Prime-valent one-regular graphs of order $20p$

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Abstract. A graph is *one-regular* and *arc-transitive* if its full automorphism group acts on its arcs regularly and transitively, respectively. In this paper, we classify connected one-regular graphs of prime valency and order $20p$ for each prime p. As a result there is only one infinite family of such graphs, that is, the cycle C_{20p} with valency two.
Keywords: symmetric graph, arc-transitive graph, one-regular graph.

1. Introduction

Throughout this paper graphs are assumed to be finite, simple, connected and undirected. For group-theoretic concepts and graph-theoretic terms not defined here we refer the reader to [20, 22] and [1, 2], respectively. Let G be a permutation group on a set Ω and $v \in \Omega$. Denote by G_v the stabilizer of v in G, that is, the subgroup of G fixing the point v. We say that G is *semiregular* on Ω if $G_v = 1$ for every $v \in \Omega$ and *regular* if G is transitive and semiregular.

For a graph X, denote by $V(X)$, $E(X)$ and $\mathrm{Aut}(X)$ its vertex set, its edge set and its full automorphism group, respectively. A graph X is said to be G-*vertex-transitive* if $G \leq \mathrm{Aut}(X)$ acts transitively on $V(X)$. X is simply called *vertex-transitive* if it is $\mathrm{Aut}(X)$-vertex-transitive. An *s-arc* in a graph is an ordered $(s+1)$-tuple $(v_0, v_1, \cdots, v_{s-1}, v_s)$ of vertices of the graph X such that v_{i-1} is adjacent to v_i for $1 \leq i \leq s$, and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s-1$. In particular, a 1-arc is just an arc and a 0-arc is a vertex. For a subgroup $G \leq \mathrm{Aut}(X)$, a graph X is said to be (G, s)-*arc-transitive* or (G, s)-*regular* if G is transitive or regular on the set of s-arcs in X, respectively. A (G, s)-arc-transitive graph is said to be (G, s)-*transitive* if it is not $(G, s+1)$-arc-transitive. In particular, a $(G, 1)$-arc-transitive graph is called G-*symmetric*. A graph X

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is simply called *s-arc-transitive*, *s-regular* or *s-transitive* if it is $(\mathrm{Aut}(X), s)$-arc-transitive, $(\mathrm{Aut}(X), s)$-regular or $(\mathrm{Aut}(X), s)$-transitive, respectively.

We denote by C_n and K_n the cycle and the complete graph of order n, respectively. Denote by D_{2n} the dihedral group of order $2n$. As we all know that there is only one connected 2-valent graph of order n, that is, the cycle C_n, which is 1-regular with full automorphism group D_{2n}. Let p and q be two primes. Classifying s-transitive and s-regular graphs has received considerable attention. The classification of s-transitive graphs of order p and $2p$ was given in [5] and [7], respectively. Liu [15] characterized prime-valent arc-transitive basic graphs of order $4p$ or $4p^2$. Li [14] and Chen [6] classified prime-valent one-regular graph of order $8p$ and $12p$, respectively. Pan [19] and Huang [13] classified the pentavalent s-transitive graphs of order $4pq$ and $4p^n$ for n a positive integer, respectively. Pan [18] determined heptavalent symmetric graph of order four times an odd square-free integer. Zhou [24] gave a complete classification of cubic one-regular graphs of order twice a square-free integer.

For 2-valent case, s-transitivity always means 1-regularity, and for cubic case, s-transitivity always means s-regularity by Miller [17]. However, for the other prime-valent case, this is not true, see for example [10] for pentavalent case and [11] for heptavalent case. Thus, the characterization and classification of prime-valent s-regular graphs is very interesting and also reveals the s-regular global and local actions of the permutation groups on s-arcs of such graphs. In particular, 1-regular action is the most simple and typical situation. In this paper, we classify prime-valent one-regular graph of order $20p$ for each prime p.

2. Preliminary results

Let X be a connected G-symmetric graph with $G \leq \mathrm{Aut}(X)$, and let N be a normal subgroup of G. The *quotient graph* X_N of X relative to N is defined as the graph with vertices the orbits of N on $V(X)$ and with two orbits adjacent if there is an edge in X between those two orbits. In view of [16, Theorem 9], we have the following:

Proposition 2.1. *Let X be a connected G-symmetric graph with $G \leq \mathrm{Aut}(X)$ and prime valency $q \geq 3$, and let N be a normal subgroup of G. Then one of the following holds:*

(1) *N is transitive on $V(X)$;*

(2) *X is bipartite and N is transitive on each part of the bipartition;*

(3) *N has $r \geq 3$ orbits on $V(X)$, N acts semiregularly on $V(X)$, and the quotient graph X_N is a connected q-valent G/N-symmetric graph.*

To extract a classification of connected prime-valent symmetric graphs of order $2p$ for a prime p from Cheng and Oxley [7], we introduce the graphs $G(2p, q)$. Let V and V' be two disjoint copies of $\mathbb{Z}_p$, say $V = \{0, 1, \cdots, p-1\}$

and $V' = \{0', 1', \cdots, (p-1)'\}$. Let q be a positive integer dividing $p-1$ and $H(p,q)$ the unique subgroup of Z_p^* of order q. Define the graph $G(2p,q)$ to have vertex set $V \cup V'$ and edge set $\{xy' \mid x-y \in H(p,q)\}$.

Proposition 2.2. *Let X be a connected q-valent symmetric graph of order $2p$ with p, q primes. Then X is isomorphic to K_{2p} with $q = 2p-1$, $K_{p,p}$ or $G(2p,q)$ with $q \,\big|\, (p-1)$. Furthermore, if $(p,q) \neq (11,5)$ then $\mathrm{Aut}(G(2p,q)) = (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$; if $(p,q) = (11,5)$ then $\mathrm{Aut}(G(2p,q)) = \mathrm{PGL}(2,11)$.*

Let $p \neq 5$ be an odd prime. Then $20p = 4{\cdot}5{\cdot}p$ is four times a square-free integer. From [18, Theorem 1.1], we have the following characterization about the full automorphism groups of connected heptavalent symmetric graphs of order $20p$ with $p \neq 5$.

Proposition 2.3. *Let p be an odd prime different from 5 and X a connected heptavalent symmetric graphs of order $20p$. Then, the full automorphism group $\mathrm{Aut}(X) \cong \mathrm{PSL}(2,p)$, $\mathrm{PGL}(2,p)$, $\mathrm{PSL}(2,p)\times\mathbb{Z}_2$ or $\mathrm{PGL}(2,p)\times\mathbb{Z}_2$ with $p \equiv 1 (\mathrm{mod}\ 7)$.*

The following proposition is the famous "N/C-Theorem", see for example [12, Chapter I, Theorem 4.5]).

Proposition 2.4. *The quotient group $N_G(H)/C_G(H)$ is isomorphic to a subgroup of the automorphism group $\mathrm{Aut}(H)$ of H.*

From [8, p.12-14] and [21, Theorem 2], we can deduce the non-abelian simple groups whose orders have at most three different prime divisors.

Proposition 2.5. *Let G be a non-abelian simple group. If the order $|G|$ has exactly three different prime divisors, then G is called K_3-simple group and isomorphic to one of the following groups.*

Table 1: **Non-abelian simple $\{2,3,p\}$-groups**

Group	Order	Group	Order
A_5	$2^2 \cdot 3 \cdot 5$	$\mathrm{PSL}(2,17)$	$2^4 \cdot 3^2 \cdot 17$
A_6	$2^3 \cdot 3^2 \cdot 5$	$\mathrm{PSL}(3,3)$	$2^4 \cdot 3^3 \cdot 13$
$\mathrm{PSL}(2,7)$	$2^3 \cdot 3 \cdot 7$	$\mathrm{PSU}(3,3)$	$2^5 \cdot 3^3 \cdot 7$
$\mathrm{PSL}(2,8)$	$2^3 \cdot 3^2 \cdot 7$	$\mathrm{PSU}(4,2)$	$2^6 \cdot 3^4 \cdot 5$

The next proposition is proved originally by John Thompson, but now an easy consequence of the Classification of Finite Simple Groups (see for example [23, Chapter 1, Section 2]), that all finite non-abelian simple $3'$-groups (whose order is not divisible by 3) are Suzuki groups. This is well known in the sense that it is mentioned frequently in the literature.

Proposition 2.6. *Any non-abelian finite simple group whose order is not divisible by 3 is isomorphic to a Suzuki group* $\mathrm{Sz}(q)$ *with* $q = 2^{2n+1}$ *and* $n \geq 1$. *In particular, the order of* $\mathrm{Sz}(q)$ *is* $q^2(q^2+1)(q-1)$.

3. Classification

This section is devoted to classifying prime-valent one-regular graphs of order $20p$ for each prime p. Let q be a prime. In what follows, we always denote by X a connected q-valent one-regular graph of order $20p$. Set $A = \mathrm{Aut}(X)$, $v \in V(X)$. Then the vertex stabilizer $A_v \cong \mathbb{Z}_q$ and hence $|A| = 20pq$.

Now, we first deal with the special case $q \leq 5$. Clearly, any connected graph of order $20p$ and valency two is isomorphic to the cycle C_{20p}. Thus, for $q = 2$, $X \cong C_{20p}$ and $A \cong D_{40p}$. Let $q = 3$. By [24, Corollary 3.3], there exists no cubic one-regular graph of order $4{\cdot}5{\cdot}p$. Let $q = 5$. By [19, Theorem 3.1], [15, Theorem 1.1] and [13], there exists no pentavalent one-regular graph of order $4{\cdot}5{\cdot}p$. The next lemma is about the case $q = 7$.

Lemma 3.1. *There exists no heptavalent one-regular graph of order* $20p$

Proof. Let X be a heptavalent one-regular graph of order $20p$. Then $q = 7$ and $|A| = 4{\cdot}5{\cdot}7{\cdot}p = 140p$. By Proposition 2.3, $A \cong \mathrm{PSL}(2,p)$, $\mathrm{PGL}(2,p)$, $\mathrm{PSL}(2,p){\times}\mathbb{Z}_2$ or $\mathrm{PGL}(2,p){\times}\mathbb{Z}_2$, where $p \equiv 1 (\mathrm{mod}\ 7)$.

Suppose that $A = \mathrm{PSL}(2,p)$. Then $|A| = 140p = \frac{1}{2}p(p^2-1)$. It forces that $p^2 = 281$. Note that p^2 is a prime square. However, 281 is a prime, a contradiction.

Suppose that $A = \mathrm{PGL}(2,p)$ or $\mathrm{PSL}(2,p){\times}\mathbb{Z}_2$. Then $|A| = 140p = p(p^2-1)$. This implies that $p^2 = 141 = 3{\cdot}47$, a contradiction.

Suppose that $A = \mathrm{PGL}(2,p){\times}\mathbb{Z}_2$. Then $|A| = 140p = 2p(p^2-1)$. An easy calculation implies that $p^2 = 71$, contrary to the fact that p^2 is a prime square. □

To finish the classification, we treat the general case $q > 7$.

Lemma 3.2. *Let* $q > 7$. *Then there exists no* q*-valent one-regular graph of order* $20p$

Proof. Let X be a q-valent one-regular graph of order $20p$. Then $|A| = 20pq$, $|V(X)| = 20p$ and $A_v \cong \mathbb{Z}_q$. If $p = 2$, then $|V(X)| = 40$. By [14, Theorem 3.3], there exists no q-valent one-regular graph of order 40 with $q > 7$. If $p = 3$, then $|V(X)| = 60 = 12{\cdot}5$. By [6, Theorem 3.1], there exists no q-valent one-regular graph of order 60 with $q > 7$. Next, we deal with $p \geq 5$ and separate them into two cases: $p = 5$ and $p \geq 7$.

Case 1. Suppose that $p = 5$. Then $|V(X)| = 20{\cdot}5$ and $|A| = 2^2{\cdot}5^2{\cdot}q$.

Note that $q > 7$. If A is not solvable, then A has a composition factor isomorphic to a K_3-simple group. By Proposition 2.5, every K_3-simple group has divisor 3. It forces that $3 \mid |A|$, a contradiction. Thus, A is solvable. Let N be a minimal normal subgroup of A. Then N is also solvable and hence isomorphic to $\mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_5$, $\mathbb{Z}_5^2$ or $\mathbb{Z}_q$. An easy calculation implies that the number of the orbits of N acting on $V(X)$ is at least 4. By Proposition 2.1, N is semiregular on $V(X)$ and so $N \ncong \mathbb{Z}_q$. Since there exists no regular graph of odd order and odd valency, we have $N \ncong \mathbb{Z}_2^2$, and since there is no q-valent graph of order 4 with $q > 7$, we have that $N \ncong \mathbb{Z}_5^2$. It follows that $N \cong \mathbb{Z}_2$ or $\mathbb{Z}_5$.

Assume that $N \cong \mathbb{Z}_2$. Then X_N is a q-valent symmetric graph of order $2{\cdot}5^2$. Recall that A is solvable. Thus, A/N is also solvable. Since $q > 7$, we have that A/N has no normal subgroup of order q by Proposition 2.1, and since there exists no regular graph of odd order and odd valency, we have that A/N has no normal subgroup of order 2. The solvability of A/N implies that A/N has a normal 5-subgroup, say M/N. With an easy calculation, we have that $|M/N| = 5^2$ or 5.

Let $|M/N| = 5^2$. Then by Proposition 2.4, $M/C_M(N) \lesssim \mathrm{Aut}(N) \cong \mathbb{Z}_1$ because $N \cong \mathbb{Z}_2$. It forces that $M = C_M(N)$. Let P be a Sylow 5-subgroup of M. Then $|P| = 5^2$ and $M = P \times N$. Since P is a normal Sylow 5-subgroup of M, we have that P is characteristic in M. The normality of M in A implies that P is also normal in A. By Proposition 2.1, X_P is a q-valent symmetric graph of order 4. However, any symmetric graphs of order 4 is isomorphic to either the cycle C_4 with valency 2 or the complete graph K_4 with valency 3. This is contrary to the fact that X_P has valency $q > 7$.

Let $|M/N| = 5$. Then M has order 10. By elementary group theory, any group of order 10 is isomorphic to either a cyclic group $\mathbb{Z}_{10}$ or a dihedral group D_{10}. Clearly, the former group has a normal subgroup of order 2 and the latter group has no normal subgroup of order 2. Since M has a normal subgroup $N \cong \mathbb{Z}_2$, we have that $M \cong \mathbb{Z}_{10}$ and M has a normal subgroup $P \cong \mathbb{Z}_5$. Clearly, P is a Sylow 5-subgroup of M. The normality of P in M forces that P is characteristic in M, and since M is normal in A, we have that P is also normal in A. By Proposition 2.1, X_P is a q-valent symmetric graph of order 20 with $q > 7$. Checking the list of symmetric graph of order up to 30 in [9], we have that $X_P \cong K_{20}$ with $q = 19$ and $A/P \lesssim \mathrm{Aut}(K_{20}) \cong \mathrm{S}_{20}$. An easy calculation implies that $|A/P| = 2^2{\cdot}5{\cdot}19$. However, by Magma [3], S_{20} has no subgroup of order $2^2{\cdot}5{\cdot}19$, a contradiction.

Assume that $N \cong \mathbb{Z}_5$. Then by Proposition 2.1, X_N is a q-valent symmetric graph of order 20, and by [9], $X_N \cong K_{20}$ with $q = 19$. A similar argument as the above paragraph we can deduce a contradiction.

Case 2. Suppose that $p \geq 7$. Then $|A| = 2^2{\cdot}5{\cdot}p{\cdot}q$ with $q > 7$.

If A is non-solvable, then A must have a composition factor isomorphic to a non-abelian simple group. Note that the order of A has exactly four different prime divisors. Thus, this non-solvable composition factor is either K_3-simple group or K_4-simple group. Since $p \geq 7$ and $q > 7$, we have that 3 is not a divisor of $|A|$. By Propositions 2.5 and 2.6, the only possibilities are the Suzuki groups $\mathrm{Sz}(2^{2n+1})$ with $n \geq 1$. This forces that

$$|\mathrm{Sz}(2^{2n+1})| = (2^{2n+1})^2((2^{2n+1})^2+1)(2^{2n+1}-1) \mid |A|.$$

This is contrary to the fact that $|A| = 2^2{\cdot}5{\cdot}p{\cdot}q$ with $p \geq 7$ and $q > 7$. Thus, A is solvable. For convenience we still use N to denote a minimal normal subgroup of A. Clearly, N is also solvable and $N \cong \mathbb{Z}_2$, $\mathbb{Z}_2^2$, $\mathbb{Z}_5$, $\mathbb{Z}_p$, $\mathbb{Z}_q$ or $\mathbb{Z}_p^2$ with $q = p$. Since $|V(X))| = 20p$, we have that N is not transitive and has at least 20 orbits on $V(X)$. By Proposition 2.1, N is semiregular on $V(X)$, and so $N \ncong \mathbb{Z}_p^2$ with $p = q$. Since there exists no connected regular graph of odd order and odd valency, we have that A has no normal subgroup of order 4 and so $N \ncong \mathbb{Z}_2^2$. Thus, $N \cong \mathbb{Z}_2$, $\mathbb{Z}_5$ or $\mathbb{Z}_p$.

Assume that $N \cong \mathbb{Z}_p$. Then X_N is a q-valent symmetric graph of order 20 and $A/N \lesssim \mathrm{Aut}(X_N)$ by Proposition 2.1. Since $q > 7$, we have that $X_N \cong K_{20}$ with $q = 19$ by [9] and $A/N \lesssim \mathrm{S}_{20}$. Note that $|A/N| = 2^2{\cdot}5{\cdot}19$. However, by Magma [3], S_{20} has no subgroup of order $2^2{\cdot}5{\cdot}19$, a contradiction.

Assume that $N \cong \mathbb{Z}_5$. Then X_N is a q-valent symmetric graph of order $4p$ and $A/N \lesssim \mathrm{Aut}(X_N)$ by Proposition 2.1. The solvability of A forces that A/N is also solvable. Recall that A has no normal subgroup of order 4, q or p^2 with $p = q$. Thus, A/N has normal subgroup $M/N \cong \mathbb{Z}_p$ or $\mathbb{Z}_2$. It follows that M is a normal subgroup of A and has order $5p$ or 10. Again by Proposition 2.1, X_M is a q-valent symmetric graph of order 4 or $2p$. For the former, $X_M \cong C_4$ or K_4. Clearly, this is impossible because X_M has valency $q > 7$. For the latter, $X_M \cong K_{2p}$, $K_{p,p}$ or $G(2p, q)$.

Let $X_M \cong K_{2p}$. Then $q = 2p - 1$ and $A/M \lesssim \mathrm{S}_{2p}$. Clearly, A/M has order $2{\cdot}p{\cdot}q$ and is 2-transitive on $V(X_M)$ because $q = 2p - 1$. By Burnside's Theorem [4, p.192, Theorem IX], any 2-transitive permutation group is either almost simple or affine. The solvability of A forces that A/M is also solvable and hence affine. It follows that A/M has a normal subgroup $K/M \cong \mathbb{Z}_p$. Since $|M| = 10$ and $p \geq 7$, we have that K has a normal Sylow p-subgroup $P \cong \mathbb{Z}_p$ by Sylow Theorem. The normality of the Sylow p-subgroup P in K forces that P is characteristic in K. Thus, P is normal in A. Again by Proposition 2.1, X_P is a q-valent symmetric graph of order 20 and by [9], $X_P \cong K_{20}$. A similar argument as the above, we deduce that A/P has order $2^2{\cdot}5{\cdot}19$ and can not be embedded in S_{20}, a contradiction.

Let $X_M \cong K_{p,p}$. Then $p = q$ and $|A/M| = 2p^2$. Since $q = p > 7$, we have that A/M has a normal Sylow p-subgroup K/M by Sylow Theorem and $|K/M| = p^2$. It follows that $|K| = 10{\cdot}p^2$. Let P be a Sylow p-subgroup of K. Then P has order p^2 and again by Slow Theorem, P is normal and hence characteristic in K. The normality of K in A forces that P is also normal in A. Since $|V(X)| = 20{\cdot}p$, we have that P acting on $V(X)$ has 20 orbits and $P_v \cong \mathbb{Z}_p$. However, by Proposition 2.1, P must be semiregular on $V(X)$, that is, $P_v = 1$, a contradiction.

Let $X_M \cong G(2p, q)$. Then $|A/M| = 2{\cdot}p{\cdot}q$ and by Proposition 2.2, $A/M \cong (\mathbb{Z}_p \rtimes \mathbb{Z}_q) \rtimes \mathbb{Z}_2$ with $q \mid (p-1)$. This implies that A/M has a normal subgroup $K/M \cong \mathbb{Z}_p$. Since M has order 10, we have that K has a normal Sylow p-subgroup $P \cong \mathbb{Z}_p$ by Sylow Theorem. Thus, P is characteristic in K and hence normal in A. By Proposition 2.1, X_P is a q-valent symmetric graph of order 20 and by [9], $X_P \cong K_{20}$. Similarly, A/P has order $2^2{\cdot}5{\cdot}19$ and by Magma [3], S_{20} has no subgroup of order $2^2{\cdot}5{\cdot}19$, a contradiction.

Assume that $N \cong \mathbb{Z}_2$. Then X_N is a q-valent symmetric graph of order $2{\cdot}5{\cdot}p$, $|A/N| = 2{\cdot}5{\cdot}p{\cdot}q$ and $A/N \lesssim \mathrm{Aut}(X_N)$. Since A/N has order twice an odd number, we have that A/N must have a normal subgroup M/N of index two. Thus, $|M/N| = 5{\cdot}p{\cdot}q$ and M also has order twice an odd number. It follows that M is also has a normal subgroup K of index two and so $|K| = 5{\cdot}p{\cdot}q$. This implies that $|A : K| = 4$ and K is also normal in A. Recall that A has no normal subgroup of order 4, q or p^2 with $q = p$. If $p = q$, then $|K| = 5{\cdot}p^2$. Since $q > 7$, we have that K must a normal Sylow p-subgroup P by Sylow Theorem. It forces that P is characteristic in K and hence normal in A. Clearly, P has order p^2, this is impossible. Thus, $p \neq q$. Since K is solvable, we have that K must have a normal subgroup $H \cong \mathbb{Z}_5$, $\mathbb{Z}_p$ or $\mathbb{Z}_q$. Note that $5, p$ and q are different primes. Thus, H is characteristic in K and hence normal in A. Since $A_v \cong \mathbb{Z}_q$, we have that $H \ncong \mathbb{Z}_q$. This implies that A has a normal subgroup $H \cong \mathbb{Z}_5$ or $\mathbb{Z}_p$. Similar arguments as the above, we can deduce that this is impossible. □

Combining the above arguments with the cases $q = 2, 3, 5$, and Lemmas 3.1-3.2, we have the following result.

Theorem 3.1. *Let p, q be two primes and X a connected q-valent one-regular graph of order $20p$. Then X is isomorphic to the cycle C_{20p} with valency 2 and* $\mathrm{Aut}(X) \cong D_{40p}$.

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