Solvability in fuzzy multigroup context

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Abstract. The adventure of fuzzy sets has witnessed myriad of group's theoretical concepts being studied as fuzzy algebraic structures. In this present work, the notion of solvable fuzzy multigroups is considered for the first time as an algebraic structure in fuzzy multigroup context. Solvable series for a fuzzy multigroup is defined in such a way that the family of the fuzzy submultigroups of the considered fuzzy multigroup has the same support. Some precursory results in normality and quotient of fuzzy multigroups are considered. It is established that there exists an if and only if condition between the solvability of a fuzzy multigroup and its support. Finally, certain results on solvable fuzzy multigroups are obtained.

Keywords: fuzzy algebra, fuzzy multigroup, normality in fuzzy multigroup, quotient fuzzy multigroup, solvable fuzzy multigroup.

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1. Introduction

The introduction of fuzzy sets by Zadeh [32] was a boost to the solution of uncertainties. With fuzzy sets as foundation, Rosenfeld [27] proposed fuzzy group as an algebraic structure of fuzzy set. Many authors have extended some group's theoretic notions to fuzzy sets as seen in [4, 23, 24]. By a way of generalization, Yager [31] introduced the notion of fuzzy multiset as a special fuzzy set that allows the repetitions of membership functions of elements of a set. Many works have been extensively carried out on fuzzy multisets and applied to many real-life problems [3, 20, 21, 22, 30].

In a continuation of the study of fuzzy algebra, Shinoj et al. [29] proposed fuzzy multigroup as an application of group theory to fuzzy multisets and deduced some related results. As a follow up, the analog of subgroups in fuzzy multigroup context was studied [5]. The notion of commutative fuzzy multigroups has been studied and a number of results were presented [2, 6]. Some group's analog notions have been investigated in fuzzy multigroup context [1, 7, 8, 12, 14, 16, 17, 18, 15, 19]. The concept of direct product of fuzzy multigroups and its generalization have been discussed [9, 13]. To show the connection between fuzzy multigroup and group, the notion of alpha-cuts of fuzzy multigroups was proposed [10, 11].

Though several concepts of group theory have been extended to fuzzy multigroups via fuzzy multisets, the notion of solvable/soluble fuzzy multigroups has not been investigated in fuzzy multigroup context. This paper seeks to introduce solvable fuzzy multigroup. The concept of solvable groups has been studied in fuzzy group setting [28, 26]. The rest of the paper is delineated as follows: Section 2 presents the notions of fuzzy multisets, fuzzy multigroups and certain existing results, Section 3 presents the concept of solvable fuzzy multigroups and discusses certain of its properties, and Section 4 summarizes and gives recommendations for future studies.

2. Preliminaries

We denote a non-empty set as X and a group as G throughout the paper.

Definition 2.1 ([32]). *A fuzzy subset F of X is an object characterized by the form*

$$F = \{\langle x, \mu_F(x)\rangle \mid x \in X\}, \tag{1}$$

where the function $\mu_F\colon X \to [0,1]$ defines the membership grade of x in X.

Definition 2.2 ([31]). *A fuzzy multiset A of X is a structure of a form*

$$A = \{\langle x, CM_A(x)\rangle \mid x \in X\} \tag{2}$$

characterized by a count membership function

$$CM_A\colon X \to N^I \text{ or } CM_A\colon X \to [0,1] \to N, \tag{3}$$

where $I = [0,1]$, $N = \{0,1,2,\cdots\}$ *and* $CM_A(x) = (\mu_A^1(x), \mu_A^2(x), \cdots, \mu_A^n(x))$ *such that* $\mu_A^1(x) \geq \mu_A^2(x) \geq \cdots \geq \mu_A^n(x)$.

Definition 2.3 ([31]). *Let* A *and* B *be fuzzy multisets of* X. *Then*

(*i*) $A = B \Longleftrightarrow CM_A(x) = CM_B(x)$, $\forall x \in X$,

(*ii*) $A \subseteq B \Longleftrightarrow CM_A(x) \leq CM_B(x)$, $\forall x \in X$,

(*iii*) $A \cap B \Longrightarrow CM_{A\cap B}(x) = \min(CM_A(x), CM_B(x))$, $\forall x \in X$,

(*iv*) $A \cup B \Longrightarrow CM_{A\cup B}(x) = \max(CM_A(x), CM_B(x))$, $\forall x \in X$,

(*v*) $A \oplus B \Longrightarrow CM_{A\oplus B}(x) = CM_A(x) \oplus CM_B(x)$, $\forall x \in X$.

Definition 2.4 ([29]). *A fuzzy multiset* A *of* G *is a fuzzy multigroup if we have (i)* $CM_A(xy) \geq \min(CM_A(x), CM_A(y))$, *and (ii)* $CM_A(x^{-1}) = CM_A(x)$, $\forall x, y \in G$. *Because*

$$\begin{aligned} CM_A(e) &= CM_A(xx^{-1}) \geq \min(CM_A(x), CM_A(x)) \\ &= CM_A(x), \forall x \in G, \end{aligned}$$

where e *is the identity element of* G, *then* $CM_A(e)$ *is the upper bound of* A, *which is called the tip of* A.

Definition 2.5 ([29]). *Let* A *be a fuzzy multigroup of* G. *Then, the support of* A *is the set* $supp(A) = \{x \in G \mid CM_A(x) \geq 0\}$.

Proposition 2.6 ([29]). *The support of a fuzzy multigroup* A *of* G *is a subgroup of* G.

Definition 2.7 ([5]). *Let* A *and* B *be fuzzy multigroups of* G. *Then, the product* $A \circ B$ *is defined to be a fuzzy multiset of* G *as follows:*

$$CM_{A\circ B}(x) = \begin{cases} \bigvee_{x=yz} \min(CM_A(y), CM_B(z)), & \text{if there exist } y, z \in G \text{ such that } x = yz \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.8 ([6]). *A fuzzy multigroup* A *of* G *is said to be commutative if* $CM_A(xy) = CM_A(yx)$, $\forall x, y \in G$. *Certainly, if* G *is a commutative group, then a fuzzy multigroup* A *of* G *is commutative.*

Definition 2.9 ([5]). *Let* A *and* B *be fuzzy multigroups of* G. *We say* A *is a fuzzy submultigroup of* B *if* $A \subseteq B$. *Again,* A *is a proper fuzzy submultigroup of* B *if* $A \subseteq B$ *and* $A \neq B$.

Definition 2.10 ([7]). *Let* A *be a fuzzy submultigroup of a fuzzy multigroup* B *of* G. *We say* A *is normal in* B *if* $CM_A(xy) = CM_A(yx) \Longleftrightarrow CM_A(y) = CM_A(x^{-1}yx)$, $\forall x, y \in G$, *and we write* $A \lhd B$.

Remark 2.11. Certainly, every normal fuzzy submultigroup is self-normal and abelian.

Definition 2.12 ([14])**.** *Suppose A is a fuzzy submultigroup of a fuzzy multigroup B of G. Then, the fuzzy submultiset yA of B for $y \in G$ defined by $CM_{yA}(x) = CM_A(y^{-1}x)$, $\forall x \in G$ is called the left fuzzy comultiset of A. Similarly, the fuzzy submultiset Ay of B for $y \in G$ defined by $CM_{Ay}(x) = CM_A(xy^{-1})$, $\forall x \in G$ is called the right fuzzy comultiset of A.*

Definition 2.13 ([14])**.** *Suppose A and B are fuzzy multigroups of G and $A \triangleleft B$. Then, the union of the set of left/right fuzzy comultisets of A such that $xA \circ yA = xyA$, $\forall x, y \in G$ is called a quotient fuzzy multigroup of B by A, denoted by B/A.*

3. On solvable fuzzy multigroups

Before investigating the notion of solvability in fuzzy multigroup, we first present the following results which are helpful in establishing solvable fuzzy multigroups.

Theorem 3.1. (*i*) *Every abelian fuzzy multigroup is self-normal.* (*ii*) *If A and B are fuzzy multigroups of G such that $A \triangleleft B$, then A is self-normal.*

Proof. Suppose A is an abelian fuzzy multigroup of G. Then

$$CM_A(xy) = CM_A(yx),\ \forall x, y \in G,$$

and so $CM_A(y) = CM_A(x^{-1}yx)$. Hence $A \triangleleft A$, which proves (i). Again, if $A \triangleleft B$ then $CM_A(x) \leq CM_B(x)$ for all $x \in G$ and $CM_A(x) > 0 < CM_A(y) \Longrightarrow CM_A(xy) = CM_A(yx)$, $\forall x, y \in G$. Thus, (ii) holds from (i). □

Theorem 3.2. *Let A be a fuzzy multigroup of G. Then supp(A) is abelian iff A is abelian.*

Proof. Let $x, y \in \text{supp}(A)$. If $\text{supp}(A)$ is abelian then $xy = yx$, and so $CM_A(xy) = CM_A(yx)$, $\forall x, y \in G$.

Conversely, if A is abelian then $CM_A(xy) = CM_A(yx)$, $\forall x, y \in G$. Thus $\text{supp}(A)$ is an abelian group because $CM_A(xy) > 0 < CM_A(yx) \Longrightarrow xy = yx$, $\forall x, y \in \text{supp}(A)$. □

Theorem 3.3. *Let A, B and C be fuzzy multigroups of G such that:* (*i*) *B/A and C/A are both in the canonical form,* (*ii*) *$B/A \triangleleft C/A$, and (iii) $(C/A)/(B/A)$ is abelian. Then, $B \triangleleft C$ and C/B is abelian.*

Proof. Let H, H_1 and H_2 be the supports of A, B and C, respectively, and let H' be the zone of A and C. Then B/A and C/A are both fuzzy multigroups of H'/H. If $x \in H_1$, then $CM_B(x) = CM_{B/A}(xH) \leq CM_{C/A}(xH) = CM_C(x)$, and $B \subseteq C$. Thus $CM_C(x) > 0 < CM_C(y) \Longrightarrow CM_{C/A}(xH) = CM_C(x) > 0 < CM_C(y) = CM_{C/A}(yH) \Longrightarrow CM_{B/A}(xyH) = CM_{B/A}(yxH) \Longrightarrow CM_B(xy) = CM_B(yx)$ for all $x, y \in G$, and so $B \triangleleft C$.

Again, we see that, $\text{supp}(C/B) = H_2/H_1$ is equivalent to $(H_2/H)/(H_1/H) = \text{supp}(C/A)/\text{supp}(B/A)$, and it is abelian. By Theorem 3.2, it follows that C/B is abelian. $\square$

Now, we define the notion of solvability of fuzzy multigroup as follows:

Definition 3.4. *If A is a fuzzy multigroup of G, then there must exist a chain of successive fuzzy submultigroups of A:*

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = A, \tag{4}$$

such that $supp(A_0) = supp(A_1) = \cdots = supp(A_n) = supp(A)$.

Thus (4) can be rewritten as

$$CM_{A_0}(x) \leq CM_{A_1}(x) \leq \cdots \leq CM_{A_n}(x) = CM_A(x) \text{ for all } x \in G. \tag{5}$$

Albeit, if A is a trivial fuzzy multigroup, we have $A_0 = A$.

Definition 3.5. *A fuzzy multigroup A of G is solvable/soluble if there exists a chain of successive fuzzy submultigroups*

$$A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = A, \tag{6}$$

where $A_i \lhd A_{i+1}$ and A_{i+1}/A_i is abelian for all $0 \leq i \leq n-1$.

Thus, such a finite chain of successive fuzzy submultigroups of A is a solvable/soluble series for A denoted by A_i. Without contradiction, the solvable series for A can be written as

$$A_0 \lhd A_1 \lhd \cdots \lhd A_n = A. \tag{7}$$

Theorem 3.6. *Let A be a fuzzy multigroup of a group G. Then A is solvable iff $supp(A)$ is a solvable group.*

Proof. Let A be a solvable fuzzy multigroup of G. Then there exists a solvable series of A as follows:

$$A_0 \lhd A_1 \lhd \cdots \lhd A_n = A.$$

Set $H = \text{supp}(A)$, i.e., H is a subgroup of G. Then

$$\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = H$$

is a solvable series for H since $\text{supp}(A) = \text{supp}(A_i)$. Thus H is a solvable group.

Conversely, let $H = \text{supp}(A)$ be solvable. Then

$$\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = H$$

is a solvable series for H. Consequently, we have

$$A_0 \lhd A_1 \lhd \cdots \lhd A_n = A,$$

which is a solvable series for A. Hence A is a solvable fuzzy multigroup of G $\square$

Theorem 3.7. *Let A and B be fuzzy multigroups of G with the same support H such that $A \subseteq B$ and A is self-normal. If A is solvable, then B is a solvable fuzzy multigroup of G.*

Proof. Let $A_0 \lhd A_1 \lhd \cdots \lhd A_n = A$ be a solvable series for A. Because A is self-normal and $\mathrm{supp}(A) = \mathrm{supp}(B) = H$, we get $A \lhd B$. Consequently, we have $\mathrm{supp}(B/A) = H/H = H$ and is abelian. Thus

$$A_0 \lhd A_1 \lhd \cdots \lhd A_n = A \lhd B$$

is a solvable series for B. Hence B is a solvable fuzzy multigroup of G. □

Theorem 3.8. *Let B be a solvable fuzzy multigroup of G and let A be a self-normal fuzzy submultigroup of B with $A \subseteq B_i$. Then A is solvable.*

Proof. Let $B_0 \lhd B_1 \lhd \cdots \lhd B_n = B$ be a solvable series for B since B is solvable. Because $A \subseteq B_i$, we have

$$B_0 \cap A \subseteq B_1 \cap A \subseteq \cdots \subseteq B_n \cap A = A.$$

Clearly, $CM_{B_1 \cap A}(x) > 0 < CM_{B_1 \cap A}(y) \Longrightarrow CM_{B_1}(x) > 0 < CM_{B_1}(y)$ and $CM_A(x) > 0 < CM_A(y) \Longrightarrow CM_{B_1}(xy) > 0 < CM_{B_1}(yx)$ and $CM_A(xy) > 0 < CM_A(yx) \Longrightarrow CM_{B_1 \cap A}(xy) = CM_{B_1 \cap A}(yx)$ for all $x, y \in G$. Thus

$$B_0 \cap A \lhd B_1 \cap A \lhd \cdots \lhd B_n \cap A = A.$$

Again, let $H_i = \mathrm{supp}(B_i)$ and $H = \mathrm{supp}(A)$. Then we get a quotient $(H_2 \cap H)/(H_1 \cap H)$, which is abelian because H_2/H_1 is abelian. The same logic holds for the other quotients, and thus

$$B_0 \cap A \subseteq B_1 \cap A \subseteq \cdots \subseteq B_n \cap A = A$$

is the solvable series for A. Hence A is solvable. □

Theorem 3.9. *Let A be a normal fuzzy submultigroup of a fuzzy multigroup B of G, and let B be self-normal. If A and B/A are solvable, then B is a solvable fuzzy multigroup of G.*

Proof. Let B'/A' be the canonical form of B/A. Then $A \subseteq A'$, $B \subseteq B'$, $\mathrm{supp}(A') = \mathrm{supp}(A) = H_1$ and $\mathrm{supp}(B') = \mathrm{supp}(B) = H_2$. Thus, there exists a solvable series

$$C_0 \lhd C_1 \lhd \cdots \lhd C_n = B'/A'.$$

Set $A' = A_m$ and $B' = B_n$. Assume there exist fuzzy multigroups B_i of G such that $A' \lhd B_i \lhd B_{i+1}$ and $C_i = B_i/A'$ in the canonical form for $0 \leq i \leq n-1$. Thus

$$B_0/A' \lhd B_1/A' \lhd \cdots \lhd B_n/A' = B'/A'$$

is a solvable series for B'/A'. Since B_0/A' is a trivial fuzzy submultigroup of B'/A', then it is meet to say that $B_0 = A'$. By Theorem 3.3, we have

$$A' = B_0 \lhd B_1 \lhd \cdots \lhd B_n = B', \tag{8}$$

where B_{i+1}/B_i is abelian for $0 \leq i \leq n-1$.

Next, A is self-normal by Theorem 3.1, and A' is solvable by Theorem 3.7. Thus, there exists a solvable series for A' as follows:

$$A_0 \lhd A_1 \lhd \cdots \lhd A_m = A'. \tag{9}$$

By juxtaposing (8) and (9), we have a solvable series for B'. Hence B is solvable by Theorem 3.8. □

4. Conclusion

In this paper, we have defined solvable fuzzy multigroup for the first time as an algebraic structure in fuzzy multigroup context and obtained some results. Solvable series of a fuzzy multigroup was defined in such a way that the family of the fuzzy submultigroups of the considered fuzzy multigroup has the same support. Certain results in normality and quotient of fuzzy multigroups were considered. Some results on solvable fuzzy multigroups could still be exploited and the notion of nilpotency is an interest area to consider in fuzzy multigroup context.

References

[1] J. A. Awolola, M. A. Ibrahim, *On homomorphism of fuzzy multigroups*, Ratio Mathematica, 39 (2020), 229-236.

[2] A. Baby, T. K. Shinoj, J. J. Sunil, *On abelian fuzzy multigroups and order of fuzzy multigroups*, Journal of New Theory, 5 (2015), 80-93.

[3] R. Biswas, *An application of Yager's bag theory in multicriteria based decision making problems*, International Journal of Intelligent Systems, 14 (1999), 1231-1238.

[4] P. Bhattacharya, N. P. Mukherjee, *Fuzzy groups: some group theoretic analogs II*, Information Science, 41 (1987), 77-91.

[5] P. A. Ejegwa, *On fuzzy multigroups and fuzzy submultigroups*, Journal of Fuzzy Mathematics, 26 (2018), 641-654.

[6] P. A. Ejegwa, *On abelian fuzzy multigroups*, Journal of Fuzzy Mathematics, 26 (2018), 655-668.

[7] P. A. Ejegwa, *On normal fuzzy submultigroups of a fuzzy multigroup*, Theory and Applications of Mathematics and Computer Science, 8 (2018), 64-80.

[8] P. A. Ejegwa, *Homomorphism of fuzzy multigroups and some of its properties*, Applied and Application of Mathematics, 13 (2018), 114-129.

[9] P. A. Ejegwa, *Direct product of fuzzy multigroups*, Journal of New Theory, 28 (2019), 62-73.

[10] P. A. Ejegwa, *On alpha-cuts homomorphism of fuzzy multigroups*, Annals of Fuzzy Mathematics and Informatics, 19 (2020), 73-87.

[11] P. A. Ejegwa, *Some properties of alpha-cuts of fuzzy multigroups*, Journal of Fuzzy Mathematics, 28 (2020), 201-222.

[12] P. A. Ejegwa, *Some group's theoretic notions in fuzzy multigroup context*, In: Jana, C., Senapati, T. Pal, M. (Eds.); Handbook of Research on Emerging Applications of Fuzzy Algebraic Structures, pp. 34-62, IGI Global Publisher, Hershey, Pennsylvania 17033-1240, USA, 2020.

[13] P. A. Ejegwa, *On generalized direct product of fuzzy multigroups*, Annals of Communications in Mathematics, 3 (2020), 35-45.

[14] P. A. Ejegwa, *Some characterizations of fuzzy comultisets and quotient fuzzy multigroups*, Ratio Mathematica, 38 (2020), 237-259.

[15] P. A. Ejegwa, *On the concept of fuzzy comultisets and its properties*, TWMS Journal of Applied and Engineering Mathematics, 10 (2020), 918-929.

[16] P. A. Ejegwa, J. M. Agbetayo, *On commutators of fuzzy multigroups*, Earthline Journal of Mathematics, 4 (2020), 189-210.

[17] P. A. Ejegwa, J. M. Agbetayo, J. A. Otuwe, *Characteristic and Frattini fuzzy submultigroups of fuzzy multigroups*, Annals of Fuzzy Mathematics and Informatics, 19 (2020), 139-155.

[18] P. A. Ejegwa, J. A. Awolola, J. M. Agbetayo, I. M. Adamu, *On the characterisation of anti-fuzzy multigroups*, Annals of Fuzzy Mathematics and Informatics, 21 (2021), 307-318.

[19] P. A. Ejegwa, Y. Feng, W. Zhang, J. Zhou, X. Ming, *On some properties of divisible and pure fuzzy multigroups*, Italian Journal of Pure and Applied Mathematics, 46 (2021), 499-512.

[20] S. Miyamoto, *Basic operations of fuzzy multisets*, Journal of Japan Society of Fuzzy Theory and Systems, 8 (1996), 639-645.

[21] S. Miyamoto, K. Mizutani, *Fuzzy multiset model and methods for nonlinear document clustering for information retrieval*, Springer-Verlag Berlin Heidelberg, 2004.

[22] K. Mizutani, R. Inokuchi, S. Miyamoto, *Algorithms of nonlinear document clustering based on fuzzy multiset model*, International Journal of Intelligent Systems, 23 (2008), 176-198.

[23] J. M. Mordeson, K. R. Bhutani, A. Rosenfeld, *Fuzzy group theory*, Springer-Verlag Berlin Heidelberg, 2005.

[24] N. P. Mukherjee, P. Bhattacharya, *Fuzzy groups: some group theoretic analogs*, Information Science, 39 (1986), 247-268.

[25] R. Rasuli, *$t-$norms over fuzzy multigroups*, Earthline Journal of Mathematics, 3 (2020), 207-228.

[26] S. Ray, *Solvable fuzzy groups*, Information Sciences, 75 (1993), 47-61.

[27] A. Rosenfeld, *Fuzzy subgroups*, Journal of Mathematical Analysis and Applications, 35 (1971), 512-517.

[28] B. K. Sarma, *Solvable fuzzy groups*, Fuzzy Sets and Systems, 106 (1999), 463-467.

[29] T. K. Shinoj, A. Baby, J. J. Sunil, *On some algebraic structures of fuzzy multisets*, Annals of Fuzzy Mathematics and Informatics, 9 (2015), 77-90.

[30] A. Syropoulos, *On generalized fuzzy multisets and their use in computation*, Iranian Journal of Fuzzy Systems, 9 (2012), 113-125.

[31] R. R. Yager, *On the theory of bags*, International Journal of General Systems, 13 (1986), 23-37.

[32] L. A. Zadeh, *Fuzzy sets*, Information and Control, 8 (1965), 338-353.

Accepted: January 22, 2022