y) - \zeta, \frac{s}{2}\right) * f_1\left(\mathcal{R}^b_l(y) - \zeta, \frac{s}{2}\right) \\ &> (1 - r) * (1 - r) > 1 - \epsilon \end{aligned}$$

and

$$\begin{aligned} f_2(\mathcal{R}^b_m - \mathcal{R}^b_l(y), s) &\leq f_2\left(\mathcal{R}^b_m - \zeta, \frac{s}{2}\right) \diamond f_2\left(\mathcal{R}^b_l(y) - \zeta, \frac{s}{2}\right) \\ &< r \diamond r < \epsilon \end{aligned}$$

which implies that

$$m \in \{j \in \mathbb{N} : f_1(\mathcal{R}^b_j(y) - \mathcal{R}^b_l(y), s) > 1 - \epsilon \text{ and } f_2(\mathcal{R}^b_j(y) - \mathcal{R}^b_l(y), s) < \epsilon\}.$$

Hence,

$$A^c \subset \{j \in \mathbb{N} : f_1(\mathcal{R}_j^b(y) - \mathcal{R}_l^b(y), s) > 1 - \epsilon \text{ and } f_2(\mathcal{R}_j^b(y) - \mathcal{R}_l^b(y), s) < \epsilon\}.$$

As $A^c \in \mathcal{F}(I)$, hence the later set belongs to $\mathcal{F}(I)$, which implies that sequence $y = (y_k)$ is Riesz I-Cauchy sequence with respect to intuitionistic fuzzy norm (f_1, f_2).

Conversely, let on contrary, $y = (y_k)$ is a sequence in X which is Riesz I-Cauchy but not Riesz I-convergent with respect to IF norm (f_1, f_2) then

$$R = \left\{ j \in \mathbb{N} : f_1\left(\mathcal{R}_j^b(y) - \zeta, \frac{s}{2}\right) > 1 - \epsilon \text{ and } f_2\left(\mathcal{R}_j^b(y) - \zeta, \frac{s}{2}\right) < \epsilon \right\} \in I$$

which implies that, $R^c \in \mathcal{F}(I)$.

Since $y = (y_k)$ is Riesz I-Cauchy with respect to IF norm (f_1, f_2), then there exists $M = M(y, \epsilon, s)$ s.t. the set

$$S = \left\{ j \in \mathbb{N} : f_1\left(\mathcal{R}_j^b(y) - \mathcal{R}_M^b(y), \frac{s}{2}\right) \leq 1 - \epsilon \textit{ or } f_2\left(\mathcal{R}_j^b(y) - \mathcal{R}_M^b(y), \frac{s}{2}\right) \geq \epsilon \right\} \in I.$$

As

$$\begin{aligned} f_1(\mathcal{R}_j^b(y) - \mathcal{R}_M^b(y), s) &\geq 2 f_1\left(\mathcal{R}_j^b(y) - \zeta, \frac{s}{2}\right) > 1 - \epsilon \text{ and} \\ f_2(\mathcal{R}_j^b(y) - \mathcal{R}_M^b(y), s) &\leq 2 f_2\left(\mathcal{R}_j^b(y) - \zeta, \frac{s}{2}\right) < \epsilon, \end{aligned}$$

if $f_1(\mathcal{R}_j^b(y) - \zeta, \frac{s}{2}) > \frac{(1-\epsilon)}{2}$ and $f_2(\mathcal{R}_j^b(y) - \zeta, \frac{s}{2}) > \frac{\epsilon}{2}$, respectively.

Hence, we have $S^c \in I$. Equivalently, $S \in \mathcal{F}(I)$, which is a contradiction, as $y = (y_k)$ is Riesz I-Cauchy with respect to IF norm (f_1, f_2).

The proof of following Theorems are straight-forward.

Theorem 2.6. *Let $y = (y_k)$ be a sequence in an IFNS $(X, f_1, f_2, *, \diamond)$ is Riesz Cauchy w.r.t. IFN (f_1, f_2) and $\mathcal{R}_j^b(y)$ clusters to ζ in X then (y_k) is Riesz I-convergent to ζ w.r.t. same IFN.*

Theorem 2.7. *Let $y = (y_k)$ be a sequence in an IFNS $(X, f_1, f_2, *, \diamond)$ is Riesz Cauchy w.r.t. IFN (f_1, f_2), then it is Riesz I-Cauchy w.r.t. IFN (f_1, f_2).*

Theorem 2.8. *Let $y = (y_k)$ be a sequence in an IFNS $(X, f_1, f_2, *, \diamond)$ is Riesz Cauchy w.r.t. IFN (f_1, f_2), then $\exists$ a subsequence (y_{k_n}) of (y_k) such that (y_{k_n}) is a Riesz Cauchy sequence w.r.t. IFN (f_1, f_2).*

Now, we define Riesz I^*-convergence in intuitionistic fuzzzy normed space and and prove some theorems about its relathionship with Riesz I-convergence in IFNS.

Definition 2.4. *A sequence $y = (y_k)$ in IFNS $(X, f_1, f_2, *, \diamond)$ is said to be Riesz I^*-convergent to $\zeta \in X$ with respect to the intuitionistic fuzzy norm (f_1, f_2) if there exists a set $A = \{j_i \in \mathbb{N} : j_i < j_{i+1}, \text{ for all } i \in \mathbb{N}\}$ such that $A \in \mathcal{F}(I)$ and $\lim \mathcal{R}^b_{j_i}(y) = \zeta$ w.r.t. IFN (f_1, f_2). In this case, we say $\mathcal{R}^b_{I^*_{(f_1,f_2)}} - \lim(y) = \zeta$*

Theorem 2.9. *Let I be an admissible ideal and a sequence $y = (y_k)$ in IFNS $(X, f_1, f_2, *, \diamond)$ is such that $\mathcal{R}^b_{I^*_{(f_1,f_2)}} - \lim(y) = \zeta$ then $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(y) = \zeta$.*

Proof of Theorem 2.6. Since $\mathcal{R}^b_{I^*_{(f_1,f_2)}} - \lim(y) = \zeta$ so there exists a subset $A = \{j_i \in \mathbb{N} : j_i < j_{i+1}, \text{ for all } i \in \mathbb{N}\}$ such that $A \in \mathcal{F}(I)$ and $\lim \mathcal{R}^b_{j_i}(y) = \zeta$ w.r.t. IFN (f_1, f_2). Hence, for each $\epsilon > 0$ and $s > 0$, there exists $m \in \mathbb{N}$ in such a way that

$$f_1(\mathcal{R}^b_{j_i}(y) - \zeta, s) > 1 - \epsilon \text{ and } f_2(\mathcal{R}^b_{j_i}(y) - \zeta, t) < \epsilon \text{ for all } i \geq m.$$

As the set

$$\{j_i \in A : f_1(\mathcal{R}^b_{j_i}(y), s) \leq 1 - \epsilon \text{ or } f_2(\mathcal{R}^b_{j_i}(y) - \zeta, s) \geq \epsilon\}$$

is contained in $\{j_1, j_2, ..., j_{m-1}\}$. Hence,

$$\{j_i \in A : f_1(\mathcal{R}^b_{j_i}(y) - \zeta, s) \leq 1 - \epsilon \text{ or } f_2(\mathcal{R}^b_{j_i}(y) - \zeta, s) \geq \epsilon\} \in I,$$

as I is an admissible ideal. Also $A \in \mathcal{F}(I)$, then by the definition of $\mathcal{F}(I)$ there exists a set $B \in I$ such that $A = \mathbb{N} \backslash B$. So,

$$\{j \in \mathbb{N} : f_1(\mathcal{R}^b_j(y) - \zeta, s) \leq 1 - \epsilon \text{ or } f_2(\mathcal{R}^b_j(y) - \zeta, s) \geq \epsilon\} \subset B \cup \{j_1, j_2, ..., j_{m-1}\}$$

Hence,

$$\{j \in \mathbb{N} : f_1(\mathcal{R}^b_j(y) - \zeta, t) \leq 1 - \epsilon \text{ or } f_1(\mathcal{R}^b_j(y) - \zeta, t) \geq \epsilon\} \in I$$

which implies that $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(y) = \zeta$.

Remark 2.2. Converse of Theorem 2.9 may not be hold in general.

Example 2.2. In the Example 1.1, let X=$\mathbb{R}$ with usual norm, then $(\mathbb{R}, f_1, f_2, *, \diamond)$ forms an IFNS.

Now, we take a decomposition of $\mathbb{N}$ as $\mathbb{N} = \cup A_i$, where every A_i is an infinite set and $A_i \cap A_l = \emptyset$, for $i \neq l$. Suppose $I = \{N \subset \mathbb{N} : N \subset \cup_{i=1}^r A_i$, for some finite natural number $r\}$ then I is a non-trivial admissible ideal.

Now, for a positive sequence of real numbers $b = (b_k)$, take a sequence (y_k) in such a way that

$$\mathcal{R}^b_j(y) = \frac{1}{i}, \text{ if } j \in A_i$$

Hence, we have for $s > 0$

$$f_1(\mathcal{R}^b_j(y), s) = \frac{s}{s + ||\mathcal{R}^b_j(y)||} \to 1 \text{ as } j \to \infty$$
$$\text{and } f_2(\mathcal{R}^b_j(y), s) = \frac{||\mathcal{R}^b_j(y)||}{s + ||\mathcal{R}^b_j(y)||} \to 0, \text{ as } j \to \infty.$$

Hence, by Proposition 2.1, $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(y) = 0$.

Let on contrary that $\mathcal{R}^b_{I^*_{(f_1,f_2)}} - \lim(y) = 0$ then $\exists$ a set $A = \{j_i \in \mathbb{N} : j_i < j_{i+1}, \text{ for all } i \in \mathbb{N}\}$ such that $A \in \mathcal{F}(I)$ and $\lim \mathcal{R}^b_{j_i}(y) = \zeta$ w.r.t. IFN (f_1, f_2). As $A \in \mathcal{F}(I)$, $\exists$ $K = \mathbb{N} \backslash A$ and $K \in I$. Then, there exists a natural numer r such that $K \subset \cup_{i=1}^{r} N_i$. Then $N_{r+1} \subset A$. Hence,

$$\mathcal{R}^b_{j_i}(y) = \frac{1}{r+1}, \text{ for infinitely many values of } j_i \text{ in } A,$$

which is a contradiction. Hence, $\mathcal{R}^b_{I^*_{(f_1,f_2)}} - \lim(y) \neq 0$.

Theorem 2.10. *Let $y = (y_k)$ be a sequence in IFNS $(X, f_1, f_2, *, \diamond)$ such that $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(y) = \zeta$ and ideal I satisfies condition (AP). Then $\mathcal{R}^b_{I^*_{(f_1,f_2)}} - \lim(y) = \zeta$.*

Proof of Theorem 2.7. As $\mathcal{R}^b_{I_{(f_1,f_2)}} - \lim(y) = \zeta$. Then $\forall \epsilon > 0$ and $s > 0$, we have

$$\{j \in \mathbb{N} : f_1(\mathcal{R}^b_j(y) - \zeta, s) \leq 1 - \epsilon \text{ or } f_2(\mathcal{R}^b_j(y) - \zeta, s) \geq \epsilon\} \in I,$$

For $r \in \mathbb{N}$ and $s > 0$, we define

$$A_r = \{j \in \mathbb{N} : 1 - \frac{1}{r} \leq f_1(\mathcal{R}^b_j(y) - \zeta, s) < 1 - \frac{1}{r+1} \text{ or } \frac{1}{r+1} < f_2(\mathcal{R}^b_j(y) - \zeta, s) \leq \frac{1}{r}\}.$$

Now, it is clear that $\{A_1, A_2, ...\}$ is a countable family of mutually disjoint sets belonging to I and therefore by the condition (AP) there is a countable family of sets $\{B_1, B_2, ...\}$ in I such that $A_i \Delta B_i$ is a finite set for each $i \in \mathbb{N}$ and $B = \cup_{i=1}^{\infty} B_i$. Since $B \in I$ so by definition of associate filter $\mathcal{F}(I)$ there is set $K \in \mathcal{F}(I)$ such that $K = \mathbb{N} - B$. Now, to prove the result it is sufficient to prove that the subsequence $(y_k)_k \in K$ is ordinary Riesz convergent to with respect to the intuitionistic fuzzy norm (ϕ, ψ). For this, let $\eta > 0$ and $s > 0$. Choose a positive integer q such that $\frac{1}{q} < \eta$. Hence, we have

$$\begin{aligned} &\{j \in \mathbb{N} : f_1(\mathcal{R}^b_j(y) - \zeta, s) \leq 1 - \eta \text{ or } f_2(\mathcal{R}^b_j(y) - \zeta, s) \geq \eta\} \\ &\subset \{j \in \mathbb{N} : f_1(\mathcal{R}^b_j(y) - \zeta, s) \leq 1 - \frac{1}{q} \text{ or } f_2(\mathcal{R}^b_j(y) - \zeta, s) \geq \frac{1}{q}\} \\ &\subset \cup_{i=1}^{q+1} A_i. \end{aligned}$$

Since $A_i \Delta B_i$ be a finite set for each $i = 1, 2, ..q + 1$, $\exists$ j_0 such that

$$(\cup_{i=1}^{q+1} B_i) \cap \{j \in \mathbb{N} : j \geq j_0\} = (\cup_{i=1}^{q+1} A_i) \cap \{j \in \mathbb{N} : j \geq j_0\}.$$

If $j \geq j_0$ and $j \in K$, then $j \notin B$. This implies that $j \in \cup_{i=1}^{q+1} B_i$ and therefore $j \notin \cup_{i=1}^{q+1} A_i$. Hence, for every $j \geq j_0$ and $j \in K$, we have

$$f_1(\mathcal{R}_j^b(y) - \zeta, s) > 1 - \eta \text{ and } f_2(\mathcal{R}_j^b(y) - \zeta, s) < \eta$$

As this holds for every $\eta > 0$ and $s > 0$, so $\mathcal{R}^b_{I^*_{(f_1,f_2)}} - \lim(y) = \zeta$.

Theorem 2.11. *For any sequence $y = (y_k)$ in IFNS $(X, f_1, f_2, *, \diamond)$, the following statements are equivalent.*

(1) $\mathcal{R}^b_{I^*_{(f_1,f_2)}} - \lim(y) = \zeta$.

(2) $\exists$ *sequences $p = (p_k)$ and $q = (q_k)$ in X such that $\mathcal{R}_j^b(y) = \mathcal{R}_j^b(p) + \mathcal{R}_j^b(q)$, $\lim_{j\to\infty} \mathcal{R}_j^b(p) = \zeta$ w.r.t. IFN (f_1, f_2) and the set $\{j \in \mathbb{N} : \mathcal{R}_j^b(q) \neq \theta\} \in I$ where zero elements of X are denoted by θ.*

Proof of Theorem 2.8. Let statement (1) holds. Then we have a subset $A = \{i_1, i_2, i_3, ... : i_1 < i_2 < ..\}$ of $\mathbb{N}$ in such a way that $A \in \mathcal{F}(I)$ and $\lim_{j\to\infty} \mathcal{R}_j^b(y) = \zeta$ w.r.t. IFN (f_1, f_2).

We define (p_k) and (q_k) such that,

$$\mathcal{R}_j^b(p) = \begin{cases} \mathcal{R}_j^b(y), & \text{if } j \in A \\ \zeta, & \text{otherwise} \end{cases}$$

and $\mathcal{R}_j^b(q) = \mathcal{R}_j^b(y) - \mathcal{R}_j^b(p)$ for $j \in \mathbb{N}$. For $j \in A^c$, for all $\epsilon > 0$ and $s > 0$,

$$f_1(\mathcal{R}_j^b(p) - \zeta, s) = 1 > 1 - \epsilon \text{ and } f_2(\mathcal{R}_j^b(p) - \zeta, s) = 0 < \epsilon$$

which implies that $\lim_{j\to\infty} \mathcal{R}_j^b(p) = \zeta$ w.r.t. IFN (f_1, f_2). As $\{j \in \mathbb{N} : \mathcal{R}_j^b(q) \neq \theta\} \subset A^c$, which implies that $\{j \in \mathbb{N} : \mathcal{R}_j^b(q) \neq \theta\} \in I$.

Now, let statement (2) holds, $A = \{j \in \mathbb{N} : \mathcal{R}_j^b(q) = 0\}$, hence $A \in \mathcal{F}(I)$ so it is an infinite set. Now, suppose $A = \{i_1, i_2, .. : i_1 < i_2 < ...\}$. As $\mathcal{R}_{i_j}^b(y) = \mathcal{R}_{i_j}^b(p)$ and $\lim_{j\to\infty} \mathcal{R}_j^b(p) = \zeta$ w.r.t. IFN (f_1, f_2), hence $\lim_{j\to\infty} \mathcal{R}_{i_j}^b(y) = \zeta$ w.r.t. IFN (f_1, f_2), which implies that $\mathcal{R}^b_{I^*_{(f_1,f_2)}} - \lim(y) = \zeta$.

Conclusion

In this article, we have defined Riesz ideal convergent and Riesz ideal Cauchy sequences in intuitionistic fuzzy normed space. We have also discussed the behavior of this notion by proving various fundamental results with counterexamples. Usage of fuzzy logic nowadays increased massively in the various fields of science and technology to tackle real-world problems. Since ideal convergence is a unified and generalized notion of various well-known notions of convergence and Riesz ideal convergence is a more generalized variant of ideal convergence. On the other hand, intuitionistic fuzzy normed space is an extension of various famous spaces and it also handles complicated situations very easily because of its inexactness of the norm, hence our study is in a more general setting than other existing studies. Therefore, these new results will further give a superior tool to tackle complex problems and also help the researchers expand their work in the area of sequence spaces in view of fuzzy theory.

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