An efficient optimal fourth-order iterative method for scalar equations

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Abstract. In the present paper, using linear combination technique, we introduce an optimal three-step iterative scheme for solving nonlinear equations. We prove the convergence of the proposed method. In order to demonstrate the performance of newly developed method, we consider some commonly used nonlinear equations for numerical as well as graphical comparisons. We also explore polynomiographs in the context of some complex polynomials.

Keywords: iterative methods, nonlinear equations, order of convergence, linear combination.

1. Introduction

Nonlinear equations and their solutions have been a scorching topic for many researchers. In this regard, vast literature is available, for examples see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and references therein. A fundamental technique for solving nonlinear equations is the well-known Newton's method, which converges quadratically:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0, \quad n = 0, 1, 2, \ldots. \tag{1}$$

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According to Kung and Turab [5] conjecture, an iterative method is called optimal if it needs $(n+1)$ functional evaluations per iteration and possesses convergence order 2^n. S. Abbasbandy [6] using modified Adomian decomposition method, proposed a fourth-order method which needs three evaluations per iteration:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{f^2(x_n)f''(x_n)}{2f'^3(x_n)} - \frac{f^3(x_n)f''^2(x_n)}{2f'^5(x_n)}, \tag{2}$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$, $f'(x_n) \neq 0$, $n = 0, 1, 2, \ldots$.

Cordero et al. [7], developed the following fourth-order method:

$$x_{n+1} = x_n - \frac{f(x_n) + f(y_n)}{f'(x_n)} - \left[\frac{f(y_n)}{f'(x_n)}\right]^2 \left[\frac{2f(x_n) + f(y_n)}{f'(x_n)}\right], \tag{3}$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$, $f'(x_n) \neq 0$, $n = 0, 1, 2, \ldots$.

A second derivative free optimal fourth-order method has been introduced by Chun et al. [8].

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}\left[1 + \frac{3}{4}\frac{f'(x_n) - f'(y_n)}{f'(x_n)} + \frac{9}{8}\left(\frac{f'(x_n) - f'(y_n)}{f'(x_n)}\right)^2\right], \tag{4}$$

where $y_n = x_n - \frac{2f(x_n)}{3f'(x_n)}$, $f'(x_n) \neq 0$, $n = 0, 1, 2, \ldots$.

In 2015, Sherma and Behl [9], also proposed a second derivative free optimal fourth-order method:

$$x_{n+1} = x_n - \left[-\frac{1}{2} + \frac{9f'(x_n)}{8f'(y_n)} + \frac{3f'(y_n)}{8f'(x_n)}\right]\frac{f(x_n)}{f'(x_n)}, \tag{5}$$

where $y_n = x_n - \frac{2f(x_n)}{3f'(x_n)}$, $f'(x_n) \neq 0$, $n = 0, 1, 2, \ldots$.

In this paper, having motivation from the above study, we propose a more effective second derivative free optimal fourth-order iterative method. The effectiveness of our method is explored by its numerical as well as graphical comparisons with some existing methods of the same class. We also investigate the dynamical behavior of newly constructed method for visualization of the roots of complex polynomials.

2. Construction of iterative method

Consider the nonlinear equation

$$f(x) = 0. \tag{6}$$

Using Taylor's expansion about γ (initial guess), equation (6) can be written in the form of the following coupled system:

$$f(x) \approx f(\gamma) + (x-\gamma)f'(\gamma) + g(x) \approx 0, \tag{7}$$

$$g(x) \approx \frac{\lambda f(x)}{f'(\gamma)} - f(\gamma) - (x-\gamma)f'(\gamma), \tag{8}$$

where $\lambda \in R$ is an auxiliary parameter.

From equation (7), we get

$$x \approx \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{g(x)}{f'(\gamma)}, \quad (9)$$
$$= c + N(x),$$

where

$$c = \gamma - \frac{f(\gamma)}{f'(\gamma)} \text{ and} \quad (10)$$

$$N(x) = -\frac{g(x)}{f'(\gamma)}. \quad (11)$$

Here, $N(x)$ is a nonlinear operator and can be approximated by using Taylor's series expansion about x_0 as follows:

$$N(x) = N(x_0) + \sum_{k=1}^{\infty} \frac{(x_i - x_0)^k}{k!} N^{(k)}(x_0). \quad (12)$$

Our aim is to find the series solution of equation (6):

$$x = \sum_{i=0}^{\infty} x_i. \quad (13)$$

Which can alternatively be expressed as

$$x = \lim_{m\to\infty} X_m, \qquad \text{where } X_m = x_0 + x_1 + \ldots + x_m. \quad (14)$$

From equations (9), (12) and (13), we get

$$x = \sum_{i=0}^{\infty} x_i = \sum_{i=0}^{\infty} (c + N(x_0) + \sum_{k=1}^{\infty} \frac{(x_i - x_0)^k}{k!} N^{(k)}(x_0), \text{ which implies}$$

$$x = c + N(x_0) + \sum_{k=1}^{\infty} \frac{(\sum_{i=0}^{k} x_i - x_0)^k}{k!} N^{(k)}(x_0). \quad (15)$$

From the last relation, we have the following scheme:

$$\left\{ \begin{array}{l} x_0 = c, \\ x_1 = N(x_0), \\ x_2 = (x_0 + x_1 - x_0) N'(x_0), \\ x_3 = \frac{(x_0 + x_1 + x_2 - x_0)^2}{2!} N''(x_0), \\ \vdots \\ x_{m+1} = \frac{(x_0 + x_1 + \ldots + x_m - x_0)^m}{m!} N^{(m)}(x_0), \ m = 0, 1, 2, \ldots. \end{array} \right\} \quad (16)$$

Thus,

$$x_1 + x_2 + \ldots + x_{m+1} = N(x_0) + (x_0 + x_1 - x_0)N'(x_0) + \ldots + \frac{(x_0 + x_1 + \ldots + x_m - x_0)^m}{m!} N^{(m)}(x_0), \tag{17}$$

where $m = 1, 2, \ldots$.

Since $x_0 = c$, therefore, equation (15) gives

$$x = c + \sum_{i=1}^{\infty} x_i. \tag{18}$$

From equation (10) and the first equation of (16), we have

$$x_0 = c = \gamma - \frac{f(\gamma)}{f'(\gamma)}. \tag{19}$$

Using equation (14) with $m = 0$ and equation (19), we have

$$x \approx X_0 = x_0 = \gamma - \frac{f(\gamma)}{f'(\gamma)}. \tag{20}$$

This formulation allows us to propose the following iterative method for solving nonlinear equation (6).

Algorithm. 2.1. For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x_n) \neq 0, \qquad n = 0, 1, 2, \ldots, \tag{21}$$

which is the well-known Newton's method.

Now, from equation (19), we have

$$x_0 - \gamma = -\frac{f(\gamma)}{f'(\gamma)}. \tag{22}$$

Using equation (11) and the second equation of (16), we get

$$x_1 = N(x_0) = -\frac{g(x_0)}{f'(\gamma)}. \tag{23}$$

Thus, using equation (8), we have

$$g(x_0) = \frac{\lambda f(x_0)}{f'(\gamma)} - f(\gamma) - (x_0 - \gamma)f'(\gamma). \tag{24}$$

From equations (22), (23) and (24), we obtain

$$x_1 = N(x_0) = -\frac{\lambda f(x_0)}{(f'(\gamma))^2} = -\frac{\lambda f(\gamma - \frac{f(\gamma)}{f'(\gamma)})}{(f'(\gamma))^2}. \tag{25}$$

Using equation (14) with $m = 1$ along with equations (19) and (25), we obtain

$$x \approx X_1 = x_0 + x_1 = \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{\lambda f(\gamma - \frac{f(\gamma)}{f'(\gamma)})}{(f'(\gamma))^2}. \tag{26}$$

This formulation allows us to propose the following iterative method for solving nonlinear equation (6).

Algorithm 2.2. For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = y_n - \frac{\lambda f(y_n)}{(f'(x_n))^2}, \tag{27}$$

where $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$, $f'(x_n) \neq 0$, $n = 0, 1, 2, \ldots$.

The last algorithm converges cubically for $\lambda = 1$ and requires 3 function evaluations per iteration.

From the third equation of (16), we get

$$x_2 = (x_1)N'(x_0) = -x_1 \frac{g'(x_0)}{f'(\gamma)}. \tag{28}$$

From equation (8), we have

$$g'(x_0) = \frac{\lambda f'(x_0)}{f'(\gamma)} - f'(\gamma). \tag{29}$$

Thus from equations (25), (28) and (29), we have

$$x_2 = \lambda^2 \frac{f(x_0) f'(x_0)}{(f'(\gamma))^4} - \frac{\lambda f(x_0)}{(f'(\gamma))^2}. \tag{30}$$

Using equation (14) with $m = 2$ along with equations (19), (25) and (30), we obtain

$$\begin{aligned} x &\approx X_3 = x_0 + x_1 + x_2 \\ &= c - 2\frac{\lambda f(x_0)}{(f'(\gamma))^2} + \lambda^2 \frac{f(x_0) f'(x_0)}{(f'(\gamma))^4} \\ &= \gamma - \frac{f(\gamma)}{f'(\gamma)} - 2\frac{\lambda f(x_0)}{(f'(\gamma))^2} + \lambda^2 \frac{f(x_0) f'(x_0)}{(f'(\gamma))^4}. \end{aligned} \tag{31}$$

This formulation allows us to propose the following iterative method for solving nonlinear equation (6).

Algorithm 2.3. For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = 2z_n - y_n - \lambda(z_n - y_n)\frac{f'(y_n)}{(f'(x_n))^2}, \tag{32}$$

where $z_n = y_n - \frac{\lambda f(y_n)}{(f'(x_n))^2}$, and $y_n = x_n - \frac{f(x_n)}{f'(x_n)}$, $f'(x_n) \neq 0$, $n = 0, 1, 2, \ldots$. The above algorithm has convergence order 3 and needs 4 function evaluations per iteration. In order to reduce the number of function evaluations by one, we make the following approximation:

$$f'(y_n) \approx \frac{f(y_n) - f(x_n)}{(y_n - x_n)}. \tag{33}$$

Algorithm 2.4. For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = 2z_n - y_n - \lambda(z_n - y_n)\frac{f(y_n) - f(x_n)}{(y_n - x_n)(f'(x_n))^2}, \tag{34}$$

where

$$z_n = y_n - \frac{\lambda f(y_n)}{(f'(x_n))^2},$$

and

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x_n) \neq 0, \qquad n = 0, 1, 2, \ldots.$$

Which is a three-step iterative method having convergence order three and needs three function evaluations per iteration.

On the basis of the linear combination of Algorithms 2.2 and 2.4, we suggest the following new optimal fourth-order iterative scheme:

$$x_{n+1} = y_n + (1 + \frac{4\theta}{3})\frac{\lambda f(y_n)}{(f'(x_n))^2} + \frac{4\theta}{3}\lambda(z_n - y_n)\frac{f(y_n) - f(x_n)}{(y_n - x_n)(f'(x_n))^2}, \tag{35}$$

where $\theta \in R$ is the adjusting parameter. Clearly, for $\theta = 0$ and $\lambda = -1$, equation (35) reduces to the method given in equation (27) and for $\theta = 3/4$ and $\lambda = -1$, it reduces to the method defined in equation (34). The performance of newly suggested method depends upon the appropriate choice of θ.

Taking $\theta = 1$ and $\lambda = -1$, the above formulation allows us to suggest the following optimal fourth-order iterative method:

Algorithm 2.5. For a given x_0, compute the approximate solution x_{n+1} by the following iterative scheme:

$$x_{n+1} = y_n - (\frac{7}{3})\frac{f(y_n)}{(f'(x_n))^2} - \frac{4}{3}(z_n - y_n)\frac{f(y_n) - f(x_n)}{(y_n - x_n)(f'(x_n))^2}, \tag{36}$$

where

$$z_n = y_n - \frac{f(y_n)}{(f'(x_n))^2},$$

and

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad f'(x_n) \neq 0, \qquad n = 0, 1, 2, \ldots.$$

To the best of our knowledge, algorithm 2.5 is a new one to solve the nonlinear equation (6).

3. Convergence analysis

In this section, convergence criteria of newly proposed method is studied in the form of the following theorem.

Theorem 3.1. *Assume that the function $f : I \subset \mathbb{R} \to \mathbb{R}$ (where I is an open interval) has a simple root $\alpha \in I$ and x_0 is sufficiently close to α. Let $f(x)$ be sufficiently differentiable in the neighborhood of α, then the algorithm 2.5 has the convergence order 4.*

Proof. Let α be a simple zero of $f(x)$. Since f is sufficiently differentiable, therefore, the Taylor's series expansions of $f(x_n)$ and $f'(x_n)$ about α are given by

$$f(x_n) = f'(\alpha)\{e_n + c_2e_n^2 + c_3e_n^3 + c_4e_n^4 + c_5e_n^5 + O\left(e_n^6\right)\}, \tag{37}$$

and

$$f'(x_n) = f'(\alpha)\{1 + 2c_2e_n + 3c_3e_n^2 + 4c_4e_4^3 + 5c_5e_n^4 + 6c_6e_n^5 + O\left(e_n^5\right)\}, \tag{38}$$

where $e_n = x_n - \alpha$ and $c_j = (\frac{1}{j!})\frac{f^{(j)}(\alpha)}{f'(\alpha)}$, $j = 1, 2, 3, \ldots$.

From equations (37) and (38), we get

$$\begin{aligned} \frac{f(x_n)}{f'(x_n)} &= e_n - c_2e_n^2 + 2(c_2^2 - c_3)e_n^3 + (-4c_2^3 + 7c_2c_3 - 3c_4)e_n^4 \\ &\quad +(8c_2^4 - 20c_2^2c_3 + 10c_2c_4 + 6c_3^2 - 4c_5)e_n^5 + O\left(e_n^6\right). \end{aligned} \tag{39}$$

Using equation (39), we find

$$\begin{aligned} y_n &= x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 + (4c_2^3 - 7c_2c_3 + 3c_4)e_n^4 \\ &\quad +(-8c_2^4 + 20c_2^2c_3 - 10c_2c_4 - 6c_3^2 + 4c_5)e_n^5 + O\left(e_n^6\right). \end{aligned} \tag{40}$$

Using equation (40), the Taylor's series of $f(y_n)$ is given as

$$\begin{aligned} f(y_n) &= c_2e_n^2 - 2(c_2^2 - c_3)e_n^3 + (5c_2^3 - 7c_2c_3 + 3c_4)e_n^4 \\ &\quad +(-12c_2^4 + 24c_2^2c_3 - 10c_2c_4 - 6c_3^2 + 4c_5)e_n^5 + O\left(e_n^6\right). \end{aligned} \tag{41}$$

Using equations (40) and (41), we get

$$\begin{aligned} z_n &= \alpha + 4c_2^2e_n^3 + (-21c_2^3 + 14c_2c_3)e_n^4 \\ &\quad + (80c_2^4 - 104c_2^2c_3 + 20c_2c_4 + c_3^2)e_n^5 + O\left(e_n^6\right). \end{aligned} \tag{42}$$

From equations (38), (40) and (41), we obtain

$$\begin{aligned} y_n - \frac{7f(y_n)}{3(f'(x_n))^2} &= \alpha - \frac{4}{3}c_2e_n^2 + (12c_2^2 - \frac{8}{3}c_3)e_n^3 + (-\frac{163}{3}c_2^3 + 42c_2c_3 - 4c_4)e_n^4 \\ &\quad + (\frac{592}{3}c_2^4 - \frac{808}{3}c_2^2c_3 + 60c_2c_4 + 36c_3^2 - \frac{16}{3}c_5)e_n^5 + O\left(e_n^6\right). \end{aligned} \tag{43}$$

Using equations (40) and (42), we have

$$4(z_n - y_n)\frac{f(y_n)-f(x_n)}{3(y_n-x_n)(f'(x_n))^2} = -\frac{4}{3}c_2e_n^2 + (12c_2^2 - \frac{8}{3}c_3)e_n^3 + (-\frac{208}{3}c_2^3 + \frac{128}{3}c_2c_3 - 4c_4)e_n^4 + (324c_2^4 - 352c_2^2c_3 + \frac{184}{3}c_2c_4 + \frac{112}{3}c_3^3 - \frac{16}{3}c_5)e_n^5 + O\left(e_n^6\right). \tag{44}$$

Using equations (43) and (44), the error term for algorithm 2.5 is given as

$$e_{n+1} = (15c_2^3 - \frac{2}{3}c_2c_3)e_n^4 + (-\frac{380}{3}c_2^4 + \frac{248}{3}c_2^2c_3 - \frac{4}{3}c_2c_4 - \frac{4}{3}c_3^2)e_n^5 + O(e_n^6). \tag{45}$$

This completes the proof. □

4. Numerical examples

In this section, we reveal the validity and efficiency of our proposed iterative method (AN1) given in algorithm 2.5 by considering the nonlinear equations from the fields of mathematical sciences. Taking x_0 as initial guess, we compare AN1 with the standard Newton's method (equation 1) (NM), Abbasbundy's method (equation 2) (AM), Cordero et al. method (equation 3) (DM), Chun's method (equation 4) (CM), and recently developed method (equation 5) (RM) by Sharma and Behl. The numerical comparison is presented in the following table and the graphical behavior is studied in Fig. 1 to Fig. 10 to demonstrate the performance of the methods. We use Maple 18 and Matlab software for comparisons, taking $|x_{n+1} - x_n|$ as stopping criteria, where $\epsilon = 10^{-15}$ represents the tolerance. Both the comparative studies show that the newly developed method i.e. algorithm 2.5 performs better.

In the following table NFE denotes the total number of functional evaluations required to reach the desired result.

Table

$f(x)$	$x_\circ$	$Method$	n	x_n	$f(x_n)$	$(\lvert x_{n+1}-x_n\rvert)$	NFE
x^3+x^2+2	1.4	NM	18	−1.6956207695598621	$2.228890e^{-16}$	$3.137243e^{-09}$	36
		AM	13	−1.6956207695598621	$2.228890e^{-16}$	$2.024970e^{-13}$	39
		DM	50	6889.4069728314899	$3.270458e^{11}$	$6.870399e^{+03}$	150
		CM	28	−1.6956207695598621	$2.228890e^{-16}$	$1.897249e^{-14}$	84
		RM	13	−1.6956207695598621	$2.228890e^{-16}$	$5.034529e^{-05}$	39
		$AN1$	6	−1.6956207695598621	$2.228890e^{-16}$	$1.071447e^{-08}$	18
$e^{sinx}+\sin(3x)-1$	−0.4	NM	6	0.0000000000000000	$1.914154e^{-27}$	$6.187332e^{-14}$	12
		AM	5	−1.8736010948989818	$6.376116e^{-17}$	$4.639035e^{-08}$	15
		DM	4	0.0000000000000000	$3.782606e^{-29}$	$8.934023e^{-08}$	12
		CM	4	3.1415926535897932	$1.538506e^{-16}$	$7.110410e^{-07}$	12
		RM	4	0.0000000000000000	$8.231301e^{-50}$	$6.173081e^{-13}$	12
		$AN1$	3	0.0000000000000000	$2.659091e^{-15}$	$1.031329e^{-07}$	9
$x^3-e^{-\sin 4x}-1$	1.03	NM	7	1.4115624181268047	$1.491129e^{-16}$	$3.531439e^{-09}$	14
		AM	12	1.4115624181268047	$1.491129e^{-16}$	$2.139755e^{-09}$	36
		DM	8	1.4115624181268047	$1.491129e^{-16}$	$8.432900e^{-15}$	24
		CM	9	1.4115624181268047	$1.491129e^{-16}$	$4.229577e^{-14}$	27
		RM	5	1.4115624181268047	$1.491129e^{-16}$	$6.641495e^{-06}$	15
		$AN1$	4	1.4115624181268047	$1.491129e^{-16}$	$7.080969e^{-09}$	12
$ln(e^{-x}+x^2)$	0.28	NM	6	−0.0000000000000000	$1.409068e^{+23}$	$3.753755e^{-12}$	12
		AM	3	0.0000000000000000	$3.426071e^{+35}$	$3.247871e^{-12}$	9
		DM	3	0.0000000000000000	$0.000000e^{+00}$	$8.553213e^{+03}$	9
		CM	5	0.7145563847430097	$1.504888e^{-18}$	$6.332712e^{-11}$	15
		RM	4	−0.0000000000000000	$4.431519e^{-28}$	$1.073788e^{-07}$	12
		$AN1$	2	0.0000000000000000	$0.000000e^{+00}$	$3.185195e^{+02}$	6
$sin^{-1}x$	0.3	NM	3	0.0000000000000000	$6.689713e^{-21}$	$2.717542e^{-07}$	6
		AM	3	−0.0000000000000000	$1.781999e^{-20}$	$3.767121e^{-07}$	9
		DM	2	0.0000000000000000	$1.673591e^{-21}$	$1.246764e^{-04}$	6
		CM	2	0.0000000000000000	$1.797242e^{-23}$	$5.576293e^{-05}$	6
		RM	2	0.0000000000000000	$1.001759e^{-22}$	$7.862795e^{-05}$	6
		$AN1$	2	0.0000000000000000	$1.362037e^{-26}$	$1.297510e^{-05}$	6
$sin(5x)$	−0.22	NM	5	0.0000000000000000	$1.384135e^{-20}$	$6.925692e^{-08}$	10
		AM	5	−1.2566370614359173	$2.307471e^{-17}$	$8.228151e^{-08}$	15
		DM	4	−0.0000000000000000	$4.183607e^{-33}$	$1.192325e^{-07}$	12
		CM	7	324.21236185046666	$1.104672e^{-14}$	$1.908201e^{-05}$	21
		RM	4	−0.0000000000000000	$1.550586e^{-69}$	$6.224901e^{-15}$	12
		$AN1$	3	−0.0000000000000013	$6.732192e^{-15}$	$6.506171e^{-06}$	9
$ln(x^2+e^X)+x$	−0.5	NM	5	0.0000000000000000	$2.761873e^{-17}$	$5.255352e^{-09}$	10
		AM	9	−0.0000000000000000	$9.183366e^{-31}$	$6.739488e^{-11}$	27
		DM	8	0.0000000000000000	$2.076011e^{-21}$	$6.099432e^{-06}$	24
		CM	5	0.0000000000000011	$2.171942e^{-15}$	$1.876283e^{-04}$	15
		RM	4	0.0000000000000000	$6.932928e^{-32}$	$1.590518e^{-08}$	12
		$AN1$	3	0.0000000000000000	$7.299088e^{-26}$	$6.617743e^{-13}$	9
$x+2x^3sin(x)-1$	1.38	NM	6	1.5523226989842709	$2.951026e^{-16}$	$2.008739e^{-11}$	12
		AM	18	4.7079470507119627	$9.955880e^{-15}$	$4.485377e^{-12}$	54
		DM	7	−32504614.839471103	$7.392443e^{+13}$	$3.474241e^{-19}$	21
		CM	4	7.0707301317015739	$2.403279e^{-14}$	$2.078366e^{-07}$	12
		RM	9	7.0707301317015739	$2.403279e^{-14}$	$2.407384e^{-14}$	27
		$AN1$	4	1.5523226989842709	$2.951026e^{-16}$	$2.345465e^{-10}$	12
$sin(5x)-sin(x)$	0.2	NM	5	−0.0000000000000000	$6.387890e^{-15}$	$5.366433e^{-06}$	10
		AM	20	−67.5442420404298480	$1.656949e^{-15}$	$1.958451e^{-08}$	60
		DM	4	0.0000000000000000	$1.387071e^{-20}$	$3.651889e^{-05}$	12
		CM	4	−0.5235987755982989	$1.598912e^{-17}$	$7.380621e^{-14}$	12
		RM	4	0.0000000000000000	$1.897089e^{-57}$	$1.559232e^{-12}$	12
		$AN1$	3	−0.0000000000000000	$2.652270e^{-18}$	$5.044085e^{-07}$	9
x^3-x^2-8	−2.3	NM	14	2.3948586738660659	$5.353746e^{-16}$	$1.947641e^{-15}$	28
		AM	14	2.3948586738660659	$5.353746e^{-16}$	$1.580128e^{-08}$	42
		DM	50	−1.7488390182525641	$1.640715e^{+01}$	$2.566501e^{+00}$	150
		CM	31	2.3948586738660659	$5.353746e^{-16}$	$4.634751e^{-12}$	93
		RM	17	2.3948586738660659	$5.353746e^{-16}$	$8.595530e^{-08}$	51
		$AN1$	7	2.3948586738660659	$5.353746e^{-16}$	$4.022589e^{-15}$	21

From the above results, it is clear that each method converges for the considered test problems but the computational cost of the proposed method is the least.

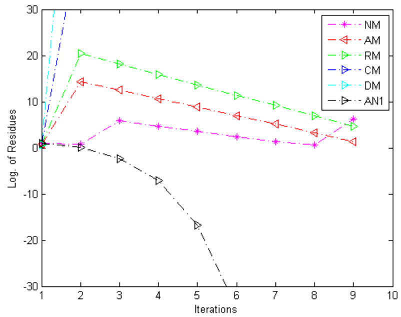

Fig. 1 ($f(x) = x^3 + x^2 + 2$)

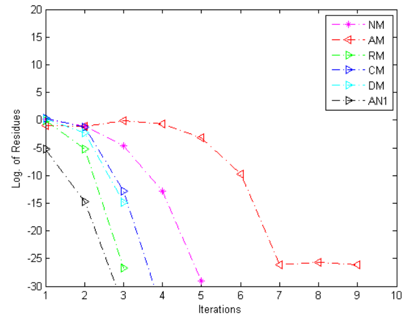

Fig. 2 ($f(x) = e^{\sin x} + \sin(3x) - 1$)

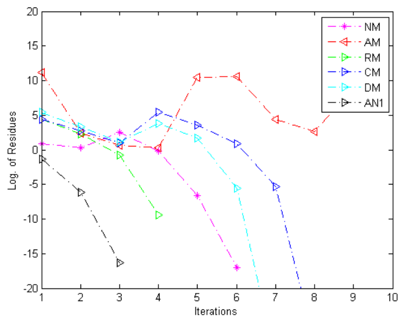

Fig. 3 ($f(x) = x^3 - e^{-\sin 4x} - 1$)

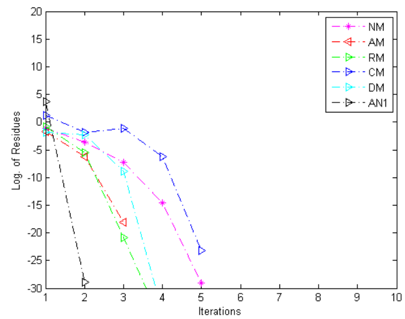

Fig. 4 ($f(x) = ln(e^{-x} + x^2)$)

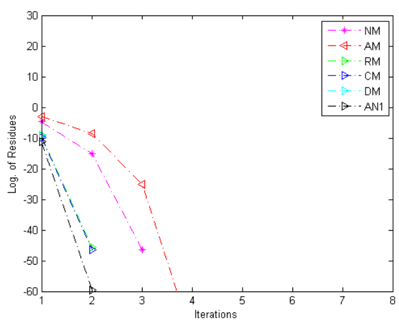

Fig. 5 ($f(x) = \sin^{-1} x$)

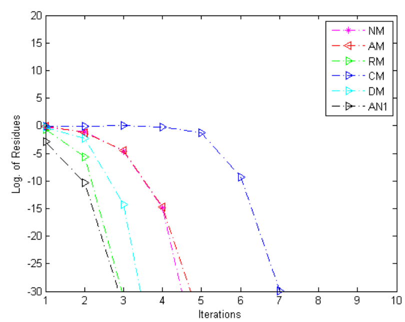

Fig. 6 ($f(x) = \sin(5x)$)

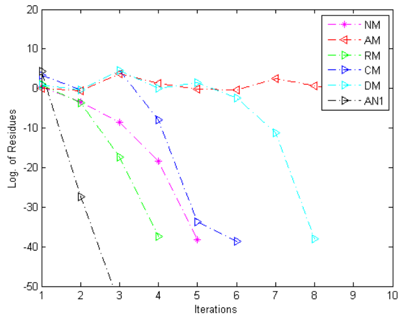

Fig. 7 ($f(x) = ln(x^2 + e^x) + x$)

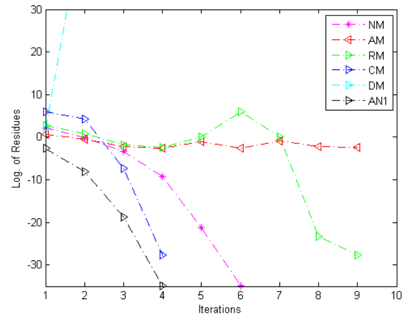

Fig. 8 ($f(x) = x + 2x^3 \sin(x) - 1$)

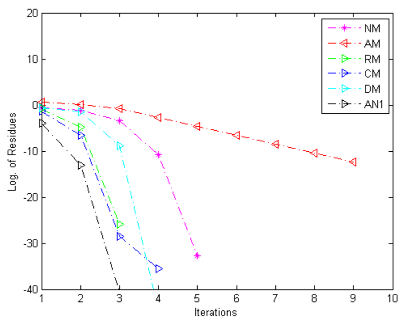

Fig. 9 ($f(x) = \sin(5x) - \sin(x)$)

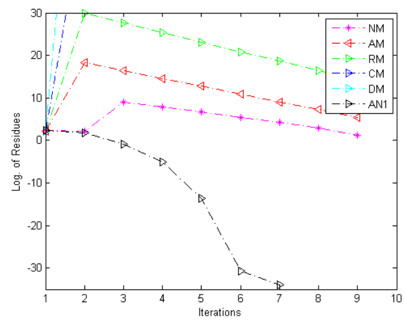

Fig. 10 ($f(x) = x^3 - x^2 - 8$)

5. Dynamical study

Polynomiography is an art and science of visualization of the zeroes of complex polynomials [12]. It has diverse applications in science, engineering, industries etc. Particularly, this art is being applied in textile industry for designing and printing.

In this section, we present some interesting polynomiographs, i.e. Fig. 11 to Fig. 18 of certain complex polynomials in the context of the newly constructed optimal fourth-order iterative method. It is obvious from these figures that we can easily identify the zeros of complex polynomials with remarkable basins of attraction.

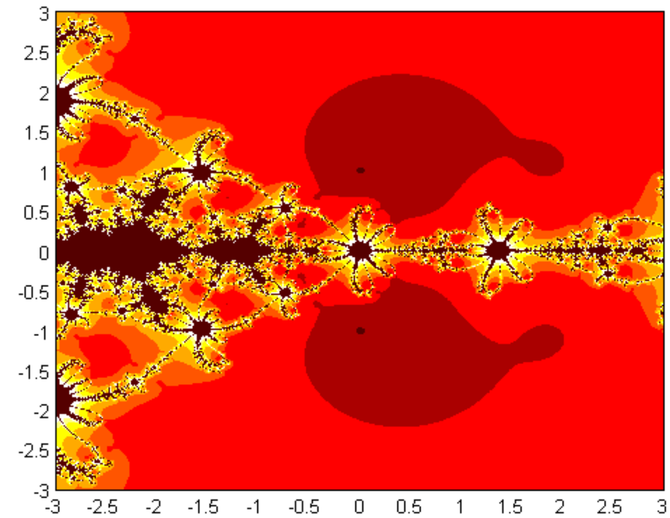

Fig. 11 Polynomiograph of $z^2 + 1$

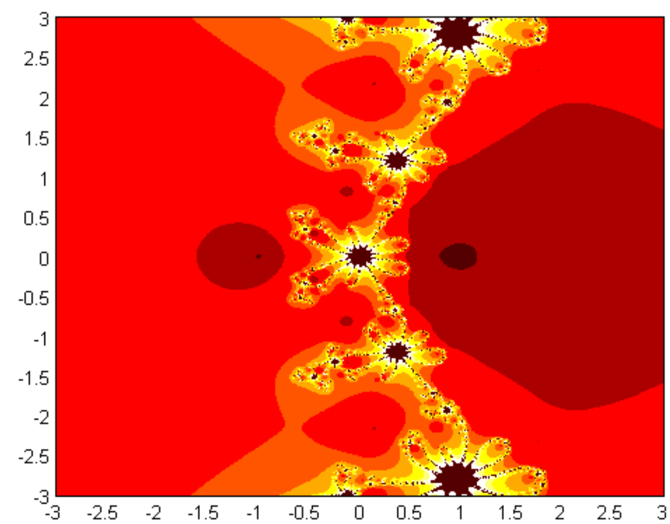

Fig. 12 Polynomiograph of $z^2 - 1$

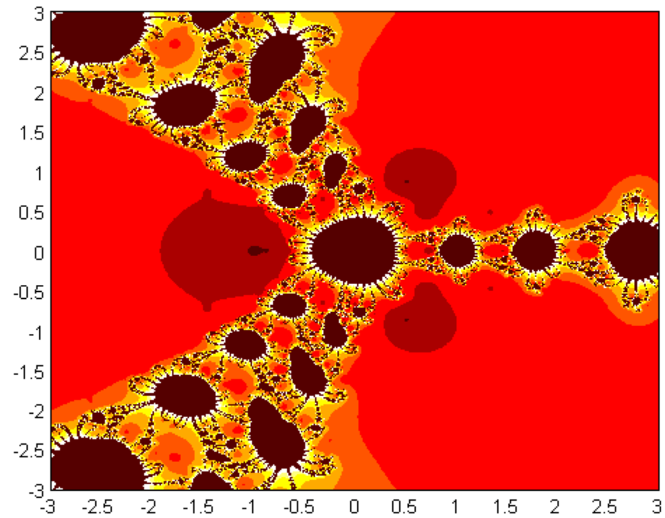

Fig. 13 Polynomiograph of $z^3 + 1$

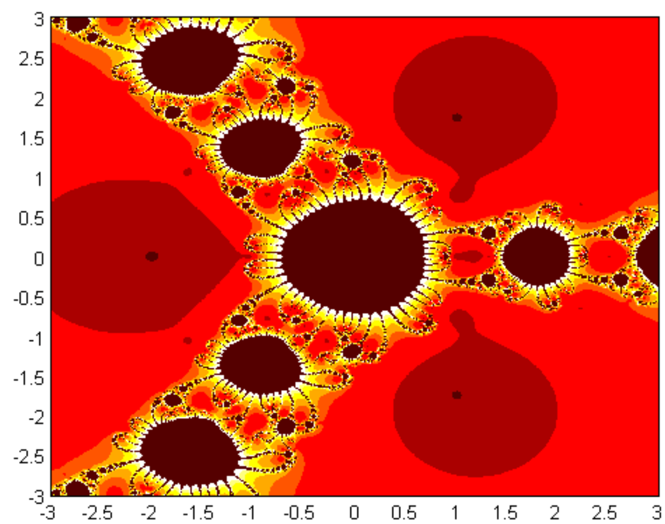

Fig. 14 Polynomiograph of $z^3 + 8$

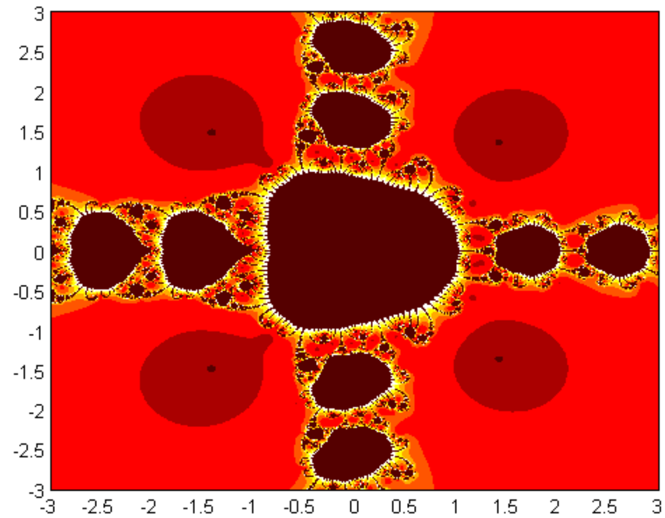

Fig. 15 Polynomiograph of $z^4 - z + 16$

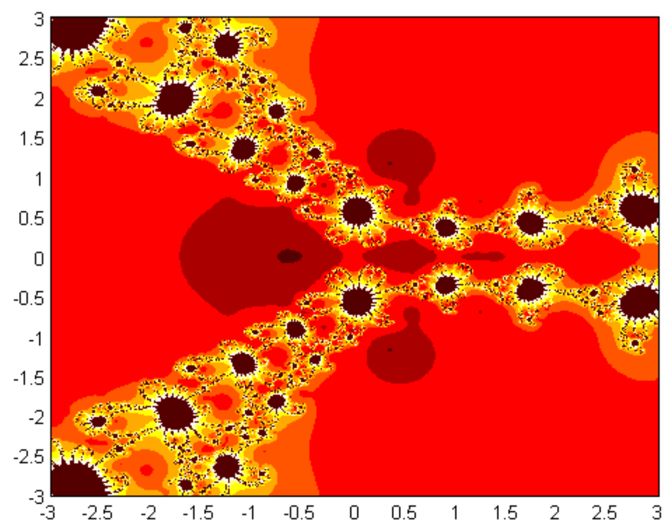

Fig. 16 Polynomiograph of $z^3 + z + 1$

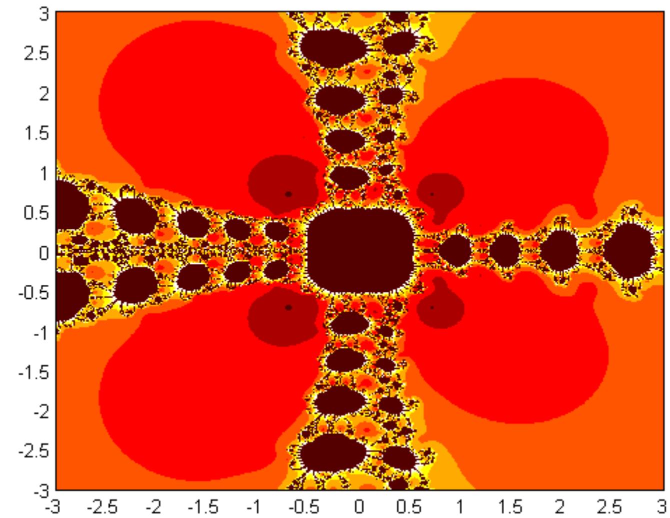

Fig. 17 Polynomiograph of $z^4 + 1$

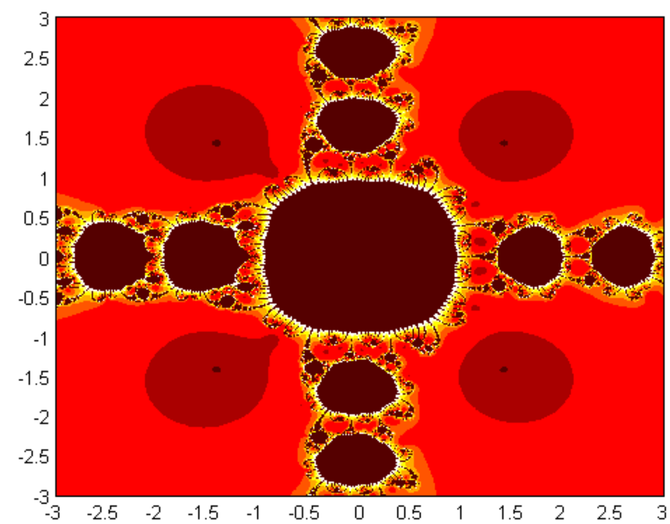

Fig. 18 Polynomiograph of $z^4 - 4$

Conclusions

A new three-step optimal fourth-order second derivative free iterative method based on the technique of linear combination has been introduced in this article. The efficiency of the newly developed method has been demonstrated both numerically and graphically by comparing the same with standard Newton's method and various other methods of the same class. In the context of the suggested method, the visualization process of the roots of certain complex polynomials has exhibited some interesting polynomiographs.

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Accepted: February 7, 2022