Total edge irregularity strength and edge irregular reflexive labeling for calendula graph

Mohamed R. Zeen El Deen*
Department of Mathematics
Faculty of Science
Suez University, Suez 42524
Egypt
mohamed.zeeneldeen@suezuniv.edu.eg

Ghada M. Elmahdy
Department of Basic science
Canal High Institute of Engineering and Technology
Suez 42524
Egypt
ghada.elmahdy91@gmail.com

Abstract. The calendula graph $Cl_{m,n}$ is a graph constructed from a cycle on m vertices C_m and m copies of C_n which are $C_{n_1}, C_{n_2}, \ldots, C_{n_m}$ and pasting the $i-$th edge of C_m to an edge of C_{n_i} for each $i \in \{1, 2, \ldots, m\}$. For a simple graph $G(V, E)$ a labeling of vertices and edges by a mapping $\Phi : V(G) \cup E(G) \to \{ 1, 2, \ldots, k \}$ providing that the weights of any two pair of edges are distinct is called *an edge irregular total $k-$labeling*, where the weight of an edge is the sum of the label of the edge itself and the labels of its two end vertices. If k is minimum and G admits an edge irregular total $k-$labeling, then k is called the total edge irregularity strength, $tes(G)$. The total $k-$ labeling is called *the reflexive edge strength* of G if the edge labeling $\Phi_e : E(G) \to \{1, 2, \ldots, k_e \}$ and a vertex labeling $\Phi_v : V(G) \to \{0, 2, 4, \ldots, 2k_v \}$, where $k = \max\{k_e, , 2k_v \}$. In the current paper, we investigate the existence of edge irregular total $k-$ labeling for the calendula graphs $Cl_{m,n}$ and precise the exact value of total edge irregularity strength of calendula graphs $Cl_{m,n}$. Besides, we explore the presence of edge reflexive irregular $r-$ labeling for calendula graphs and determine the perfect value of reflexive edge strength.

Keywords: irregular labeling, total edge irregularity strength, edge irregular reflexive labeling, reflexive edge strength, calendula graph.

1. Introduction

Graph labeling is one of the fundamental mathematical disciplines in graph theory. There are numerous applications of graph labeling in multiple areas such as coding hypothesis, computer science, physics, and astronomy. For more interesting applications of graph labeling see [1, 2]. A labeling of a graph $G = (V, E)$ is a mapping that carries graph elements (edges or vertices, or both) to

*. Corresponding author

positive integers subject to certain restrictions. If the domain is the vertex-set or the edge-set, the labeling is called vertex labeling or edge labeling respectively. Similarly if the domain is $V(G) \cup E(G)$, then the labeling is called total labeling. There are many different kinds of graph labeling (see [3, 4, 5, 6, 7]) all that kinds of labeling problem will have the following three common characteristics. A set of numbers from which vertex or edge labels are chosen, a rule that assigns a value to each edge or vertex, and a condition that these values must satisfy. A comprehensive survey of graph labeling is given in [8].

Definition 1.1 ([9]). *Let C_m be a cycle of length m with vertices $u_1, u_2, \cdots, u_m$. Let C_{n_i}, $1 \leq i \leq m$ be m copies of a cycle of length n, and v_{ij}, $1 \leq i \leq m$, $1 \leq j \leq n$ be the vertices of m copies of C_n. Let $a_i = u_i u_{i+1}$ denote to the edge of the cycle C_m for $1 \leq i \leq m-1$ and $a_m = u_m u_1$. Let $e_{ij} = v_{ij} v_{i(j+1)}$ denote to the edges of m copies of C_n for $1 \leq i \leq m$, $1 \leq j \leq n-1$ and $e_{in} = v_{in} v_{i1}$ for $1 \leq i \leq m$. The calendula graphs, denoted by $Cl_{m,n}$ obtained by pasting each edge a_i of C_m to an edge e_{in} of C_{n_i} for each $1 \leq i \leq m$, i.e., $a_i \equiv e_{in}$, $1 \leq i \leq m$ and $u_i \equiv v_{i1} \equiv v_{(i-1)n}$, $2 \leq i \leq m-1$. It is obvious that the order of $Cl_{m,n}$ is $m(n-1)$ and the size of $Cl_{m,n}$ is mn , see Fig. 1.*

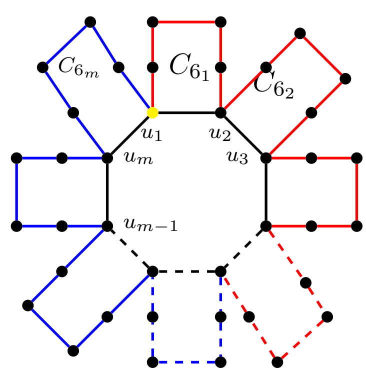

Figure 1: The Calendula graph $Cl_{m,6}$

2. Total edge irregularity strength for calendula graphs

An irregular assignment of G was defined by Chartrand et al. in [10] as a $d-$labeling of the edges $\Theta : E \longrightarrow \{1, 2, \cdots, d\}$ such that the vertex weights $wt_\Theta(x) = \sum \Theta(xy)$, where the sum is over all vertices y adjacent to x are distinctive for all vertices, i.e., $wt_\Theta(x) \neq wt_\Theta(y)$ for all vertices $x, y \in V(G)$ with $x \neq y$. The smallest d for which there is an irregular assignment is the irregularity strength, $S(G)$, this graph parameter $S(G)$ is an invariant for each graph

Bača et al. [11] defined the notion of an edge irregular total $k-$ labeling of a graph $G=(V,E)$ as a labeling of the vertices and edges of G, $\Phi : V\cup E \longrightarrow \{1,2,\cdots,k\}$ such that the edge weights $wt_{\Phi}(xy)=\Phi(x)+\Phi(y)+\Phi(xy)$ are different for all edges, i.e., $wt_{\Phi}(xy)\neq wt_{\Phi}(x^{'}y^{'})$ for all edges $xy, x^{'}y^{'}\in E$ with $xy\neq x^{'}y^{'}$. They also defined the *total edge irregularity strength* of G, $tes(G)$, to be the minimum k for which the graph G has an edge irregular total $k-$labeling. Moreover, in [11], for any graph G a lower bound on the total edge irregularity strength is given by

$$\max\left\{\left\lceil \frac{\triangle(G)+2}{3}\right\rceil, \left\lceil \frac{|E(G)|+2}{3}\right\rceil\right\} \leq tes(G) \tag{1}$$

where $\triangle(G)$ is the maximum degree of G.

Since then, many researchers try to find exact values for the total edge irregularity strength of graphs. In [12] Ivanĉo et al. proved that for any tree T $tes(T)$ is equal to its lower bound. Results on the total edge irregularity strength can be found in [13, 14, 15, 16, 17, 18, 19].

before we progress to our main result we discuss the total edge irregularity strength for a small case.

Theorem 2.1. *Let $n\geq 4$ be even positive integer and $Cl_{4,n}$ be the calendula graph. Then*

$$tes(Cl_{4,n})=\left\lceil \frac{4\ n+2}{3}\right\rceil$$

Proof. The calendula graph $Cl_{4,n}$ has $|V(Cl_{4,n})|=4(n-1)$, $|E(Cl_{4,n})|=4n$ and the maximum degree $\triangle(Cl_{4,n})=4$. Thus, the inequality (1) becomes

$$\left\lceil \frac{4\ n+2}{3}\right\rceil \leq tes(Cl_{4,n}).$$

To prove the equality, we need to show that there exist an edge irregularity total $k-$ labeling, $k=\lceil \frac{4\ n+2}{3}\rceil$, there are three cases

Case (1). When $n\equiv 2\ mod\ 3$, $n\geq 4$. Suppose that $k=\lceil \frac{4\ n+2}{3}\rceil$, we construct the total $k-$ labeling function $\Phi : V(Cl_{4,n})\cup E(Cl_{4,n}) \longrightarrow \{1,2,\cdots,k\}$ as follows:

$$\Phi(v_{1j})=\begin{cases} 1 & \text{if } j=1;\\ j-1 & \text{if } 2\leq j\leq n-2;\\ 2n-k-2 & \text{if } j=n-1;\\ k & \text{if } j=n.\end{cases}$$

$$\Phi(v_{2j})=\begin{cases} 2n-k+2 & \text{if } j=2;\\ n-2 & \text{for } 3\leq j\leq \frac{n}{2},\ if\ n\ is\ even;\\ & \text{or } 3\leq j\leq \frac{n-1}{2},\ if\ n\ is\ odd;\\ k & \text{for } \frac{n}{2}+1\leq j\leq n,\ if\ n\ is\ even;\\ & \text{or } \frac{n+1}{2}\leq j\leq n,\ if\ n\ is\ odd;\end{cases}$$

$$\Phi(v_{3j}) = \begin{cases} k & \text{for } 1 \le j \le \frac{n}{2}, \quad if \quad n \; is \; even \; ; \\ & \text{or } \; 1 \le j \le \frac{n+1}{2}, \quad if \quad n \; is \; odd \; ; \\ n-2 & \text{for } \frac{n}{2}+1 \le j \le n-2, \quad if \quad n \; is \; even \; ; \\ & \text{or } \; \frac{n+3}{2} \le j \le n-2, \quad if \quad n \; is \; odd \; ; \\ 2n-k+3 & \text{if} \quad j = \; n-1; \\ k & \text{if} \quad j = \; n. \end{cases}$$

$$\Phi(v_{4j}) = \begin{cases} 2n-k-1 & \text{if} \quad j = \; 2; \\ n-j & \text{if} \quad 3 \le j \le n-1; \\ 1 & \text{if} \quad j = \; n. \end{cases}$$

$$\Phi(e_{1j}) = \begin{cases} 1 & \text{if} \quad j = 1; \\ 2 & \text{if} \quad 2 \le j \le n-3; \\ k-n+2 & \text{if} \quad j = \; n-2; \\ 1 & \text{if} \quad j = \; n-1; \\ \dfrac{k}{2}-2 & \text{if} \quad j = \; n. \end{cases}$$

$$\Phi(e_{2j}) = \begin{cases} 1 & \text{if} \quad j = 1; \\ k-n+5 & \text{if} \quad j = \; 2; \\ 5+2j & \text{for } 3 \le j \le \frac{n}{2}-1, \quad if \quad n \; is \; even \; ; \\ & \text{or } \; 3 \le j \le \frac{n-3}{2}, \quad if \quad n \; is \; odd \; ; \\ 2k-2n-1 & \text{if} \quad j = \; \frac{n}{2}, and \quad n \quad even; \\ 2k-2n-2 & \text{if} \quad j = \; \frac{n-1}{2}, and \quad n \quad odd; \\ k-2n+2j-3 & \text{for } \frac{n}{2}+1 \le j \le n-1, \quad if \quad n \; is \; even \; ; \\ & \text{or } \; \frac{n+1}{2} \le j \le n-1, \quad if \quad n \; is \; odd \; ; \\ k-3 & \text{if} \quad j = \; n. \end{cases}$$

$$\Phi(e_{3j}) = \begin{cases} 2(2n-k-j+1) & \text{for } 1 \le j \le \frac{n}{2}-1, \quad if \quad n \; is \; even \; ; \\ & \text{or } \; 1 \le j \le \frac{n-1}{2}, \quad if \quad n \; is \; odd \; ; \\ 2n-k+4 & \text{if} \quad j = \; \frac{n}{2} \quad and \quad n \quad even; \\ 2n-k+3 & \text{if} \quad j = \; \frac{n+1}{2} \quad and \quad n \quad odd; \\ 2(n-j+3) & \text{for } \frac{n}{2}+1 \le j \le n-3, \quad if \quad n \; is \; even \; ; \\ & \text{or } \; \frac{n+3}{2} \le j \le n-3, \quad if \quad n \; is \; odd \; ; \\ k-n+5 & \text{if} \quad j = \; n-2; \\ 1 & \text{if} \quad j = \; n-1; \\ k-2 & \text{if} \quad j = \; n. \end{cases}$$

$$\Phi(e_{4j}) = \begin{cases} 1 & \text{if} \quad j = 1; \\ k-n+2 & \text{if} \quad j = 2; \\ 3 & \text{if} \quad 3 \leq j \leq n-2; \\ 2 & \text{if} \quad j = n-1; \\ 2n-k+1 & \text{if} \quad j = n. \end{cases}$$

Case (2). When $n \equiv 0 \ mod\ 3$. The labeling function $\Phi : V(Cl_{4,n}) \cup E(Cl_{4,n}) \longrightarrow \{1, 2, \cdots, k\}$ characterized as in case (1) in all vertices but with some modifications in edges given by

$$\Phi(e_{1j}) = \frac{k-3}{2} \qquad if \qquad j = n$$

$$\Phi(e_{2j}) = \begin{cases} 2k-2n & \text{if} \quad j = \frac{n}{2}, and \quad n \quad even; \\ 2k-2n-1 & \text{if} \quad j = \frac{n-1}{2}, and \quad n \quad odd; \\ k-2n+2j-2 & \text{for } \frac{n}{2}+1 \leq j \leq n-1, \quad if \quad n \ is \ even\ ; \\ & \text{or } \quad \frac{n+1}{2} \leq j \leq n-1, \quad if \quad n \ is \ odd\ ; \\ k-2 & \text{if} \quad j = n. \end{cases}$$

$$\Phi(e_{3j}) = k-1 \qquad if \qquad j = n$$

Case (3). When $n \equiv 1 \ mod\ 3$,

- If $n = 4$, the labeling $\Phi : V(Cl_{4,4}) \cup E(Cl_{4,4}) \longrightarrow \{1, 2, \cdots, k = 6\}$ characterized as in Fig. 2.

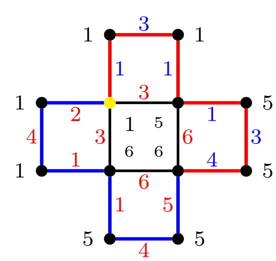

Figure 2: The calendula graph $Cl_{4,4}$ with an edge irregularity total $k = 6$ labeling

- If $n > 4$ the labeling function $\Phi : V(Cl_{4,n}) \cup E(Cl_{4,n}) \longrightarrow \{1, 2, \cdots, k\}$ defined as in case (1) in all vertices but with some modifications in edges given by

$$\Phi(e_{1j}) = \frac{k}{2} - 1 \qquad if \qquad j = n$$

$$\Phi(e_{2j}) = \begin{cases} 2k-2n+1 & \text{if } \ j = \frac{n}{2}, and \ \ n \ \ even; \\ 2k-2n & \text{if } \ j = \frac{n-1}{2}, and \ \ n \ \ odd; \\ k-2n+2j-1 & \text{for } \frac{n}{2}+1 \le j \le n-1, \ \ if \ \ n \ is \ even \ ; \\ & \quad \text{or } \frac{n+1}{2} \le j \le n-1, \ \ if \ \ n \ is \ odd \ ; \\ k-1 & \text{if } \ j = \ n. \end{cases}$$

$\Phi(e_{3j}) = \ k \qquad if \qquad j = \ n$

In all cases, all the vertex and edge labels are at most $k = \lceil \frac{4\ n+2}{3} \rceil$, also under the labeling Φ the weights of the edges are given by

$$\forall \ \ i = \ 1, \ 2 \qquad wt_{\Phi}(e_{ij}) \ = 2n(i-1)+2j+1 \ , \qquad\qquad 1 \le j \le n$$

$$\forall \ \ i = \ 3,4 \qquad wt_{\Phi}(e_{ij}) = \begin{cases} 2n(5-i)+2-2j & \text{if } \ 1 \le j \le n-1; \\ 2n(\ 5-i)+2 & \text{if } \ j = n. \end{cases}$$

We can see that the weights of edges in the first half cycles C_{n_1}, C_{n_2} form an increasing sequence of consecutive odd integers from 3 up to $4n+1$. For the second half cycles C_{n_3}, C_{n_4}, the weights of edges form a decreasing sequence of consecutive even integers from $4n+2$ up to 4. The labeling Φ is the required edge irregular total $k = \lceil \frac{4\ n+2}{3} \rceil$ labeling. This concludes the proof. □

Theorem 2.2. *Let* $n \ge 4$ *be even positive integer and* $Cl_{5,n}$ *be the calendula graph. Then*

$$tes(Cl_{5,n}) = \lceil \frac{5\ n+2}{3} \rceil \ .$$

Proof. The calendula graph $Cl_{5,n}$ has $|V(Cl_{5,n})| \ = 5(n-1)$, $|E(Cl_{5,n})\ | = 5n$ and the maximum degree $\triangle(Cl_{m,n}) = 4$. Thus, the inequality (1) becomes

$$\lceil \frac{5\ n+2}{3} \rceil \ \ \le \ tes(Cl_{5,n}).$$

To prove the inverse inequality, we define the function $\Phi : V(Cl_{5,n}) \cup E(Cl_{5,n}) \longrightarrow \{1,2,\cdots,k\}$ to be a total $k = \lceil \frac{5\ n+2}{3} \rceil$ labeling as follows:

$$\Phi(v_{1j}) = \begin{cases} 1 & \text{if } \ j = 1; \\ j-1 & \text{if } \ 2 \le j \le n. \end{cases}$$

$$\Phi(v_{2j}) = \begin{cases} n-1 & \text{for } 1 \le j \le \frac{n}{2}, \ \ if \ \ n \ \ is \ even \ ; \\ & \text{or } \ 1 \le j \le \frac{n+1}{2}, \ \ if \ \ n \ \ is \ \ odd \ ; \\ \lceil \frac{4\ n+2}{3} \rceil - 2 & \text{for } \frac{n}{2}+1 \le j \le n-1, \ \ if \ \ n \ \ is \ \ even \ ; \\ & \quad \text{or } \ \frac{n+3}{2} \le j \le n-1, \ \ if \ \ n \ \ is \ \ odd \ ; \\ k & \text{if } \ j = \ n. \end{cases}$$

$\Phi(v_{3j}) = \ \ k \qquad if \qquad 1 \le j \le n.$

$$\Phi(v_{4j}) = \begin{cases} \lceil \frac{4\,n+2}{3} \rceil - 2 & \text{for } 2 \le j \le \frac{n}{2},\ \ if\ \ n\ \ even\ ; \\ & \text{or } \ 2 \le j \le \frac{n-1}{2},\ \ if\ \ n\ \ odd\ ; \\ n-1 & \text{for } \frac{n}{2}+1 \le j \le n,\ \ if\ \ n\ \ even\ ; \\ & \text{or } \ \frac{n+1}{2} \le j \le n,\ \ if\ \ n\ \ odd. \end{cases}$$

$$\Phi(v_{5j}) = \ n-j \qquad if \qquad 1 \le j \le n-1 \quad \text{and} \qquad \Phi(v_{5n}) = \ 1\ .$$

$$\Phi(e_{1j}) = \begin{cases} 1 & \text{if} \quad j=1; \\ 2 & \text{if} \quad 2 \le j \le n-1; \\ n+1 & \text{if} \quad j = \ n. \end{cases}$$

$$\Phi(e_{2j}) = \begin{cases} 3+2j & \text{for } 1 \le j \le \frac{n}{2}-1,\ \ if\ \ n\ \ even\ ; \\ & \text{or } \ 1 \le j \le \frac{n-1}{2},\ \ if\ \ n\ \ odd\ ; \\ 2n+4-\lceil \frac{4\,n+2}{3} \rceil & \text{if} \quad j = \frac{n}{2}, and\ \ n\ \ even; \\ 2n+5-\lceil \frac{4\,n+2}{3} \rceil & \text{if} \quad j = \frac{n+1}{2}, and\ \ \ n\ \ odd; \\ 2n+2j+5-2\lceil \frac{4\,n+2}{3} \rceil & \text{for } \frac{n}{2}+1 \le j \le n-2,\ \ if\ \ n\ \ even\ ; \\ & \text{or } \ \frac{n+3}{2} \le j \le n-2,\ \ if\ \ n\ \ odd\ ; \\ 4n+1-k-\lceil \frac{4\,n+2}{3} \rceil & \text{if} \quad j = \ n-1; \\ 3n+2-k & \text{if} \quad j = \ n. \end{cases}$$

$$\Phi(e_{3j}) = \begin{cases} 4n+1+2j-2k & \text{for } 1 \le j \le \frac{n}{2},\ \ if\ \ n\ \ even\ ; \\ & \text{or } \ 1 \le j \le \frac{n+1}{2},\ \ if\ \ n\ \ odd\ ; \\ 6n+4-2j-2k & \text{for } \frac{n}{2}+1 \le j \le n,\ \ if\ \ n\ \ even\ ; \\ & \text{or } \ \frac{n+3}{2} \le j \le n,\ \ if\ \ n\ \ odd. \end{cases}$$

$$\Phi(e_{4j}) = \begin{cases} 4n+2-\lceil \frac{4\,n+2}{3} \rceil - k & \text{if} \quad j = \ 1; \\ 4n+6-2j-2\lceil \frac{4\,n+2}{3} \rceil & \text{for } 2 \le j \le \frac{n}{2}-1,\ \ if\ \ n\ \ even\ ; \\ & \text{or } \ 2 \le j \le \frac{n-3}{2},\ \ if\ \ n\ \ odd\ ; \\ 2n+5-\lceil \frac{4\,n+2}{3} \rceil & \text{if} \quad j = \frac{n}{2}\ \ and\ \ n\ \ even; \\ 2n+6-\lceil \frac{4\,n+2}{3} \rceil & \text{if} \quad j = \frac{n-1}{2}\ \ and\ \ n\ \ odd; \\ 2n+4-2j & \text{for } \frac{n}{2}+1 \le j \le n-1,\ \ if\ \ n\ \ even\ ; \\ & \text{or } \ \frac{n+1}{2} \le j \le n-1,\ \ if\ \ n\ \ odd\ ; \\ 3n+3-k & \text{if} \quad j = \ n. \end{cases}$$

$$\Phi(e_{5j}) = \begin{cases} 3 & \text{if} \quad 1 \le j \le n-2; \\ 2 & \text{if} \quad j = \ n-1; \\ n+2 & \text{if} \quad j = \ n. \end{cases}$$

Notice that all the vertex and edge labels are at most $k = \lceil \frac{5\,n+2}{3} \rceil$, moreover under the labeling Φ the weights of the edges are given by

$$\forall \ \ i = \ 1,\ 2 \qquad wt_{\Phi}(e_{ij}) \ = 2n(i-1)+2j+1\ , \qquad\qquad 1 \le j \le n$$

$$wt_\Phi(e_{3j}) = \begin{cases} 4n+2j+1 & \text{for } 1 \le j \le \frac{n}{2}, \quad and \ \ n \ \ is \ \ even \ ; \\ & \text{or } \ 1 \le j \le \frac{n+1}{2}, \quad and \ \ n \ \ is \ \ odd \ ; \\ 6n-2j+4 & \text{for } \frac{n}{2}+1 \le j \le n, \quad and \ \ n \ \ is \ \ even \ ; \\ & \text{or } \ \frac{n+3}{2} \le j \le n, \ \ and \ \ n \ \ is \ \ odd \ ; \end{cases}$$

$$\forall \ \ i = \ 4,5 \qquad wt_\Phi(e_{ij}) = \begin{cases} 2n(6-i)+2-2j & \text{if } \ \ 1 \le j \le n-1. \\ 2n(\ 6-i)+2 & \text{if } \ \ j=n. \end{cases}$$

The weights of edges in the cycles C_{n_1}, C_{n_2} form an increasing sequence of consecutive odd integers from 3 up to $4n+1$. In the cycle C_{n_3} the first $\frac{n}{2}$ edges form an increasing sequence of consecutive odd integers from $4n+3$ up to $5n+1$, while the last $\frac{n}{2}$ edges form a decreasing sequence of consecutive even integers from $5n+2$ up to $4n+4$. For the cycles C_{n_4}, C_{n_5} , the weights of edges form a decreasing sequence of consecutive even integers from $4n+2$ up to 4 . This complete the proof. □

Illustration: The graph $Cl_{5,10}$ with an edge irregularity total $k=18$ labeling is shown in Fig. 3.

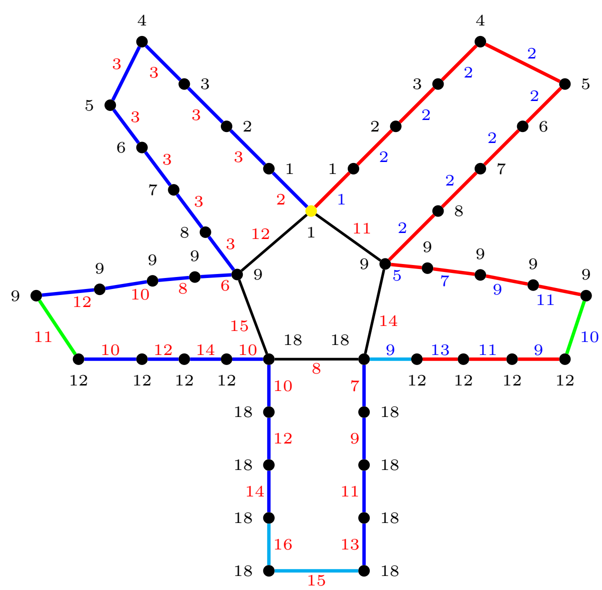

Figure 3: The calendula graph $Cl_{5,10}$ with an edge irregularity total $k=18$ labeling

Theorem 2.3. *Let* $m \ge 6$, $n \ge 4$ *be positive integers and* $Cl_{m,n}$ *be the calendula graph. Then*

$$tes(Cl_{m,n}) = \lceil \frac{m\ n+2}{3} \rceil$$

Proof. The calendula graph $Cl_{m,n}$ has $|V(Cl_{m,n})| = m(n-1)$, $|E(Cl_{m,n})| = mn$ and the maximum degree $\triangle(Cl_{m,n}) = 4$. Thus, the inequality (1) becomes

$$\lceil \frac{m\,n+2}{3} \rceil \ \leq\ tes(Cl_{m,n}).$$

To prove the equality, it suffices to prove the existence of an optimal total $k-$ labeling $\Phi : V(Cl_{m,n}) \cup E(Cl_{m,n}) \longrightarrow \{1, 2, \cdots, k\}$ is a total $k.-$ labeling, $k = \lceil \frac{m\,n+2}{3} \rceil$, we establish the labeling in the following way:

$$\Phi(v_{1j}) = \begin{cases} 1 & \text{if } \ j = 1; \\ j-1 & \text{if } \ 2 \leq j \leq n. \end{cases}$$

$$\Phi(v_{2j}) = \begin{cases} n-1 & \text{for } 1 \leq j \leq \frac{n}{2}, \ \ if \ \ n \ \ even \ ; \\ & \text{or } \ 1 \leq j \leq \frac{n+1}{2}, \ \ if \ \ n \ \ odd \ ; \\ \lceil \frac{4\,n+2}{3} \rceil - 2 & \text{for } \frac{n}{2}+1 \leq j \leq n-1, \ \ if \ \ n \ \ even \ ; \\ & \text{or } \ \frac{n+3}{2} \leq j \leq n-2\ , \ \ if \ \ n \ \ odd \ ; \\ \lceil \frac{6\,n+2}{3} \rceil & \text{if } \ j = \ n. \end{cases}$$

$\forall\ 3 \leq i \leq \frac{m}{2} - 1$ if m is even, or $\forall\ 3 \leq i \leq \frac{m-3}{2}$ if m is odd

$$\Phi(v_{ij}) = \begin{cases} \lceil \frac{2ni+2}{3} \rceil & \text{if } \ 1 \leq j \leq n-1; \\ \lceil \frac{2n(i+1)+2}{3} \rceil & \text{if } \ j = \ n. \end{cases}$$

$\forall\ i = \frac{m-1}{2}$ and m is odd

$$\Phi(v_{(\frac{m-1}{2})j}) = \begin{cases} \lceil \frac{n(m-1)+2}{3} \rceil & \text{if } \ 1 \leq j \leq n-1; \\ \lceil \frac{nm+2}{3} \rceil & \text{if } \ j = \ n. \end{cases}$$

$\forall\ i = \frac{m}{2}\ ,\ \frac{m}{2}+1$ and m is even, or $\forall\ i = \frac{m+1}{2}$ and m is odd

$$\Phi(v_{ij}) = \ k = \ \lceil \frac{nm+2}{3} \rceil \quad \text{if } \ 1 \leq j \leq n.$$

if m is odd
$$\Phi(v_{(\frac{m+3}{2})j}) = \begin{cases} \lceil \frac{nm+2}{3} \rceil & \text{if } \ j = \ 1. \\ \lceil \frac{n(m-1)+2}{3} \rceil & \text{if } \ 2 \leq j \leq n. \end{cases}$$

$\forall\ \frac{m}{2}+2 \leq i \leq m-2$ and m even, or $\forall\ \frac{m+5}{2} \leq i \leq m-2$ and m is odd

$$\Phi(v_{ij}) = \begin{cases} \lceil \frac{2n[m-(i-2)\,]+2}{3} \rceil & \text{if } \ j = \ 1\ ; \\ \lceil \frac{2n[m-(i-1)\,]+2}{3} \rceil & \text{if } \ 2 \leq j \leq n. \end{cases}$$

$$\Phi(v_{(m-1)j}) = \begin{cases} \lceil \dfrac{6n+2}{3} \rceil & \text{if } \ j=1; \\ \lceil \dfrac{4n+2}{3} \rceil - 2 & \begin{array}{l}\text{for } 2 \le j \le \frac{n}{2}, \ \ if \ \ n \ \ even \ ; \\ \text{or } \ 2 \le j \le \frac{n-1}{2}, \ \ if \ \ n \ \ odd \ ;\end{array} \\ n-1 & \begin{array}{l}\text{for } \frac{n}{2}+1 \le j \le n, \ \ if \ \ n \ \ even \ ; \\ \text{or } \ \frac{n+1}{2} \le j \le n \ , \ \ if \ \ n \ \ odd \ ;\end{array} \end{cases}$$

$$\Phi(v_{mj}) = \ n-j \qquad if \qquad 1 \le j \le n-1 \quad \text{and} \qquad \Phi(v_{mn}) = \ 1$$

$$\Phi(e_{1j}) = \begin{cases} 1 & \text{if } \ j=1; \\ 2 & \text{if } \ 2 \le j \le n-1. \\ n+1 & \text{if } \ j = \ n. \end{cases}$$

$$\Phi(e_{2j}) = \begin{cases} 3+2j & \begin{array}{l}\text{for } 1 \le j \le \frac{n}{2}-1, \ \ and \ \ n \ \ even \ ; \\ \text{or } 1 \le j \le \frac{n-1}{2}, \ \ and \ \ n \ \ odd \ ;\end{array} \\ 2n+4-\lceil \dfrac{4n+2}{3} \rceil & \text{if } \ j = \ \frac{n}{2} and \ \ n \ \ is \ even; \\ 2n+5-\lceil \dfrac{4n+2}{3} \rceil & \text{if } \ j = \ \frac{n+1}{2} and \ \ n \ \ is \ odd; \\ 2n+2j+5-2\lceil \dfrac{4n+2}{3} \rceil & \begin{array}{l}\text{for } \frac{n}{2}+1 \le j \le n-2, \ \ and \ \ n \ \ even \ ; \\ \text{or } \frac{n+3}{2} \le j \le n-2, \ \ and \ \ n \ \ odd \ ;\end{array} \\ 4n+1-\lceil \dfrac{4n+2}{3} \rceil - \lceil \dfrac{6n+2}{3} \rceil & \text{if } \ j = \ n-1; \\ 3n+2-\lceil \dfrac{6n+2}{3} \rceil & \text{if } \ j = \ n; \end{cases}$$

$\forall \ \ 3 \le i \le \frac{m}{2}-1$ and m is even, or $\forall \ \ 3 \le i \le \frac{m-3}{2}$ and m is odd

$$\Phi(e_{ij}) = \begin{cases} 2n(i-1)+2j+1-2\lceil \dfrac{2ni+2}{3} \rceil & \text{if } \ 1 \le j \le n-2; \\ 2ni-1- \ \lceil \dfrac{2ni+2}{3} \rceil - \lceil \dfrac{2n(i+1)+2}{3} \rceil & \text{if } \ j = \ n-1; \\ 2ni+1- \ \lceil \dfrac{2ni+2}{3} \rceil - \lceil \dfrac{2n(i+1)+2}{3} \rceil & \text{if } \ j = \ n. \end{cases}$$

$$\Phi(e_{\frac{m-1}{2}j}) = \begin{cases} n(m-3)+2j+1-2\lceil \dfrac{n(m-1)+2}{3} \rceil & \text{if } 1 \le j \le n-2 \text{ and } \ m \ \text{ odd;} \\ n(m-1)-1-\lceil \dfrac{n(m-1)+2}{3} \rceil - k & \text{if } \ j=n-1 \ \text{ and } \ m \ \text{ odd ;} \\ n(m-1)+1-\lceil \dfrac{n(m-1)+2}{3} \rceil - k & \text{if } \ j=n \ \text{ and } \ m \ \text{ odd.} \end{cases}$$

$$\Phi(e_{\frac{m}{2}j}) = \ n(m-2) \ +2j+1-2k \qquad \text{if } \ 1 \le j \le n \text{ and } \ m \ \text{ is even.}$$

$$\Phi(e_{\frac{m+1}{2}j}) = \begin{cases} n(m-1)+2j+1-2k & \text{for } 1 \le j \le \frac{n}{2}, \ \textit{and } n \ \textit{even}; \text{ or } 1 \le j \le \frac{n+1}{2}, \ \textit{and } n \ \textit{odd}; \\ n(m+1)-2j+4-2k & \text{for } \frac{n}{2}+1 \le j \le n, \ \textit{and } n \ \textit{even}; \text{ or } \frac{n+3}{2} \le j \le n, \ \textit{and } n \ \textit{odd}. \end{cases}$$

$$\Phi(e_{\frac{m+3}{2}j}) = \begin{cases} n(m-1) - \lceil \frac{n(m-1)+2}{3} \rceil - \lceil \frac{nm+2}{3} \rceil & \text{if } j = 1, \text{ and } m \text{ odd}; \\ n(m-1)+2(1-j) - 2\lceil \frac{n(m-1)+2}{3} \rceil & \text{if } 2 \le j \le n-1 \text{ and } m \text{ odd}; \\ n(m-1)+2 - \lceil \frac{n(m-1)+2}{3} \rceil - \lceil \frac{nm+2}{3} \rceil & \text{if } j = n \text{ and } m \text{ odd}. \end{cases}$$

$$\Phi(e_{(\frac{m}{2}+1)j}) = \begin{cases} n\ m + 2 - 2(k+j) & \text{if } 1 \le j \le n-1 \text{ and } m \text{ is even}; \\ n\ m - 2(k-1) & \text{if } j = n \text{ and } m \text{ is even}. \end{cases}$$

$\forall\ \frac{m}{2}+2 \le i \le m-2$ and m is even, or $\forall\ \frac{m+5}{2} \le i \le m-2$ and m is odd

$$\Phi(e_{ij}) = \begin{cases} 2n(m+1-i) - \lceil \frac{2n(m-i+1)+2}{3} \rceil - \lceil \frac{2n(m-i+2)+2}{3} \rceil & \text{if } j = 1; \\ 2n(m+1-i)+2(1-j) - 2\lceil \frac{2n(m-i+1)+2}{3} \rceil & 2 \le j \le n-1; \\ 2n(m+1-i)+2 - \lceil \frac{2n(m-i+1)+2}{3} \rceil - \lceil \frac{2n(m-i+2)+2}{3} \rceil & \text{if } j = n. \end{cases}$$

$$\Phi(e_{(m-1)j}) = \begin{cases} 4n+2 - \lceil \frac{4n+2}{3} \rceil - \lceil \frac{6n+2}{3} \rceil & \text{if } j = 1; \\ 4n+6-2j - 2\lceil \frac{4n+2}{3} \rceil & \text{for } 2 \le j \le \frac{n}{2}-1, \ \textit{and } n \ \textit{even}; \text{ or } 2 \le j \le \frac{n-3}{2}, \ \textit{and } n \ \textit{odd}; \\ 2n+5 - \lceil \frac{4n+2}{3} \rceil & \text{if } j = \frac{n}{2} \text{ and } n \text{ even}; \\ 2n+6 - \lceil \frac{4n+2}{3} \rceil & \text{if } j = \frac{n-1}{2} \text{ and } n \text{ odd}; \\ 2n+4-2j & \text{for } \frac{n}{2}+1 \le j \le n-1, \ \textit{and } n \ \textit{even}; \text{ or } \frac{n+1}{2} \le j \le n-1, \ \textit{and } n \ \textit{odd}; \\ 3n+3 - \lceil \frac{6n+2}{3} \rceil & \text{if } j = n. \end{cases}$$

$$\Phi(e_{mj}) = \begin{cases} 3 & \text{if } 1 \le j \le n-2; \\ 2 & \text{if } j = n-1; \\ n+2 & \text{if } j = n. \end{cases}$$

• If $n = m = 5$, the labeling $\Phi : V(Cl_{5,5}) \cup E(Cl_{5,5}) \longrightarrow \{1, 2, \cdots, k = 9\}$ defined as in Fig. 4.

It is evident that all the vertex and edge labels are at most $k = \lceil \frac{m\ n+2}{3} \rceil$. Besides, the weights of edges under the labeling Φ are given by

- If m is even number

$\forall\ 1 \le i \le \frac{m}{2}$, $\quad wt_{\Phi}(e_{ij}) = 2n(i-1)+2j+1$, $\quad 1 \le j \le n$

$\forall\ \frac{m}{2}+1 \le i \le m,$

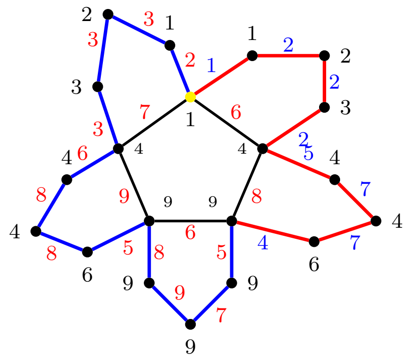

Figure 4: The calendula graph $Cl_{5,5}$ with an edge irregularity total $k = 9$ labeling

$$wt_{\Phi}(e_{ij}) = \begin{cases} 2n(m+1-i)+2-2j & \text{if} \quad 1 \leq j \leq n-1; \\ 2n(\ m+1-i)+2 & \text{if} \quad j = n. \end{cases}$$

We can see that weights of edges in the first half cycles $C_{n_1}, C_{n_2}, \cdots, C_{n_{\frac{m}{2}}}$ form an increasing sequence of consecutive odd integers from 3 up to $mn+1$. For the second half cycles $C_{n_{\frac{m}{2}+1}}, C_{n_{\frac{m}{2}+2}}, \cdots, C_{n_m}$ weights of edges form a decreasing sequence of consecutive even integers from $mn+2$ up to 4 .

- If m is odd number

$\forall \ \ 1 \leq i \leq \frac{m-1}{2} \qquad wt_{\Phi}(e_{ij}) \ = 2n(i-1)+2j+1 \ , \qquad\qquad 1 \leq j \leq n$

$$wt_{\Phi}(e_{(\frac{m+1}{2})j}) = \begin{cases} n(m-1)+2j+1 & \text{for } 1 \leq j \leq \frac{n}{2}, \quad \textit{and} \quad n \textit{ is even} ; \\ & \text{or } 1 \leq j \leq \frac{n+1}{2}, \quad \textit{and } n \textit{ is odd} ; \\ n(m+1)-2j+4 & \text{for } \frac{n}{2}+1 \leq j \leq n, \ \textit{and} \quad n \textit{ is even} ; \\ & \text{or } \frac{n+3}{2} \leq j \leq n, \ \textit{and } n \textit{ is odd.} \end{cases}$$

$\forall \ \ 1 \leq i \leq \frac{m+3}{2}, \ \ \cdots, \ m$

$$wt_{\Phi}(e_{ij}) = \begin{cases} 2n(m+1-i)+2-2j & \text{if} \quad 1 \leq j \leq n-1; \\ 2n(m+1-i)+2 & \text{if} \quad j = n. \end{cases}$$

Since all the edge weights are distinct, the labeling Φ is the required edge irregular total $k = \lceil \frac{m\ n+2}{3} \rceil$ labeling. This concludes the proof. $\square$

3. Edge irregular reflexive for calendula graphs

The concept of the edge irregular reflexive $r-$ labeling was introduced by Ryan et al. [20]. For a graph G, an edge labeling $\Psi_e : E(G) \longrightarrow \{1, 2, \cdots, r_e\}$ and a vertex labeling $\Psi_v : V(G) \longrightarrow \{0, 2, 4, \cdots, 2r_v\}$, then labeling Ψ defined by

$$\Psi(x) = \begin{cases} \Psi_v(x) & \text{if} \quad x \ \in V(G) \\ \Psi_e(x) & \text{if} \quad x \ \in E(G) \end{cases}$$

is a total $r-$ labeling where $r = \max\{r_e, 2r_v\}$. The total $r-$ labeling Ψ is called an edge irregular reflexive $r-$ labeling of the graph G if distinct edges has different weights. The smallest value of r for which such labeling exists is called the reflexive edge strength of the graph G and is denoted by $res(G)$. During the past few years, res(G) has been inspected for distinctive family of graphs (see [21, 22, 23, 24, 25, 26]). In this section, we examine the edge irregular reflexive $r-$ labeling for the calendula graphs.

Let us recall the following lemma proved in [21]

Lemma 3.1. *For every graph G,*

$$res(G) \geq \begin{cases} \lceil \dfrac{E(G)}{3} \rceil & \text{if } |E(G)| \not\equiv 2,3 \ (mod\ 6). \\ \lceil \dfrac{E(G)}{3} \rceil + 1 & \text{if } |E(G)| \equiv 2,3 \ (mod\ 6). \end{cases} \tag{2}$$

Also, Baća et al. [22] suggested the following conjecture:

Conjecture 3.1 ([22]). *Let G be a simple graph with maximum degree $\triangle = \triangle(G)$. Then*

$$res(G) = \max\{\lfloor \frac{\triangle + 2}{2} \rfloor, \lfloor \frac{E(G)}{3} \rfloor + d\},$$

where $d = 1$ for $|E(G)| \equiv 2,3\ (mod\ 6)$, and zero otherwise

Theorem 3.1. *Let $n \geq 4$ be positive integer and $Cl_{4,n}$ be the calendula graph. Then*

$$res(Cl_{4,n}) = \begin{cases} \lceil \dfrac{4\ n}{3} \rceil & \text{if } |E(G)| \equiv 0,4 \ (mod\ 6). \\ \lceil \dfrac{4\ n}{3} \rceil + 1 & \text{if } |E(G)| \equiv 2 \ (mod\ 6). \end{cases}$$

Proof. Since $|V(Cl_{4,n})| = 4(n-1)$, $|E(Cl_{4,n})| = 4n$ and $\triangle(Cl_{m,n}) = 4$, the inequality (2) becomes

$$res(Cl_{4,n}) \geq \begin{cases} \lceil \dfrac{4\ n}{3} \rceil & \text{if } |E(G)| \equiv 0,4 \ (mod\ 6). \\ \lceil \dfrac{4\ n}{3} \rceil + 1 & \text{if } |E(G)| \equiv 2 \ (mod\ 6). \end{cases}$$

To demonstrate the equality, we have to appear that there exists an edge irregularity reflexive $r = \lceil \frac{4\ n}{3} \rceil$ labeling, for the calendula graph $Cl_{4,n}$, there are three cases:

Case (1). When $n \equiv 0\ mod\ 3$, $n \geq 4$, then $|E(G)| \equiv 0\ (mod\ 6)$. Suppose that $r = \lceil \frac{4\ n}{3} \rceil$, we build the labeling function $\Psi_v : V(Cl_{4,n})) \longrightarrow \{0, 2, \cdots, 2r_v = r\}$ as follows:

$$\Psi_v(v_{1j}) = \begin{cases} j-1 & \text{for } j = 1,3,5,\cdots, n-3, \ \ if \ \ n \ \ is \ \ even; \\ & \text{or } j = 1,3,5,\cdots, n-2, \ \ if \ \ n \ \ is \ \ odd; \\ j-2 & \text{for } j = 2,4,6,\cdots, n-2, \ \ if \ \ n \ \ is \ \ even; \\ & \text{or } j = 2,4,6,\cdots, n-3, \ \ if \ \ n \ \ is \ \ odd; \\ 2n-4-r & \text{if } \ j = n-1; \\ r & \text{if } \ j = n. \end{cases}$$

$$\Psi_v(v_{2j}) = \begin{cases} r & for \ \text{j=1} \ \text{ and } \ \text{for } \frac{n}{2}+1 \le j \le n, \ if \ \ n \ \ is \ \ even; \\ & \text{or } \frac{n+3}{2} \le j \le n, \ if \ \ n \ \ is \ \ odd; \\ 2n-r & \text{if } \ j = 2; \\ n-2 & \text{if } \ 3 \le j \le \frac{n}{2}, \ \ n \ \ is \ \ even; \\ n-1 & \text{if } \ 3 \le j \le \frac{n+1}{2}, \ \ n \ \ is \ \ odd; \end{cases}$$

$$\Psi_v(v_{3j}) = \begin{cases} r & \text{for } 1 \le j \le \frac{n}{2}, \ \ if \ \ n \ \ is \ \ even; \ \ and \ \ \text{j= n}; \\ & \text{or } \ 1 \le j \le \frac{n-1}{2}, \ \ if \ \ n \ \ is \ \ odd; \\ n-2 & \text{if } \ \frac{n}{2}+1 \le j \le n-2, \ \ n \ \ is \ \ even; \\ n-1 & \text{if } \ \frac{n+1}{2} \le j \le n-2, \ \ n \ \ is \ \ odd; \\ 2n-r & \text{if } \ j = n-1; \end{cases}$$

$$\Psi_v(v_{4j}) = \begin{cases} r & \text{if } \ j = 1; \\ 2n-4-r & \text{if } \ j = 2; \\ n-j-1 & \text{for } j = 3,5,\cdots, n-1, \ \ if \ \ n \ \ is \ \ even; \\ & \text{or } \ j = 4,6,\cdots, n-1, \ \ if \ \ n \ \ is \ \ odd; \\ n-j & \text{for } j = 4,6,\cdots, n, \ \ if \ \ n \ \ is \ \ even; \\ & \text{or } \ j = 3,5,\cdots, n, \ \ if \ \ n \ \ is \ \ odd; \end{cases}$$

For edges, we construct the labeling function $\Psi_e : E(Cl_{4,n})) \longrightarrow \{1,2,\cdots,r\}$ as follows:

$$\Psi_e(e_{1j}) = \begin{cases} 1 & \text{if } \ 1 \le j \le n-3, \ \ and \ \ j = n-1; \\ r-n+3 & \text{if } \ j = n-2, \ \ n \ \ is \ \ even; \\ r-n+2 & \text{if } \ j = n-2, \ \ n \ \ is \ \ odd; \\ 2n-r-1 & \text{if } \ j = n. \end{cases}$$

$$\Psi_e(e_{2j}) = \begin{cases} 1 & \text{if} \quad j = 1; \\ r-n+5 & \text{if} \quad j = \ 2, \quad n \ \ is \ \ even \ ; \\ r-n+4 & \text{if} \quad j = \ 2, \quad n \ \ is \ \ odd \ ; \\ 3+2j & \text{if} \quad 3 \le j \le \frac{n}{2} - 1, \quad n \ \ is \ \ even \ ; \\ 1+2j & \text{if} \quad 3 \le j \le \frac{n-1}{2}, \quad n \ \ is \ \ odd \ ; \\ 2n+1-r & \begin{array}{l} \text{for } j = \ \frac{n}{2}, \quad n \ \ is \ \ even \ \ ; \\ \text{and} \ \ j = \ \frac{n+1}{2}, \quad n \ \ is \ \ odd \ \ ; \end{array} \\ 2n-1-2r+2j & \begin{array}{l} \text{for } \frac{n}{2}+1 \le j \le n, \ \ if \ \ n \ \ is \ \ even \ ; \\ \text{or} \ \ \frac{n+3}{2} \le j \le n, \ \ if \ \ n \ \ is \ \ odd \ ; \end{array} \end{cases}$$

$$\Psi_e(e_{3j}) = \begin{cases} 4n-2r-2j & \begin{array}{l} \text{for } 1 \le j \le \frac{n}{2} - 1, \ \ if \ \ n \ \ is \ \ even \ ; \\ \text{or} \ \ 1 \le j \le \frac{n-3}{2}, \ \ if \ \ n \ \ is \ \ odd \ ; \end{array} \\ 2n+2-r & \begin{array}{l} \text{for } j = \ \frac{n}{2}, \quad n \ \ is \ \ even \ \ ; \\ \text{and} \ \ j = \ \frac{n-1}{2}, \quad n \ \ is \ \ odd \ \ ; \end{array} \\ 2n+4-2j & \text{if} \quad \frac{n}{2}+1 \le j \le n-3 \ \ and \ \ n \ \ even; \\ 2n+2-2j & \text{if} \quad \frac{n+1}{2} \le j \le n-3 \ \ and \ \ n \ \ odd; \\ r-n+6 & \text{if} \quad j = \ n-2 \ \ and \ \ n \ \ is \ \ even; \\ r-n+5 & \text{if} \quad j = \ n-2 \ \ and \ \ n \ \ is \ \ odd; \\ 2 & \text{if} \quad j = \ n-1; \\ r & \text{if} \quad j = \ \ n. \end{cases}$$

$$\Psi_e(e_{4j}) = \begin{cases} 2 & \text{if} \quad j = 1 \ \ and \ \ 3 \le j \le n-1; \\ r-n+4 & \text{if} \quad j = \ 2, \quad n \ \ is \ \ even \ ; \\ r-n+3 & \text{if} \quad j = \ 2, \quad n \ \ is \ \ odd \ ; \\ 2n-r & \text{if} \quad j = \ \ n. \end{cases}$$

Case (2). When $n \equiv \ 1 \, mod \, 3$, then $|E(G)| \equiv 4 \; (mod \; 6)$. Let $r = \lceil \frac{4 \ n}{3} \rceil$, the labeling functions Ψ_v and Ψ_e defined as in case (1) in all vertices and edges but with some modifications which are given by

$$\Psi_v(v_{1(n-1)}) = \ 2n-2-r \ , \qquad \Psi_v(v_{1n}) = \ r-2 \ ,$$

$$\Psi_v(v_{21}) = \ r-2 \ , \qquad \Psi_v(v_{22}) = \ 2n-r+2 \ ,$$

$$\Psi_v(v_{3(n-1)}) = \ 2n+2-r \ , \qquad \Psi_v(v_{3n}) = \ r-2 \ ,$$

$$\Psi_v(v_{41}) = \ r-2 \ , \qquad \Psi_v(v_{42}) = \ 2n-r-2 \ ,$$

$$\Psi_e(e_{1j}) = \begin{cases} r-n+1 & \text{if} \quad j = \ \ n-2, \quad n \ \ even \ ; \\ r-n & \text{if} \quad j = \ n-2, \quad n \ \ odd \ ; \\ 2n-r+1 & \text{if} \quad j = \ \ n. \end{cases}$$

$$\Psi_e(e_{2j}) = \begin{cases} r-n+3 & \text{if } \ j = \ 2, \ \ n \ \ even \ ; \\ r-n+2 & \text{if } \ j = \ 2, \ \ n \ \ odd \ ; \\ 2n-1-2r+2j & \text{for } \frac{n}{2}+1 \le j \le n-1, \ \ if \ \ n \ \ even \ ; \\ & \text{or } \frac{n+3}{2} \le j \le n-1, \ \ if \ \ n \ \ odd \ ; \\ 4n-2r+1 & \text{if } \ j = \ n. \end{cases}$$

$$\Psi_e(e_{3j}) = \begin{cases} r-n+4 & \text{if } \ j = \ n-2 \ \ and \ \ n \ \ even; \\ r-n+3 & \text{if } \ j = \ n-2 \ \ and \ \ n \ \ odd; \end{cases}$$

$$\Psi_e(e_{4j}) = \begin{cases} r-n+2 & \text{if } \ j = \ 2, \ \ n \ \ even \ ; \\ r-n+1 & \text{if } \ j = \ 2, \ \ n \ \ odd \ ; \\ 2n-r+2 & \text{if } \ j = \ n. \end{cases}$$

Case (3). When $n \equiv \ 2 \ mod \ 3,$ then $|E(G)| \equiv 2 \ (mod \ 6)$.suppose that $r = \lceil \frac{4\,n}{3} \rceil + 1$, the labeling functions Ψ_v and Ψ_e defined as in case (1) in all vertices and edges but with some modifications which are given by

$\Psi_v(v_{1(n-1)}) = \ 2n-r \ ,$ $\Psi_v(v_{1n}) = \ r-4 \ ,$

$\Psi_v(v_{21}) = \ r-4 \ ,$ $\Psi_v(v_{22}) = \ 2n-r+4 \ ,$

$\Psi_v(v_{3(n-1)}) = \ 2n+4-r \ ,$ $\Psi_v(v_{3n}) = \ r-4 \ ,$

$\Psi_v(v_{41}) = \ r-4 \ ,$ $\Psi_v(v_{42}) = \ 2n-r \ ,$

$$\Psi_e(e_{1j}) = \begin{cases} r-n-1 & \text{if } \ j = \ n-2, \ \ n \ \ even \ ; \\ r-n-2 & \text{if } \ j = \ n-2, \ \ n \ \ odd \ ; \\ 2n-r+3 & \text{if } \ j = \ n. \end{cases}$$

$$\Psi_e(e_{2j}) = \begin{cases} r-n+1 & \text{if } \ j = \ 2, \ \ n \ \ even \ ; \\ r-n & \text{if } \ j = \ 2, \ \ n \ \ odd \ ; \\ 2n-1-2r+2j & \text{for } \frac{n}{2}+1 \le j \le n-1, \ \ if \ \ n \ \ even \ ; \\ & \text{or } \frac{n+3}{2} \le j \le n-1, \ \ if \ \ n \ \ odd \ ; \\ 4n-2r+3 & \text{if } \ j = \ n. \end{cases}$$

$$\Psi_e(e_{3j}) = \begin{cases} r-n+2 & \text{if } \ j = \ n-2 \ \ and \ \ n \ \ even; \\ r-n+1 & \text{if } \ j = \ n-2 \ \ and \ \ n \ \ odd; \end{cases}$$

$$\Psi_e(e_{4j}) = \begin{cases} r-n & \text{if } \ j = \ 2, \ \ n \ \ even \ ; \\ r-n-1 & \text{if } \ j = \ 2, \ \ n \ \ odd \ ; \\ 2n-r+4 & \text{if } \ j = \ n. \end{cases}$$

In all cases, notice that all the vertex and edge labels are at most $r = \lceil \frac{4\,n}{3} \rceil$ or $r = \lceil \frac{4\,n}{3} \rceil + 1$. Moreover under the labeling Ψ the weights of the edges are given by

$$\forall \ \ i = \ 1, \ 2 \qquad wt_{\Psi}(e_{ij}) \ = 2n(i-1)+2j-1 \ , \qquad 1 \le j \le n.$$

$$\forall \;\; i = \; 3,4 \qquad wt_{\Psi}(e_{ij}) = \begin{cases} 2n(5-i)-2j & \text{if} \quad 1 \leq j \leq n-1; \\ 2n(5-i) & \text{if} \quad j = n. \end{cases}$$

We can see that weights of edges in the first half cycles C_{n_1}, C_{n_2} form an increasing sequence of consecutive odd integers from 1 up to $4n-1$. For the second half cycles C_{n_3}, C_{n_4} weights of edges form a decreasing sequence of consecutive even integers from $4n$ up to 2 . In this way the labeling Ψ is the required edge irregularity reflexive $r-$ labeling. This completes the proof. □

Illustration: The calendula graph $Cl_{4,12}$ with an edge irregularity total $k = 17$ labeling and an edge irregularity reflexive $r = 16$ labeling are shown in Fig. 5.

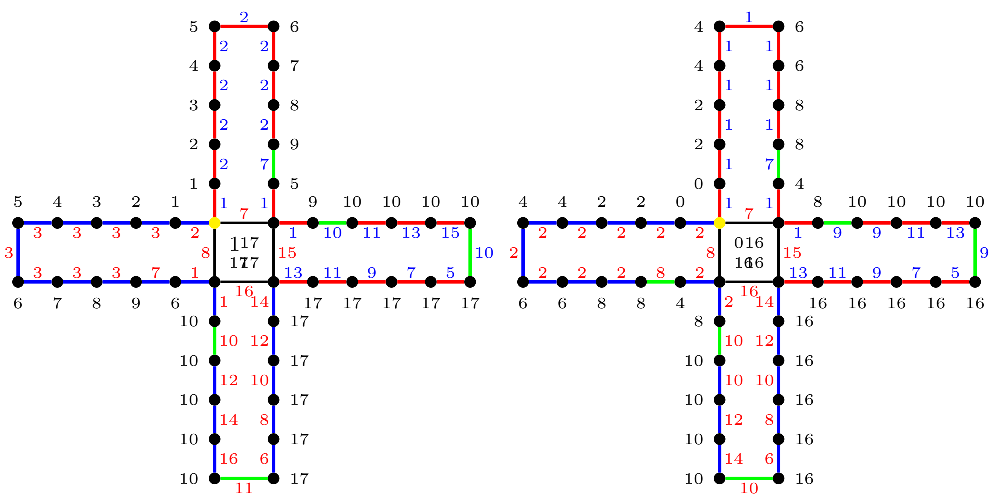

Figure 5: (a) $Cl_{4,12}$ with $k = 17$ (b) $Cl_{4,12}$ with $r = 16$

Theorem 3.2. *Let* $n \geq 6$ *and* m *be an even positive integer,* $m \geq 6$ *. Then the calendula graph* $Cl_{m,n}$ *have .*

$$res(Cl_{m,n}) = \begin{cases} \lceil \dfrac{mn}{3} \rceil & \textit{for } n \equiv 0 \textit{ mod } 3; \\ & \textit{or} \;\; n \equiv \;1\;,2 \textit{ mod } 3 \textit{ and } |m\;n| \equiv 0,4 \textit{ mod } 6; \\ \lceil \dfrac{m\;n}{3} \rceil + 1 & \textit{if } n \equiv 1,2 \textit{ mod } 3 \textit{ and } |mn| \equiv 2(\textit{mod } 6). \end{cases}$$

Proof. Since $|V(Cl_{m,n})| \;= m(n-1)$, $|E(Cl_{m,n})\;| = mn$ and $\triangle(Cl_{m,n}) = 4$. To illustrate the equality in (2), we have to show up that there exists an edge edge irregularity reflexive $r-$ labeling, $r = \lceil \frac{m\;n}{3} \rceil$, or $r = \lceil \frac{m\;n}{3} \rceil + 1$, for the calendula graph $Cl_{m,n}$, there are three cases

Case (1). When $n \equiv \; 0 \; mod \; 3$ or $n \equiv \; 1 \; mod \; 3$ and $|m\;n| \not\equiv 2 \; (mod \; 6)$, we take $r = \lceil \frac{m\;n}{3} \rceil$.

When $n \equiv \; 1 \; mod \; 3$ and $|m\;n| \equiv 2 \; (mod \; 6)$, we put $r = \lceil \frac{m\;n}{3} \rceil + 1$.

We construct the labeling function $\Psi_v : V(Cl_{m,n})) \longrightarrow \{0, 2, \cdots, r\}$ as follows:

$$\Psi_v(v_{1j}) = \begin{cases} j-1 & \text{for } j = 1,3,5,\cdots, n-1, \;\; if \;\; n \;\; even \; ; \\ & \text{or } j = 1,3,5,\cdots, n, \;\; if \;\; n \;\; odd \; ; \\ j-2 & \text{for } j = 2,4,6,\cdots, n, \;\; if \;\; n \;\; even \; ; \\ & j = 2,4,6,\cdots, n-1, \;\; if \;\; n \;\; odd. \end{cases} \tag{3}$$

$$\Psi_v(v_{2j}) = \begin{cases} n-2 & \text{if } \; 1 \le j \le \frac{n}{2}, \;\; n \;\; even \; ; \\ n-1 & \text{if } \; 1 \le j \le \frac{n+1}{2}, \;\; n \;\; odd \; ; \\ \lceil \frac{4\,n}{3} \rceil & \text{for } \frac{n}{2}+1 \le j \le n-1, \;\; if \;\; n \;\; even \; ; \\ & \text{or } \frac{n+3}{2} \le j \le n-1, \;\; if \;\; n \;\; odd \; ; \\ \lceil \frac{6\,n}{3} \rceil & \text{if } \; j = n. \end{cases} \tag{4}$$

$\forall \quad 3 \le i \le \frac{m}{2} - 1$

If $n \equiv 0 \; mod\; 3$, or $n \equiv 1 \; mod\; 3, \;\; and \;\; i \equiv 2 \; mod\; 3,$

$$\Psi_v(v_{ij}) = \begin{cases} \lceil \frac{2\,n\,i}{3} \rceil & \text{if } \; 1 \le j \le n-1; \\ \lceil \frac{2\,n\,(i+1)}{3} \rceil & \text{if } \; j = n, \end{cases} \tag{5}$$

If $n \equiv 1 \; mod\; 3, \;\; and \;\; i \equiv 0 \; mod\; 3,$

$$\Psi_v(v_{ij}) = \begin{cases} \lceil \frac{2\,n\,i}{3} \rceil & \text{if } \; 1 \le j \le n-1; \\ \lceil \frac{2\,n\,(i+1)}{3} \rceil + 1 & \text{if } \; j = n, \end{cases} \tag{6}$$

If $n \equiv 1 \; mod\; 3, \;\; and \;\; i \equiv 1 \; mod\; 3,$

$$\Psi_v(v_{ij}) = \begin{cases} \lceil \frac{2\,n\,i}{3} \rceil + 1 & \text{if } \; 1 \le j \le n-1; \\ \lceil \frac{2\,n\,(i+1)}{3} \rceil & \text{if } \; j = n, \end{cases} \tag{7}$$

$$\forall \quad i = \frac{m}{2}, \frac{m}{2}+1 , \qquad \Psi_v(v_{ij}) = r \quad for \quad 1 \le j \le n \tag{8}$$

$\forall \quad \frac{m}{2} + 2 \le i \le m - 2$

If $n \equiv 0 \ mod\ 3$ or $n \equiv 1 \ mod\ 3$, *and* $i \equiv 1 \ mod\ 3$,

$$\Psi_v(v_{ij}) = \begin{cases} \lceil \dfrac{2\ n\ [m-(i-2)]}{3} \rceil & \text{if} \quad j = 1; \\ \lceil \dfrac{2\ n\ [m-(i-1)]}{3} \rceil & \text{if} \quad 2 \le j \le n, \end{cases} \tag{9}$$

If $n \equiv 1 \ mod\ 3$, *and* $i \equiv 2 \ mod\ 3$,

$$\Psi_v(v_{ij}) = \begin{cases} \lceil \dfrac{2\ n\ [m-(i-2)]}{3} \rceil & \text{if} \quad j = 1; \\ \lceil \dfrac{2\ n\ [m-(i-1)]}{3} \rceil + 1 & \text{if} \quad 2 \le j \le n, \end{cases} \tag{10}$$

If $n \equiv 1 \ mod\ 3$, *and* $i \equiv 0 \ mod\ 3$,

$$\Psi_v(v_{ij}) = \begin{cases} \lceil \dfrac{2\ n\ [m-(i-2)]}{3} \rceil + 1 & \text{if} \quad j = 1; \\ \lceil \dfrac{2\ n\ [m-(i-1)]}{3} \rceil & \text{if} \quad 2 \le j \le n, \end{cases} \tag{11}$$

$$\Psi_v(v_{(m-1)j}) = \begin{cases} \lceil \dfrac{6\ n}{3} \rceil & \text{if} \quad j = 1; \\ \lceil \dfrac{4\ n}{3} \rceil & \text{for } 2 \le j \le \frac{n}{2}, \ \ if \ \ n \ \ even\ ; \\ & \text{or } \ 2 \le j \le \frac{n-1}{2}, \ if \ \ n \ \ odd\ ; \\ n-2 & \text{if} \quad \frac{n}{2}+1 \le j \le n, \quad n \ \ even\ ; \\ n-1 & \text{if} \quad \frac{n+1}{2} \le j \le n, \quad n \ \ odd, \end{cases} \tag{12}$$

$$\Psi_v(v_{mj}) = \begin{cases} n-j-1 & \text{for } j = 1, 3, \cdots, n-1, \ if \ \ n \ \ even\ ; \\ & \text{or } \ j = 2, 4, \cdots, n-1, \ if \ \ n \ \ odd\ ; \\ n-j & \text{for } j = 2, 4, \cdots, n, \ if \ \ n \ \ even\ ; \\ & j = 1, 3, \cdots, n, \ \ if \ \ n \ \ odd, \end{cases} \tag{13}$$

For the edges, we define the labeling function $\Psi_e : E(Cl_{m,n})) \longrightarrow \{1, 2, \cdots, r\}$ as follows:

$$\Psi_e(e_{1j}) = \begin{cases} 1 & \text{if} \quad 1 \le j \le n-1; \\ n+1 & \text{if} \quad j = n\ , \ \ n \ \ even\ ; \\ n & \text{if} \quad j = n, \quad n \ \ odd, \end{cases} \tag{14}$$

$$(15)\qquad \Psi_e(e_{2j}) = \begin{cases} 2j+3 & \text{if } 1 \le j \le \frac{n}{2}-1,\ n\ even; \\ 2j+1 & \text{if } 1 \le j \le \frac{n-1}{2},\ n\ odd\ ; \\ 2n+1-\lceil\frac{4n}{3}\rceil & \text{for } j=\frac{n}{2}\ if\ n\ even; \text{ or } j=\frac{n+1}{2},\ if\ n\ odd; \\ 2n-1+2j-2\lceil\frac{4n}{3}\rceil & \text{for } \frac{n}{2}+1 \le j \le n-2\ if\ n\ even; \text{ or } \frac{n+3}{2} \le j \le n-2\ if\ n\ odd; \\ 4n-3-\lceil\frac{4n}{3}\rceil-\lceil\frac{6n}{3}\rceil & \text{if } j=n-1. \\ 3n+1-\lceil\frac{6n}{3}\rceil & \text{if } j=n,\ n\ is\ even. \\ 3n-\lceil\frac{6n}{3}\rceil & \text{if } j=n,\ n\ is\ odd, \end{cases}$$

$\forall \quad 3 \le i \le \frac{m}{2}-1$

If $n \equiv 0\ mod\ 3$ or $n \equiv 1\ mod\ 3,\ and\ i \equiv 2\ mod\ 3,$

$$(16)\qquad \Psi_e(e_{ij}) = \begin{cases} 2n(i-1)+2j-1-2\lceil\frac{2ni}{3}\rceil & \text{if } 1 \le j \le n-2; \\ 2ni-3-\lceil\frac{2ni}{3}\rceil-\lceil\frac{2n(i+1)}{3}\rceil & \text{if } j=n-1, \\ 2ni-1-\lceil\frac{2ni}{3}\rceil-\lceil\frac{2n(i+1)}{3}\rceil & \text{if } j=n, \end{cases}$$

If $n \equiv 1\ mod\ 3,\ and\ i \equiv 0\ mod\ 3,$

$$(17)\qquad \Psi_e(e_{ij}) = \begin{cases} 2n(i-1)+2j-1-2\lceil\frac{2ni}{3}\rceil & \text{if } 1 \le j \le n-2; \\ 2ni-4-\lceil\frac{2ni}{3}\rceil-\lceil\frac{2n(i+1)}{3}\rceil & \text{if } j=n-1, \\ 2ni-2-\lceil\frac{2ni}{3}\rceil-\lceil\frac{2n(i+1)}{3}\rceil & \text{if } j=n, \end{cases}$$

If $n \equiv 1\ mod\ 3,\ and\ i \equiv 1\ mod\ 3,$

$$(18)\qquad \Psi_e(e_{ij}) = \begin{cases} 2n(i-1)+2j-3-2\lceil\frac{2ni}{3}\rceil & \text{if } 1 \le j \le n-2; \\ 2ni-4-\lceil\frac{2ni}{3}\rceil-\lceil\frac{2n(i+1)}{3}\rceil & \text{if } j=n-1, \\ 2ni-2-\lceil\frac{2ni}{3}\rceil-\lceil\frac{2n(i+1)}{3}\rceil & \text{if } j=n, \end{cases}$$

$$\Psi_e(e_{\frac{m}{2}j}) = n(m-2) + 2j - 1 - 2r, \qquad for \quad 1 \le j \le n \tag{19}$$

$$\Psi_e(e_{(\frac{m}{2}+1)j}) = \begin{cases} nm - 2(r+j) & \text{if} \quad 1 \le j \le n-1 \ ; \\ nm - 2r & \text{if} \quad j = \ n \ . \end{cases} \tag{20}$$

$\forall \quad \frac{m}{2} + 2 \ \le \ i \ \le \ m-2$

If $n \equiv 0 \ mod\ 3$ or $n \equiv 1 \ mod\ 3$, *and* $i \equiv 1 \ mod\ 3$,

(21)
$$\Psi_e(e_{ij}) = \begin{cases} 2n(m+1-i) - 2 - \lceil \frac{2n[m-i+1]}{3} \rceil - \lceil \frac{2n[m-i+2]}{3} \rceil & \text{if} \ \ j = 1; \\ 2n(m+1-i) - 2j - 2\lceil \frac{2n[m-i+1]}{3} \rceil & \text{if} \ \ 2 \le j \le n-1, \\ 2n(m+1-i) - \lceil \frac{2n[m-i+1]}{3} \rceil - \lceil \frac{2n[m-i+2]}{3} \rceil & \text{if} \ \ j = n, \end{cases}$$

If $n \equiv 1 \ mod\ 3$, *and* $i \equiv 2 \ mod\ 3$,

(22)
$$\Psi_e(e_{ij}) = \begin{cases} 2n(m+1-i) - 3 - \lceil \frac{2n[m-i+1]}{3} \rceil - \lceil \frac{2n[m-i+2]}{3} \rceil & \text{if } j = 1; \\ 2n(m+1-i) - 2 - 2j - 2\lceil \frac{2n[m-i+1]}{3} \rceil & \text{if} \ \ 2 \le j \le n-1, \\ 2n(m+1-i) - 1 - \lceil \frac{2n[m-i+1]}{3} \rceil - \lceil \frac{2n[m-i+2]}{3} \rceil & \text{if} \ \ j = \ n, \end{cases}$$

If $n \equiv 1 \ mod\ 3$, *and* $i \equiv 0 \ mod\ 3$,

(23)
$$\Psi_e(e_{ij}) = \begin{cases} 2n(m+1-i) - 3 - \lceil \frac{2n[m-i+1]}{3} \rceil - \lceil \frac{2n[m-i+2]}{3} \rceil & \text{if} \ \ j = 1; \\ 2n(m+1-i) - 2j - 2\lceil \frac{2n[m-i+1]}{3} \rceil & \text{if } 2 \le j \le n-1; \\ 2n(m+1-i) - 1 - \lceil \frac{2n[m-i+1]}{3} \rceil - \lceil \frac{2n[m-i+2]}{3} \rceil & \text{if} \ \ j = n, \end{cases}$$

$$(24)\qquad \Psi_e(e_{(m-1)j}) = \begin{cases} 4n-2-\lceil\frac{4n}{3}\rceil-\lceil\frac{6n}{3}\rceil & \text{if } j=1; \\ 4n-2j-2\lceil\frac{4n}{3}\rceil & \text{for } 2\le j\le \frac{n}{2}-1, if\ n\ even; \\ & \text{or } 2\le j\le \frac{n-3}{2},\ if\ n\ odd; \\ 2n+2-\lceil\frac{4n}{3}\rceil & \text{for } j=\frac{n}{2},\ if\ n\ even; \\ & \text{or } j=\frac{n-1}{2},\ if\ n\ odd. \\ 2n+4-2j & \text{if } \frac{n}{2}+1\le j\le n-1,\ n\ is\ even; \\ 2n+2-2j & \text{if } \frac{n+1}{2}\le j\le n-1,\ n\ is\ odd; \\ 3n+2-\lceil\frac{6n}{3}\rceil & \text{if } j=n,\ n\ is\ even; \\ 3n+1-\lceil\frac{6\,n}{3}\rceil & \text{if } j=n,\ n\ is\ odd, \end{cases}$$

$$(25)\qquad \Psi_e(e_{mj}) = \begin{cases} 2 & \text{if } 1\le j\le n-1; \\ n+2 & \text{if } j=n,\quad n\ is\ even\ ; \\ n+1 & \text{if } j=n,\quad n\ is\ odd \end{cases}$$

Case (2). When $n\equiv 2\,mod\,3$ and $|m\,n|\equiv 0\,,4\ (mod\,6)$, we choose $r=\lceil\frac{m\,n}{3}\rceil$, when $n\equiv 2\ mod\ 3$ and $|m\ n|\equiv 2\ (mod\ 6)$, we choose $r=\lceil\frac{m\ n}{3}\rceil+1$.

The labeling function $\Psi : V(Cl_{m,n})\cup E(Cl_{m,n}) \longrightarrow \{1,2,\cdots,r\}$ defined as in case (1) but with some modifications which are given by

$$\Psi_v(v_{2j}) = \lceil\frac{4\ n}{3}\rceil+1 \qquad \text{for } \frac{n}{2}+1\le j\le n-1,\ \ if\ \ n\ \ is\ \ even\ ; \quad \text{or } \frac{n+3}{2}\le j\le n-1,\ \ if\ \ n\ \ is\ \ odd,$$

$\forall \quad 3\ \le\ i\ \le\ \frac{m}{2}-1$,

If $i\equiv\ 0\ mod\ 3$, in this case, the labeling of $\Psi_v(v_{ij})$ will match as within Eq.(5).

If $i\equiv\ 1\ mod\ 3$, in this case, the labeling of $\Psi_v(v_{ij})$ will match as within Eq.(6).

If $i\equiv\ 2\ mod\ 3$, in this case, the labeling of $\Psi_v(v_{ij})$ will match as within Eq. (7).

$\forall \quad \frac{m}{2}+2\ \le\ i\ \le\ m-2$,

If $i\equiv\ 0\ mod\ 3$, in this case, the labeling of $\Psi_v(v_{ij})$ will match as within Eq.(9).

If $i\equiv\ 1\ mod\ 3$, in this case, the labeling of $\Psi_v(v_{ij})$ will match as within Eq.(10).

If $i\equiv\ 2\ mod\ 3$, in this case, the labeling of $\Psi_v(v_{ij})$ will match as within Eq.(11).

$$\Psi_v(v_{(m-1)j}) = \lceil \frac{4\,n}{3} \rceil + 1 \qquad \text{for } 2 \le j \le \tfrac{n}{2},\ \ if\ \ n\ \ is\ \ even\ ; \text{ or } 2 \le j \le \tfrac{n-1}{2},\ \ if\ \ n\ \ is\ \ odd,$$

$$\Psi_e(e_{2j}) = \begin{cases} 2n - \lceil \frac{4n}{3} \rceil & \text{for } j = \frac{n}{2},\ \ if\ \ n\ \ is\ \ even; \text{ or } j = \frac{n+1}{2},\ \ if\ \ n\ \ is\ \ odd; \\ 2n - 3 + 2j - 2\lceil \frac{4n}{3} \rceil & \text{for } \frac{n}{2} + 1 \le j \le n-2,\ if\ \ n\ \ is\ \ even; \text{ or } \frac{n+3}{2} \le j \le n-2,\ \ if\ \ n\ \ is\ \ odd; \\ 4n - 4 - \lceil \frac{4n}{3} \rceil - \lceil \frac{6n}{3} \rceil & \text{if } \ j = \ n-1. \end{cases}$$

$\forall$, $3 \le i \le \frac{m}{2} - 1$

If $i \equiv 0 \ mod\ 3$ in this case, the labeling of $\Psi_e(e_{ij})$ will match as within Eq.(16).

If $i \equiv 1 \ mod\ 3$ in this case, the labeling of $\Psi_e(e_{ij})$ will match as within the Eq. (17).

If $i \equiv 2 \ mod\ 3$ in this case, the labeling of $\Psi_e(e_{ij})$ will match as within Eq.(18).

$\forall$, $\frac{m}{2} + 2 \le i \le m-2$

If $i \equiv 0 \ mod\ 3$ in this case, the labeling of $\Psi_e(e_{ij})$ will match as within Eq.(21).

If $i \equiv 1 \ mod\ 3$ in this case, the labeling of $\Psi_e(e_{ij})$ will match as within Eq.(22).

If $i \equiv 2 \ mod\ 3$ in this case, the labeling of $\Psi_e(e_{ij})$ will match as within Eq.(23).

$$\Psi_e(e_{(m-1)j}) = \begin{cases} 4n - 3 - \lceil \frac{4n}{3} \rceil - \lceil \frac{6\,n}{3} \rceil & \text{if } \ j = \ 1; \\ 4n - 2j - 2\lceil \frac{4n}{3} \rceil - 2 & \text{for } 2 \le j \le \frac{n}{2} - 1,\ \ if\ \ n\ \ is\ \ even; \text{ or } 2 \le j \le \frac{n-3}{2},\ \ if\ \ n\ \ is\ \ odd; \\ 2n + 1 - \lceil \frac{4n}{3} \rceil & \text{for } j = \frac{n}{2},\ \ if\ \ n\ \ is\ \ even\ ; \text{ or } j = \frac{n-1}{2},\ \ if\ \ n\ \ is\ \ odd, \end{cases}$$

In all cases, we can see that all the vertex and edge labels are at most $r = \lceil \frac{m\,n}{3} \rceil$ or $r = \lceil \frac{m\,n}{3} \rceil + 1$. Besides, under the labeling Ψ the weights of the edges are given by

- If m is even number

$$\forall \ 1 \le i \le \tfrac{m}{2} \qquad wt_\Psi(e_{ij}) = 2n(i-1) + 2j - 1\ , \qquad 1 \le j \le n$$

$\forall \ \frac{m}{2} + 1 \le i \le m$

$$wt_\Psi(e_{ij}) = \begin{cases} 2n(m+1-i) - 2j & \text{if } \ 1 \le j \le n-1; \\ 2n(\ m+1-i) & \text{if } \ j = n. \end{cases}$$

We can see that weights of edges in the first half cycles $C_{n_1}, C_{n_2}, \cdots, C_{n_{\frac{m}{2}}}$ form an increasing sequence of consecutive odd integers from 1 up to $mn-1$. For the second half cycles $C_{n_{\frac{m}{2}+1}}, C_{n_{\frac{m}{2}+2}}, \cdots, C_{n_m}$ weights of edges form a decreasing sequence of consecutive even integers from mn up to 2. This concludes the proof. □

Theorem 3.3. *Let $n \geq 6$ and m be an odd positive integer, $m \geq 5$. Then the calendula graph $Cl_{m,n}$ have .*

$$res(Cl_{m,n}) = \begin{cases} \lceil \frac{m\ n}{3} \rceil & \text{for } n \equiv 0 \ mod\ 6; \\ & or\ n \equiv 2,4\ mod\ 6\ and\ |mn| \equiv 0,4\ mod\ 6; \\ & or\ n \equiv 1,5\ mod\ 6\ and\ |mn| \equiv 1,5\ mod\ 6; \\ \lceil \frac{mn}{3} \rceil + 1 & for\ n \equiv 3\ mod\ 6\ ; \\ & or\ n \equiv 2\ ,4\ mod\ 6\ and\ |mn| \equiv 2\ mod\ 6; \\ & or\ n \equiv 1\ ,5\ mod\ 6\ and\ |mn| \equiv 3\ mod\ 6. \end{cases}$$

Proof. Since $|V(Cl_{m,n})| = m(n-1)$, $|E(Cl_{m,n})| = mn$ and the maximum degree $\triangle(Cl_{m,n}) = 4$. Thus, the inequality (2) becomes

$$res(Cl_{m,n}) \geq \begin{cases} \lceil \frac{m\ n}{3} \rceil & \text{if } |m\ n| \not\equiv 2,3\ (mod\ 6).\ ; \\ \lceil \frac{m\ n}{3} \rceil + 1 & \text{if } |m\ n| \equiv 2,3\ (mod\ 6). \end{cases}$$

To prove the inverse inequality, we need to show that there exist an edge irregularity reflexive $r-$ labeling, $r = \lceil \frac{m\ n}{3} \rceil$, or $r = \lceil \frac{m\ n}{3} \rceil + 1$, for the calendula graph $Cl_{m,n}$. The labeling in the case for m odd is the same as the case in Theorem 3.2 when m is even, but with some modifications, we list it in the following:

For vertices and edges when $3 \leq i \leq \frac{m}{2} - 1$, m is even, the labeling is the same when $3 \leq i \leq \frac{m-3}{2}$, m is odd.

$$\forall \quad 1 \leq j \leq n-1$$

$$\Psi_v(v_{(\frac{m-1}{2})j}) = \begin{cases} \lceil \frac{(m-1)n}{3} \rceil & \text{for } n \equiv 0\ mod\ 3; \\ & \text{or } n \equiv 1\ mod\ 3\ \ and\ m \equiv 1,5\ mod\ 6; \\ & \text{or } n \equiv 2\ mod\ 3\ \ and\ m \equiv 1,3\ mod\ 6; \\ \lceil \frac{(m-1)n}{3} \rceil + 1 & \text{for } n \equiv 1\ mod\ 3\ \ and\ m \equiv 3\ mod\ 6; \\ & \text{or } n \equiv 2\ mod\ 3\ \ and\ m \equiv 5\ mod 6, \end{cases}$$

$$\Psi_v(v_{(\frac{m-1}{2})n}) = \begin{cases} r & \text{for } n \equiv 0\ mod\ 3,\ \ or\ \ n \equiv\ 2,\ 4\ mod\ 6; \\ & \text{or } n \equiv 1\ mod\ 6\ \ and\ m \equiv 3,5\ mod\ 6; \\ & \text{or } n \equiv 5\ mod\ 6\ \ and\ m \equiv 1,3\ mod\ 6; \\ r-1 & \text{for } n \equiv 1\ mod\ 6\ \ and\ m \equiv 1\ mod\ 6; \\ & \text{or } n \equiv 5\ mod\ 6\ and\ m \equiv 5\ mod\ 6, \end{cases}$$

$\forall \quad 1 \leq j \leq n$

$$\Psi_v(v_{(\frac{m+1}{2})j}) = \begin{cases} r & \text{for } n \equiv 0,2 \; mod\; 3, \quad or \quad n \equiv 4 \; mod\; 6; \\ & \text{or } n \equiv 1 \; mod\; 3 \;\; and\; m \equiv 3,5 \; mod\; 6; \\ & \text{or } n \equiv 5 \; mod\; 6 \;\; and\; m \equiv 1,3 \; mod\; 6; \\ r-1 & \text{for } n \equiv 1 \; mod\; 3 \;\; and\; m \equiv 1 \;\; mod\; 6; \\ & \text{or } n \equiv 5 \; mod\; 6 \;\; and\; m \equiv 5 \; mod\; 6, \end{cases}$$

$$\Psi_v(v_{(\frac{m+3}{2})1}) = \begin{cases} r & \text{for } n \equiv 0 \; mod\; 3, \quad or \quad n \equiv 2,\; 4 \; mod\; 6; \\ & \text{or } n \equiv 1 \; mod\; 3 \;\; and\; m \equiv 3,5 \; mod\; 6; \\ & \text{or } n \equiv 5 \; mod\; 6 \;\; and\; m \equiv 1,3 \; mod\; 6; \\ r-1 & \text{for } n \equiv 1 \; mod\; 3 \;\; and\; m \equiv 1 \; mod\; 6; \\ & \text{or } n \equiv 5 \; mod\; 6 \;\; and\; m \equiv 5 \; mod\; 6, \end{cases}$$

$\forall \quad 2 \leq j \leq n$

$$\Psi_v(v_{(\frac{m+3}{2})j}) = \begin{cases} \lceil \frac{(m-1)n}{3} \rceil & \text{for } n \equiv 0 \; mod\; 3 \;\; ; \\ & \text{or } n \equiv 1 \; mod\; 3 \;\; and\; m \equiv 1,5 \; mod\; 6; \\ & \text{or } n \equiv 2 \; mod\; 3 \;\; and\; m \equiv 1,3 \; mod\; 6; \\ \lceil \frac{(m-1)\; n}{3} \rceil + 1 & \text{for } n \equiv 1 \; mod\; 3 \;\; and\; m \equiv 3 \;\; mod\; 6; \\ & \text{or } n \equiv 2 \; mod\; 3 \;\; and\; m \equiv 5 \; mod\; 6, \end{cases}$$

$\forall \quad \frac{m+5}{2} \leq i \leq m-2$, there are three cases

Case (1).

If $n \equiv 0 \; mod\; 3$, or $n \equiv 1 \; mod\; 3$, $m \equiv 1 \; mod\; 6$, and $i \equiv 0 \; mod\; 3$,
or $n \equiv 1 \; mod\; 3$, $m \equiv 5 \; mod\; 6$, and $i \equiv 1 \; mod\; 3$,
or $n \equiv 1 \; mod\; 3$, $m \equiv 3 \; mod\; 6$, and $i \equiv 2 \; mod\; 3$,
or $n \equiv 2 \; mod\; 3$, $m \equiv 1 \; mod\; 6$, and $i \equiv 2 \; mod\; 3$,
or $n \equiv 2 \; mod\; 3$, $m \equiv 5 \; mod\; 6$, and $i \equiv 0 \; mod\; 3$,
or $n \equiv 2 \; mod\; 3$, $m \equiv 3 \; mod\; 6$, and $i \equiv 1 \; mod\; 3$.

In this case, the labeling of vertices $\Psi_v(v_{ij})$ will match as within Eq.(9). Also, the labeling of edges $\Psi_e(e_{ij})$ will match as within Eq.(21).

Case (2).

If $n \equiv 1 \; mod\; 3$, $m \equiv 1 \; mod\; 6$, and $i \equiv 1 \; mod\; 3$,
or $n \equiv 1 \; mod\; 3$, $m \equiv 5 \; mod\; 6$, and $i \equiv 2 \; mod\; 3$,
or $n \equiv 1 \; mod\; 3$, $m \equiv 3 \; mod\; 6$, and $i \equiv 0 \; mod\; 3$,
or $n \equiv 2 \; mod\; 3$, $m \equiv 1 \; mod\; 6$, and $i \equiv 0 \; mod\; 3$,
or $n \equiv 2 \; mod\; 3$, $m \equiv 5 \; mod\; 6$, and $i \equiv 1 \; mod\; 3$,
or $n \equiv 2 \; mod\; 3$, $m \equiv 3 \; mod\; 6$, and $i \equiv 2 \; mod\; 3$.

In this case, the labeling of vertices $\Psi_v(v_{ij})$ will match as within Eq.(10), and the labeling of edges $\Psi_e(e_{ij})$ will match as within Eq.(22).

Case (3).

If $n \equiv 1 \; mod\; 3$, $m \equiv 1 \; mod\; 6$, and $i \equiv 2 \; mod\; 3$,
or $n \equiv 1 \; mod\; 3$, $m \equiv 5 \; mod\; 6$, and $i \equiv 0 \; mod\; 3$,

or $n \equiv 1\ mod\ 3$, $m \equiv 3\ mod\ 6$, and $i \equiv 1\ mod\ 3$,

or $n \equiv 2\ mod\ 3$, $m \equiv 1\ mod\ 6$, and $i \equiv 1\ mod\ 3$,

or $n \equiv 2\ mod\ 3$, $m \equiv 5\ mod\ 6$, and $i \equiv 2\ mod\ 3$,

or $n \equiv 2\ mod\ 3$, $m \equiv 3\ mod\ 6$, and $i \equiv 0\ mod\ 3$.

In this case, the labeling of vertices $\Psi_v(v_{ij})$ will match as within Eq.(11) and the labeling of edges $\Psi_e(e_{ij})$ will match as within Eq.(23).

$\forall \quad 1 \le j \le n-2$

$$\Psi_e(e_{(\frac{m-1}{2})j}) = \begin{cases} n(m-3)+2j-1-2\lceil\frac{(m-1)n}{3}\rceil & \begin{array}{r}\text{for } n \equiv 0\ mod\ 3;\\ \text{or } n \equiv 1\ mod\ 3\ and\ m \equiv 1,5\ mod\ 6;\\ \text{or } n \equiv 2\ mod\ 3\ and\ m \equiv 1,3\ mod\ 6;\end{array} \\ n(m-3)+2j-3-2\lceil\frac{(m-1)\ n}{3}\rceil & \begin{array}{r}\text{for } n \equiv 1\ mod\ 3\ and\ m \equiv 3\ mod\ 6;\\ \text{or } n \equiv 2\ mod\ 3\ and\ m \equiv 5\ mod\ 6,\end{array} \end{cases}$$

$$\Psi_e(e_{(\frac{m-1}{2})(n-1)}) = \begin{cases} n(m-1)-3-\lceil\frac{(m-1)n}{3}\rceil-r & \begin{array}{r}\text{for } n \equiv 0\ mod\ 3,\ or\ n \equiv 5\ mod\ 6;\\ \text{or } n \equiv 1\ mod\ 6\ and\ m \equiv 5\ mod\ 6;\\ \text{or } n \equiv 2\ mod\ 6\ and\ m \equiv 1,3\ mod\ 6;\\ \text{or } n \equiv 4\ mod\ 6\ and\ m \equiv 1,5\ mod\ 6;\end{array} \\ n(m-1)-4-\lceil\frac{(m-1)n}{3}\rceil-r & \begin{array}{r}\text{for } n \equiv 1\ mod\ 6\ and\ m \equiv 3\ mod\ 6;\\ \text{or } n \equiv 2\ mod\ 6\ and\ m \equiv 5\ mod\ 6;\\ \text{or } n \equiv 4\ mod\ 6\ and\ m \equiv 3\ mod\ 6;\end{array} \\ n(m-1)-2-\lceil\frac{(m-1)n}{3}\rceil-r & \text{if } n,m \equiv 1\ mod\ 6 \end{cases}$$

$$\Psi_e(e_{(\frac{m-1}{2})n}) = \begin{cases} n(m-1)-1-\lceil\frac{(m-1)n}{3}\rceil-r & \begin{array}{r}\text{for } n \equiv 0\ mod 3,\ or\ n \equiv 5\ mod\ 6;\\ \text{or } n \equiv 1\ mod\ 6\ and\ m \equiv 5\ mod\ 6;\\ \text{or } n \equiv 2\ mod\ 6\ and\ m \equiv 1,3\ mod\ 6;\\ \text{or } n \equiv 4\ mod\ 6\ and\ m \equiv 1,5\ mod\ 6;\end{array} \\ n(m-1)-2-\lceil\frac{(m-1)n}{3}\rceil-r & \begin{array}{r}\text{for } n \equiv 1\ mod\ 6\ and\ m \equiv 3\ mod\ 6;\\ \text{or } n \equiv 2\ mod\ 6\ and\ m \equiv 5\ mod\ 6;\\ \text{or } n \equiv 4\ mod\ 6\ and\ m \equiv 3\ mod\ 6;\end{array} \\ n(m-1)-\lceil\frac{(m-1)n}{3}\rceil-r & \text{if } n,m \equiv 1\ mod\ 6, \end{cases}$$

$\forall \quad 1 \le j \le \frac{n+1}{2}$ and n is odd, or $1 \le j \le \frac{n}{2}$ and n is even

$$\Psi_e(e_{(\frac{m+1}{2})j}) = \begin{cases} n(m-1)+2j-1-2r & \begin{array}{r}\text{for } n \equiv 0\ mod\ 3,\ or\ n \equiv 2,4\ mod\ 6\ ;\\ \text{or } n \equiv 1\ mod\ 6\ and\ m \equiv 3,5\ mod\ 6\ ;\\ \text{or } n \equiv 5\ mod\ 6\ and\ m \equiv 1,3\ mod\ 6\ ;\end{array} \\ n(m-1)+2j+1-2r & \begin{array}{r}\text{for } n \equiv 1\ mod\ 6\ and\ m \equiv 1\ mod\ 6;\\ \text{or } n \equiv 5\ mod\ 6\ and\ m \equiv 5\ mod\ 6,\end{array} \end{cases}$$

$\forall \quad \frac{n+3}{2} \le j \le n$ and n is odd, or $\frac{n}{2}+1 \le j \le n$ and n is even

$$\Psi_e(e_{(\frac{m+1}{2})j}) = \begin{cases} & \text{for } n \equiv 0 \ mod\ 3\ ; \\ n(m+1)-2j+2-2r & \text{for } n \equiv 2,4 \ mod\ 6\ ; \\ & \text{or } n \equiv 1 \ mod\ 6 \ and\ m \ \equiv 3,5 \ mod\ 6; \\ & \text{or } n \equiv 5 \ mod\ 6 \ and\ m \ \equiv 1,3 \ mod\ 6; \\ n(m+1)-2j+4-2r & \text{for } \ n \equiv 1 \ mod\ 6 \ \ and\ m \equiv \ 1 \ \ mod\ 6; \\ & \text{or } \ n \equiv 5 \ mod\ 6 \ \ and\ m \equiv \ 5 \ mod\ 6, \end{cases}$$

$$\Psi_e(e_{(\frac{m+3}{2})1}) = \begin{cases} & \text{for } n \equiv 0 \ mod\ 3, \ or \ \ n \equiv 5 \ mod\ 6; \\ n(m-1)-\lceil\frac{(m-1)n}{3}\rceil-2-r & \text{or } n \equiv 1 \ mod\ 6 \ and\ m \equiv 5 \ mod\ 6; \\ & \text{or } n \equiv 2 \ mod\ 6 \ and\ m \equiv 1,3 \ mod\ 6; \\ & \text{or } n \equiv 4 \ mod\ 6 \ and\ m \equiv 1,5 \ mod\ 6; \\ & forn \equiv 1 \ mod\ 6 \ and\ m \equiv 3 \ mod\ 6; \\ n(m-1)-\lceil\frac{(m-1)n}{3}\rceil-3-r & \text{or } n \equiv 2 \ mod\ 6 \ and\ m \equiv 5 \ mod\ 6; \\ & \text{or } n \equiv 4 \ mod\ 6 \ and\ m \equiv 3 \ mod\ 6; \\ n(m-1)-\lceil\frac{(m-1)n}{3}\rceil-1-r & \text{if } \ n,m \equiv 1 \ mod\ 6, \end{cases}$$

$\forall \qquad 2 \le j \le n-1$

$$\Psi_e(e_{(\frac{m+3}{2})j}) = \begin{cases} & \text{for } n \equiv 0 \ mod\ 3; \\ n(m-1)-2j-2\lceil\frac{(m-1)n}{3}\rceil & \text{or } n \equiv 1 \ mod\ 3 \ and\ m \equiv 1,5 \ mod\ 6; \\ & \text{or } n \equiv 2 \ mod\ 3 \ and\ m \equiv 1,3 \ mod\ 6; \\ n(m-1)-2j-2\lceil\frac{(m-1)n}{3}\rceil-2 & \text{for } n \equiv 1 \ mod\ 3 \ and\ m \ \equiv \ 3 \ mod\ 6; \\ & \text{or } n \equiv 2 \ mod\ 3 \ and\ m \ \equiv \ 5 \ mod\ 6, \end{cases}$$

$$\Psi_e(e_{(\frac{m+3}{2})n}) = \begin{cases} & \text{for } n \equiv 0 \ mod\ 3, \ or\ n \equiv 5 \ mod\ 6; \\ n(m-1)-\lceil\frac{(m-1)n}{3}\rceil-r & \text{or } n \equiv 1 \ mod\ 6 \ and\ m \equiv 5 \ mod\ 6; \\ & \text{or } n \equiv 2 \ mod\ 6 \ and\ m \equiv 1,3 \ mod\ 6; \\ & \text{or } n \equiv 4 \ mod\ 6 \ and\ m \equiv 1,5 \ mod\ 6; \\ & \text{for } n \equiv 1 \ mod\ 6 \ and\ m \equiv 3 \ mod\ 6; \\ n(m-1)-\lceil\frac{(m-1)n}{3}\rceil-1-r & \text{or } n \equiv 2 \ mod\ 6 \ and\ m \equiv 5 \ mod\ 6; \\ & \text{or } n \equiv 4 \ mod\ 6 \ and\ m \equiv 3 \ mod\ 6; \\ n(m-1)-\lceil\frac{(m-1)n}{3}\rceil+1-r & \text{if } \ n,m \equiv 1 \ mod\ 6. \end{cases}$$

In all cases, we are able see that all the vertex and edge names are at most $r = \lceil \frac{m\,n}{3} \rceil$ or $r = \lceil \frac{m\,n}{3} \rceil + 1$.

When m is odd number, the weight of the edges of the calendula graphs $Cl_{m,n}$ are given by

$$\forall \ \ 1 \le i \le \tfrac{m-1}{2} \qquad wt_{\Phi}(e_{ij}) \ = 2n(i-1)+2j-1\ , \qquad\qquad 1 \le j \le n$$

$$wt_{\Phi}(e_{(\frac{m+1}{2})j}) = \begin{cases} n(m-1)+2j-1 & \text{for } 1 \le j \le \frac{n}{2}, \ n \ \ is \ \ even; \\ & \text{or } 1 \le j \le \frac{n+1}{2}, \ \ n \ \ is \ \ odd, \\ n(m+1)-2j+2 & \text{for } \ \frac{n}{2}+1 \le j \le n, \ \ n \ \ is \ \ even; \\ & \text{or } \ \frac{n+3}{2} \le j \le n, \ \ n \ \ is \ \ odd, \end{cases}$$

$$\forall \ \ \tfrac{m+3}{2} \le i \le m \qquad wt_{\Phi}(e_{ij}) = \begin{cases} 2n(m+1-i)-2j & \text{if} \quad 1 \le j \le n-1; \\ 2n(m+1-i) & \text{if} \quad j = n. \end{cases}$$

We can see that weights of edges in all cycles $C_{n_1}, C_{n_2}, \cdots, C_{n_m}$ are all different, this concludes the proof. □

Table 1 illustrates the different cases of an edge irregularity reflexive labeling for the calendula $Cl_{m,n}$ graph.

Table 1: The different cases of an edge irregularity reflexive labeling for the calendula $Cl_{m,n}$

Graph	n	m	$E(Cl_{m,n})$	r
$Cl_{m,n}$	$n \equiv 0 \; mod \; 6$	$\forall \; m$	$E \equiv 0 \; mod \; 6$	$\lceil \frac{m\,n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 4 \; mod \; 6$	$m \equiv 0, 3 \; mod \; 6$	$E \equiv 0 \; mod \; 6$	$\lceil \frac{m\,n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 4 \; mod \; 6$	$m \equiv 1, 4 \; mod \; 6$	$E \equiv 4 \; mod \; 6$	$\lceil \frac{m\,n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 4 \; mod \; 6$	$m \equiv 2, 5 \; mod \; 6$	$E \equiv 2 \; mod \; 6$	$\lceil \frac{m\,n}{3} \rceil + 1$
$Cl_{m,n}$	$n \equiv 2 \; mod \; 6$	$m \equiv 0, 3 \; mod \; 6$	$E \equiv 0 \; mod \; 6$	$\lceil \frac{m\,n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 2 \; mod \; 6$	$m \equiv 2, 5 \; mod \; 6$	$E \equiv 4 \; mod \; 6$	$\lceil \frac{m\,n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 2 \; mod \; 6$	$m \equiv 1, 4 \; mod \; 6$	$E \equiv 2 \; mod \; 6$	$\lceil \frac{m\,n}{3} \rceil + 1$
$Cl_{m,n}$	$n \equiv 3 \; mod \; 6$	$m \equiv 0, 2, 4 \; mod \; 6$	$E \equiv 0 \; mod \; 6$	$\lceil \frac{m\,n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 3 \; mod \; 6$	$m \equiv 1, 3, 5 \; mod \; 6$	$E \equiv 3 \; mod \; 6$	$\lceil \frac{m\,n}{3} \rceil + 1$
$Cl_{m,n}$	$n \equiv 1 \; mod \; 6$	$m \equiv 0 \; mod \; 6$	$E \equiv 0 \; mod \; 6$	$\lceil \frac{m\,n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 1 \; mod \; 6$	$m \equiv 1 \; mod \; 6$	$E \equiv 1 \; mod \; 6$	$\lceil \frac{m\,n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 1 \; mod \; 6$	$m \equiv 4 \; mod \; 6$	$E \equiv 4 \; mod \; 6$	$\lceil \frac{m\,n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 1 \; mod \; 6$	$m \equiv 5 \; mod \; 6$	$E \equiv 5 \; mod \; 6$	$\lceil \frac{m\,n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 1 \; mod \; 6$	$m \equiv 2 \; mod \; 6$	$E \equiv 2 \; mod \; 6$	$\lceil \frac{m\,n}{3} \rceil + 1$
$Cl_{m,n}$	$n \equiv 1 \; mod \; 6$	$m \equiv 3 \; mod \; 6$	$E \equiv 3 \; mod \; 6$	$\lceil \frac{m\,n}{3} \rceil + 1$
$Cl_{m,n}$	$n \equiv 5 \; mod \; 6$	$m \equiv 1 \; mod \; 6$	$E \equiv 5 \; mod \; 6$	$\lceil \frac{m\,n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 5 \; mod \; 6$	$m \equiv 2 \; mod \; 6$	$E \equiv 0 \; mod \; 6$	$\lceil \frac{m\,n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 5 \; mod \; 6$	$m \equiv 4 \; mod \; 6$	$E \equiv 4 \; mod \; 6$	$\lceil \frac{m\,n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 5 \; mod \; 6$	$m \equiv 5 \; mod \; 6$	$E \equiv 1 \; mod \; 6$	$\lceil \frac{m\,n}{3} \rceil$
$Cl_{m,n}$	$n \equiv 5 \; mod \; 6$	$m \equiv 0 \; mod \; 6$	$E \equiv 2 \; mod \; 6$	$\lceil \frac{m\,n}{3} \rceil + 1$
$Cl_{m,n}$	$n \equiv 5 \; mod \; 6$	$m \equiv 3 \; mod \; 6$	$E \equiv 3 \; mod \; 6$	$\lceil \frac{m\,n}{3} \rceil + 1$

Illustration: The calendula graphs $Cl_{10,12}$ with an edge irregularity total $k = 41$ labeling and $Cl_{10,12}$ with an edge irregularity reflexive $r = 40$ labeling are shown in Fig. 6.

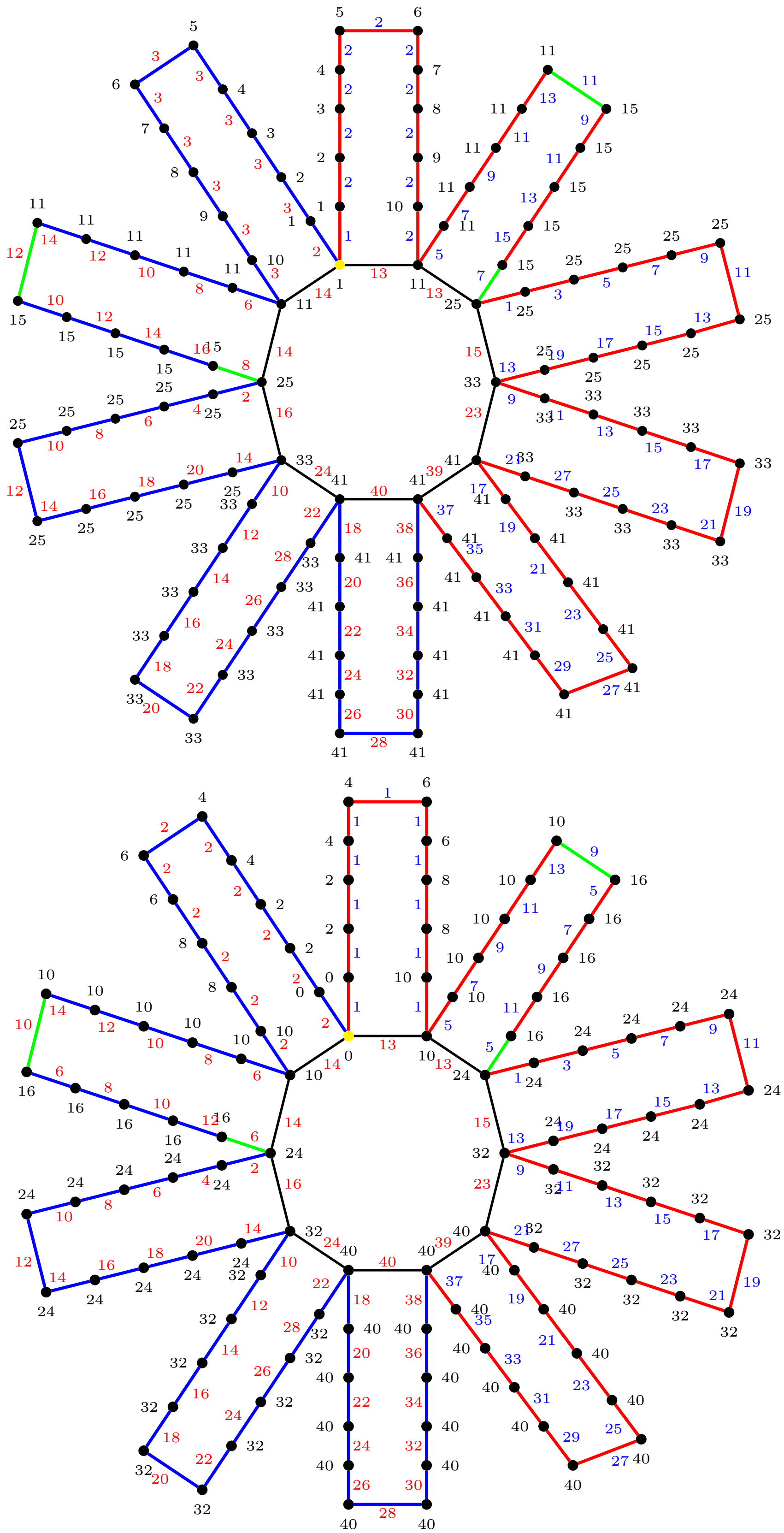

Figure 6: The calendula graphs $Cl_{10,12}$ with an edge irregularity total $k = 41$ labeling and $Cl_{10,12}$ with an edge irregularity reflexive $r = 40$ labeling

Illustration: The calendula graphs $Cl_{9,11}$ with an edge irregularity total $k = 34$ labeling and $Cl_{9,11}$ with an edge irregularity reflexive $r = 34$ labeling are shown in Fig. 7.

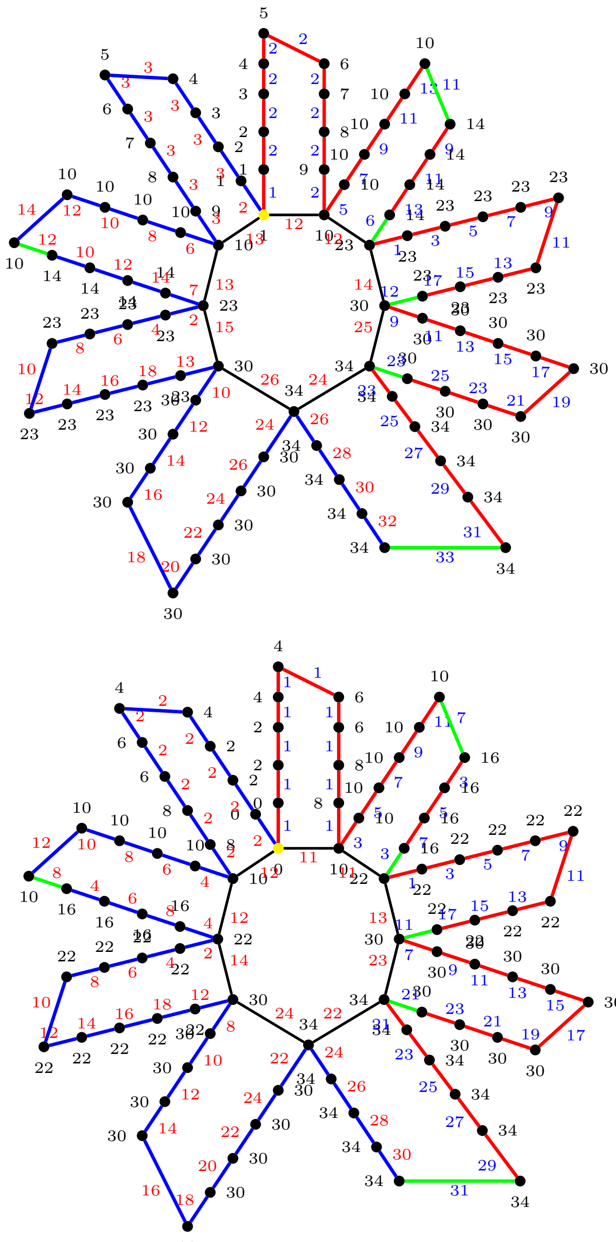

Figure 7: The calendula graphs $Cl_{9,11}$ with an edge irregularity total $k = 34$ labeling and $Cl_{9,11}$ with an edge irregularity reflexive $r = 34$ labeling

4. Conclusion

lately, graph labeling has become a fruitful branch of several research studies in graph theory. It has massive applications in many disciplines, like coding theory, X-rays, radar, communication networks, and astronomy. In this work, we have determined the total edge irregular strength $k-$ and the edge irregularity reflexive $r-$ for calendula $Cl_{4,n}$, $n \geq 4$, $Cl_{5,n}$, $n \geq 4$. Furthermore, the exact value of total edge irregular strength $k-$ and the edge irregularity reflexive $r-$ for a generalized calendula $Cl_{m,n}$ was defined.

References

[1] J. Gross, J. Yellen, *Graph theory and its applications,* CRC Press: London, UK, 1999.

[2] G. Uma, G. M. Jebarani Maheswari, V. Balaji, *Coding techniques through Fibonacci webs, difference cordial labeling and GMJ code method*, Journal of Physics: Conf. Series, 1139 (2018).

[3] M. R. Zeen El Deen, *Edge-even graceful labeling of some graphs*, Journal of Egyptian Mathematical Society, 27 (2019).

[4] M. R. Zeen El Deen, N. Omar, *Further results on edge even graceful labeling of the join of two graphs*, Journal of Egyptian Mathematical Society, 28 (2020), 21.

[5] M. R. Zeen El Deen, N. Omar, *Extending of edge even graceful labeling of graphs to strong r-edge even graceful labeling*, Journal of Mathematics, vol. 2021, Article ID 6643173, (2021).

[6] M. R. Zeen El Deen, *Edge-$\delta-$ graceful labeling for some cyclic-related graphs*, Advances in Mathematical Physics, 2020, Article ID 6273245, (2020).

[7] M. R. Zeen El Deen, G. Elmahdy, *New classes of graphs with edge $\delta-$ graceful labeling*, AIMS Math., 7 (2022), 3554-3589.

[8] J. A. Gallian, *A Dynamic survey of graph labeling*, The Electronic Journal of Combinatorics, 2018.

[9] M. Basher, *On even vertex odd mean labeling of the calendula graphs*, Proyecciones (Antofagasta), 39 (2020), 1515-1535.

[10] G. Chartrand, M. S. Jacobson, J. Lehek. O. R. Oellermann, S. Ruiz, F. Saba, *Irregular networks*, Congr. Number, 64 (1988), 187-192.

[11] M. Bača, S. Jendrol, M. Miller, J. Ryan, *On irregular total labellings*, Discrete Mathematics, 307 (2007), 1378-1388.

[12] J. Ivanĉo, S. Jendrol, *Total edge irregularity strength of trees*, Discussiones Mathematicae Graph Theory, 26 (2006), 449-456.

[13] J. Miškuf, R. Sotăk, *Total edge irregularity strength of complete graphs and complete bipartite graphs,* Discrete Math., 310 (2010), 400-407.

[14] O. B. S. Al-Mushayt, A. Ahmad, M. K. Siddiqui, *On the total edge irregularity strength of hexagonal grid graphs,* Australas. J. Comb. , 53 (2012), 263-272.

[15] M. Bača , M. K. Siddiqui, *Total edge irregularity strength of generalized prism*, Applied Mathematics and Computation, 235 (2014), 168-173.

[16] A. Ahmad, O. B. S. Al-Mushayt, M. Bača, *On edge irregularity strength of graphs,* Appl. Math. Comput., 243 (2014), 607-610.

[17] M. Bača, M. K. Siddiqui, *Total edge irregularity strength of generalized prism,* Appl. Math. Comput., 235 (2014), 168-173.

[18] F. Salama, *On total edge irregularity strength of polar grid graph*, Journal of Taibah University for Science, 13 (2019), 912-916.

[19] I. Tarawneh, R. Hasni, A. Ahmad, M. A. Asim, *On the edge irregularity strength for some classes of plane graphs*, AIMS Math., 6 (2021), 2724-2731.

[20] J. Ryan, B. Munasinghe, D. Tanna, *Reflexive irregular labelings*, preprint.

[21] M. Bača, M. Irfan, J. Ryan, A. Semaničovă-Feňnovčkovă, D. Tanna, *On edge irregular reflexive labelings for the generalized friendship graphs,* Mathematics, 67 (2017), 2-11.

[22] D. Tanna, J. Ryan, A. Semaničovă-Feňnovčkovă, *Reflexive edge irregular labeling of prisms and wheels,* Australas. J. Combin., 69 (2017), 394-401.

[23] X. Zhang, M. Ibrahim, M. K. Siddiqui, *Edge irregular reflexive labeling for the disjoint union of gear graphs and prism graphs,* Mathematics, 6 (2018), 142.

[24] M. Bača, M. Irfan, J. Ryan, D. Tanna, *Note on edge irregular reflexive labelings of graphs*, AKCE International Journal of Graphs and Combinatorics, 16 (2019), 145-157.

[25] I. H. Agustin, I. Utoyo, M. Venkatachalam, *Edge irregular reflexive labeling of some tree graphs,* J. Phys.: Conf. Ser., 1543 (2020), 012008.

[26] M. Basher, *On the reflexive edge strength of the circulant graphs*, AIMS Math., 6 (2021), 9342-9365.

Accepted: March 28, 2022