Inequalities of DVT-type-the one-dimensional case continued

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Abstract. In this note, the investigation of particular inequalities of DVT-type in integer numbers is continued.
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1. Introduction

In [2], A. Drápal and V. Valent proved that in a finite quasigroup Q of order n the number of associative triples $a(Q) \geq 2n - i(Q) + (\delta_1 + \delta_2)$, where $i(Q)$ is the number of idempotents in Q, i.e., $i(Q) = |\{ x \in Q \,|\, xx = x \}|$, $\delta_1 = |\{ z \in Q \,|\, zx \neq x \text{ for all } x \in Q \}|$ and $\delta_2 = |\{ z \in Q \,|\, xz \neq x \text{ for all } x \in Q \}|$ (Theorem 2.5). This important result is an easy consequence of the inequality

$$\sum_{i=1}^{n}(a_i^2 + b_i^2 + a_i b_i) - \sum_{i=1}^{k}(a_i + b_i) \geq 3n - 2k + (r + s),$$

where $n \geq k \geq 0, a_1, \dots, a_n, b_1 \dots, b_n$ are non-negative integers such that $\sum a_i = n = \sum b_i$, $a_i \geq 1$ and $b_i \geq 1$ for $1 \leq i \leq k$, r is the number of i with $a_i = 0$

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and s is the number of i with $b_i = 0$ (Proposition 2.4(ii)). The lengthy and complicated proof of this DVT-inequality (inequality of Drápal-Valent type) in [2] is based on highly semantically involved insight.

In [3], a very short elementary arithmetical proof of a more general inequality of this type was found. This inequality is two-dimensional in the sense that it works with two n-tuples of integers. The approach in [3] opens a road to investigation of similar inequalities of DVT-type which could be useful in further investigations of estimates in non-associative algebra and they are also of independent interest. Hence they deserve a thorough examination, however the research is only at its beginning. In [1], the investigation of the one-dimensional case working with one n-tuple of real numbers was started. This note is an immediate continuation of [1].

2. Second concepts

Let $n \geq 1$ and let $\alpha = (a_1, \dots, a_n)$ be an ordered n-tuple of integers. Let I be any subset (whether empty or non-empty) of the interval $\{1, \dots, n\}$. We put

(1) $z(\alpha, a) = |\{\, i \mid 1 \leq i \leq n, a_i = a\}|$ for every $a \in \mathbb{R}$;

(2) $z(\alpha) = z(\alpha, 0)$;

(3) $z(\alpha, +) = \sum_{a>0} z(\alpha, a)$;

(4) $z(\alpha, -) = \sum_{a<0} z(\alpha, a)$;

(5) $s(\alpha) = \sum_{i=1}^{n} a_i$;

(6) $r(\alpha) = \sum_{i=1}^{n} a_i^2$;

(7) $q(\alpha) = r(\alpha) - s(\alpha)$;

(8) $t(\alpha) = q(\alpha) - z(\alpha)$.

(9) $I^{\perp} = \{1, \dots, n\} \setminus I$;

(10) $s(\alpha, I) = -|I| + \sum_{i \in I} a_i \; (= \sum_{i \in I} (a_i - 1)\,)$;

(11) $r(\alpha, I, +) = \sum_{i=1}^{n} a_i^2 + \sum_{i \in I} a_i$;

(12) $r(\alpha, I, -) = \sum_{i=1}^{n} a_i^2 - \sum_{i \in I} a_i$;

(13) $t(\alpha, I) = \sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{n} a_i + |I| - \sum_{i \in I} a_i - z(\alpha)$.

Lemma 2.1. (i) $r(\alpha, I, +) \geq z(\alpha, +)$.

(ii) $r(\alpha, I, +) = z(\alpha, +)$ *if and only if* $a_i \in \{0, -1\}$ *for* $i \in I$ *and* $a_i \in \{0, 1\}$ *for* $i \in I^{\perp}$.

Proof. Put $K_1 = \{ i \,|\, a_i \geq 2 \}$, $K_2 = \{ i \,|\, a_i = 1 \}$, $K_3 = \{ i \,|\, a_i = 0 \}$, $K_4 = \{ i \,|\, a_i = -1 \}$, $K_5 = \{ i \,|\, a_i \leq -2 \}$. Then $r(\alpha) + \sum_{i \in I} a_i \geq 4|K_1 \setminus I| + 6|k_1 \cap I| + |k_2 \setminus I| + 2|K_2 \cap I| + |K_4 \setminus I| + 4|K_5 \setminus I| + 2|K_5 \cap I| \geq |K_1 \setminus I| + |K_1 \cap I| + |K_2 \setminus I| + |K_2 \cap I| = |K_1| + |K_2| = z(\alpha, +)$. If $r(\alpha, I, +) = z(\alpha, +)$ then $K_1 = \emptyset$, $K_\cap I = \emptyset$, $K_4 \subseteq I$, $K_5 = \emptyset$, and therefore $a_i \in \{0, 1, -1\}$ for every i. Moreover, $a_i \neq 1$ for every $i \in I$ and $a_i \neq -1$ for every $i \in I^\perp$. These arguments are reversible. □

Lemma 2.2. (i) $r(\alpha, I, -) \geq z(\alpha, -)$.

(ii) $r(\alpha, I, -) = z(\alpha, -)$ *if and only if* $a_i \in \{0, 1\}$ *for* $i \in I$ *and* $a_i \in \{0, -1\}$ *for* $i \in I^\perp$.

Proof. This follows from 2.1 ($a_i \leftrightarrow -a_i$). □

Lemma 2.3. *Let* $s(\alpha) \geq 0$. *Then:*

(i) $r(\alpha, I, +) \geq z(\alpha, -) + s(\alpha) \geq z(\alpha, -)$.

(ii) $r(\alpha, I, +) = z(\alpha, -) + s(\alpha)$ *if and only if* $a_i \in \{0, -1\}$ *for* $i \in I$ *and* $a_i \in \{0, -1\}$ *for* $i \notin I$.

Proof. We have $r(\alpha, I, +) = r(\alpha) + s(a) - \sum_{i \notin I} a_i = r(\alpha, I^\perp, -) + s(\alpha) \geq z(\alpha, -) + s(\alpha) \geq r(\alpha, -)$ by 2.2(i). The rest is clear. □

Lemma 2.4. *Let* $s(\alpha) \leq 0$. *Then:*

(i) $r(\alpha, I, -) \geq z(\alpha, +) + s(\alpha) \geq z(\alpha, +)$.

(ii) $r(\alpha, I, -) = z(\alpha, +) + s(\alpha)$ *if and only if* $a_i \in \{0, 1\}$ *for* $i \in I$ *and* $a_i \in \{0, -1\}$ *for* $i \notin I$.

Proof. This follows from 2.3 ($a_i \leftrightarrow -a_i$). □

Lemma 2.5. (i) *If* $s(\alpha) \geq 0$ *then* $r(\alpha, I, +) \geq \max(z(\alpha, +), z(\alpha, -))$.

(ii) *If* $s(\alpha) \leq 0$ *then* $r(\alpha, I, -) \geq \max(z(\alpha, +), z(\alpha, -))$.

Proof. Just combine 2.1(i), 2.3(i), 2.2(i) and 2.4(i). □

Lemma 2.6. (i) $s(\alpha, I) + s(\alpha, I^\perp) = s(\alpha) - n$.

(ii) $r(\alpha, I, +) + r(\alpha, I, -) = 2r(\alpha)$.

(iii) $r(\alpha, I, +) + r(\alpha, I^\perp, +) = 2r(\alpha) + s(\alpha)$.

(iv) $r(\alpha, I, -) + r(\alpha, I^\perp, -) = 2r(\alpha) - s(\alpha)$.

(v) $t(\alpha, I) + t(\alpha, I^\perp) = 2r(\alpha) - 3s(\alpha) + n - 2z(\alpha) = 2t(\alpha) + n - s(\alpha)$.

Proof. All is obvious. □

Lemma 2.7. (i) $r(\alpha, I, +) - r(\alpha, I, -) = 2\sum_{i \in I} a_i$.

(ii) $r(\alpha, I, +) - r(\alpha, I^\perp, +) = \sum_{i \in I} a_i - \sum_{i \in I^\perp} a_i$.

(iii) $r(\alpha, I, -) - r(\alpha, I^\perp, -) = \sum i \in I^\perp a_i - \sum_{i \in I} a_i$.

(iv) $r(\alpha, I, +) - r(\alpha, I^\perp, -) s(\alpha)$.

(v) $r(\alpha, I, -) - r(\alpha, I^\perp, +) = -s(\alpha)$.

(vi) $t(\alpha, I) - t(\alpha, I^\perp) = |I| - |i^\perp| + \sum_{i \in I^\perp} a_i - \sum_{i \in I} a_i = 2|I| - n + s(\alpha) - 2\sum_{i \in I} a_i = s(\alpha) - n - 2s(\alpha, I)$.

Proof. All is obvious. □

Lemma 2.8. (i) $s(\alpha, \emptyset) = 0$.
(ii) $s(\alpha, \{1, \dots, n\}) = s(\alpha) - n$.
(iii) $r(\alpha, \emptyset, +) = r(\alpha)$.
(iv) $r(\alpha, \{1, \dots, n\}, +) = r(a) + s(\alpha)$.
(v) $r(\alpha, \emptyset, -) = r(\alpha)$.
(vi) $r(\alpha, \{1, \dots, n\}, -) = r(\alpha) - s(\alpha)$
(vii) $t(\alpha, \emptyset) = t(\alpha)$.
(viii) $t(\alpha, \{1, \dots, n\}) = t(\alpha) + n - s(\alpha)$.

Proof. It is obvious. □

3. Technical results

Lemma 3.1. *Put* $\beta = \alpha - 1 = (a_1 - 1, \dots, a_n - 1)$. *Then:*
(i) $s(\beta, I) = s(\alpha, I) - |I|$.
(ii) $r(\beta, I, +) = r(\alpha, I, +) - 2s(\alpha) + n - |I|$.
(iii) $r(\beta, I, +) = t(\alpha, I^{\perp}) + z(\alpha)$.
(iv) $r(\beta, I, -) = r(\alpha, I, -) - 2s(\alpha) + n + |I|$.
(v) $t(\beta, I) = t(\alpha, I) - 2s(\alpha) + 2n + |I| + z(\alpha) - z(\alpha, 1)$.
(vi) $t(\alpha, I) = r(\beta) + s(\beta) - \sum_{i \in I} b_i - z(\beta, -1)$.

Proof. (i), (ii), (iii) and (iv) are obvious.

(v) We have $t(\beta, I) = \sum_{i=1}^{n} (a_i - 1)^2 - \sum_{i=1}^{n} (a_i - 1) + |I| - \sum_{i \in I} (a_i - 1) - z(\beta) = \sum_{i=1}^{n} a_i^2 - 2\sum_{i=1}^{n} a_i + n - \sum_{i=1}^{n} a_i + n + |I| - \sum_{i \in I} a_i + |I| - z(\alpha, 1) = t(\alpha, I) - 2s(\alpha) + 2n + |I| + z(\alpha) - z(\alpha, 1)$.

(vi) We have $r(\beta) + s(\beta) - \sum_{i \in I} b_i - z(\beta, -1) = \sum_{i=1}^{n} a_i^2 - 2\sum_{i=1}^{n} a_i + n + \sum_{i=1}^{n} a_n - n + |I| - \sum_{i \in I} a_i - z(\alpha) = \sum_{i=1}^{n} a_i^2 - \sum_{i=1}^{n} a_i + |I| - \sum_{i \in I} a_i - z(\alpha) = t(\alpha, I)$. □

Put $\tau(\alpha) = 2r(\alpha) + 2\sum_{i \in J} a_i - 3z(\alpha, -)$, where $J = \{\, i \mid 1 \le i \le n, a_i < 0 \,\}$.

Lemma 3.2. *Let* $\max(\alpha) = 1$ *and* $\min(\alpha) = -1$. *Then:*
(i) $s(\alpha) = z(\alpha, +) - z(\alpha, -)$.
(ii) $s(\alpha) \ge 0$ *if and only if* $z(\alpha, +) \ge z(\alpha, -)$.
(iii) $r(\alpha) = z(\alpha, +) + z(\alpha, -)$.
(iv) $\tau(\alpha) = 2z(\alpha, +) - 3z(\alpha, -)$.

Proof. All is obvious. □

Lemma 3.3. *Let* $\max(\alpha) = 1$, $\min(\alpha) \le -2$. *Then:*
(i) $s(\alpha) \le z(\alpha, +) - z(\alpha, -) + 1 + \min(\alpha)$.
(ii) *If* $s(\alpha) \ge 0$ *then* $z(\alpha, +) \ge z(\alpha, -) + 1$.

Proof. It is obvious. □

Lemma 3.4. *Let* $\max(\alpha) = 1$, $\min(\alpha) \le -2$, $s(\alpha) \ge 0)$ *and* $z(\alpha,+) = z(\alpha,-)+1$. *Then:*
(i) $\min(\alpha) = -2$ *and* $z(\alpha, \min(\alpha)) = 1$.
(ii) $s(\alpha) = 0$.
(iii) $r(\alpha) = 2z(\alpha,-) + 4$.
(iv) $\tau(\alpha) = 6 - z(a,-)$.
(v) $\tau(\alpha) = 0$ *if and only if* $z(\alpha,+) = 7$ *and* $z(\alpha,-) = 6$.

Proof. It follows from 3.3 that $0 \le s(\alpha) \le 2 + \min(\alpha)$, and so $\min(\alpha) = -2$ and $s(\alpha) = 0$. The rest is easy. □

Lemma 3.5. *Let* $\max(\alpha) \ge 2$, $\min(\alpha) = -1$. *Then:*
(i) $s(\alpha) \ge \max(\alpha) + z(\alpha,+) - 1 - z(\alpha,-)$.
(ii) *If* $z(\alpha,-) \le z(\alpha,+) + 1$ *then* $s(\alpha) \ge 0$.
(iii) $r(\alpha) \ge \max(\alpha)^2 + z(a,+) + z(\alpha,-) - 1$.
(iv) $r(\alpha) = \max(\alpha)^2+z(\alpha,+)+z(\alpha,-)-1$ *if and only if* $z(\alpha,+)-z(\alpha,1) = 1$.
(v) $\tau(\alpha) \ge 2\max(\alpha)^2 + 2z(\alpha,+) - 3z(\alpha,-) - 2$.
(vi) $\tau(\alpha) = 2\max(\alpha)^2 + 2z(\alpha,+) - 3z(\alpha,-) - 2$ *if and only if* $z(a,+) - z(\alpha,1) = 1$.
(vii) *If* $2z(\alpha,+) \ge 3z(\alpha,-)$ *then* $\tau(\alpha) \ge 6$.

Proof. It is easy. □

Now, assume that $n \ge 2$ and choose $j,k \in \{1,\dots,n\}$ such that $j \ne k$ and $a_j = \max(\alpha)$, $a_k = \min(\alpha)$. Consider the n-tuple $\beta = (b_1,\dots,b_n)$ such that $b_i = a_i$ for $i \ne j,k$, $b_j = a_j - 1$ and $b_k = a_k + 1$.

Lemma 3.6. *Let* $a_j \ge 1$ *and* $a_k \le -2$. *Then:*
(i) $z(\alpha,-) = z(\beta,-)$.
(ii) *If* $a_j \ge 2$ *then* $z(\alpha,+) = z(\beta,+)$.
(iii) *If* $a_j = 1$ *then* $z(\beta,+) = z(\alpha,+) - 1$.
(iv) *If* $z(\alpha,\max(\alpha)) \ge 2$ *then* $\max(\beta) = \max(\alpha)$.
(v) *If* $z(\alpha,\max(\alpha)) = 1$ *then* $\max(\beta) = \max(\alpha) - 1$.
(vi) *If* $z(\alpha,\min(\alpha)) = 1$ *then* $\min(\beta) = \min(\alpha) + 1$.
(vii) *If* $z(\alpha,\min(\alpha)) \ge 2$ *then* $\min(\beta) = \min(\alpha)$.
(viii) $\tau(\alpha) - \tau(\beta) = 4a_j - 4a_k - 6 \ge 6$.

Proof. We have $r(\alpha) - r(\beta) = 2(a_j - a_k - 1)$ (see [1, 2.8]) and the rest is easy. □

Lemma 3.7. *Let* $a_j \ge 1$ *and* $a_k = -1$. *Then:*
(i) $z(\beta,-) = z(\alpha,-) - 1$.
(ii) *If* $a_j \ge 2$ *then* $z(\beta,+) = z(\alpha,+)$.
(iii) *If* $a_j = 1$ *then* $z(\beta,+) = z(\alpha,+) - 1$.
(iv) *If* $z(\alpha,\max(\alpha)) \ge 2$ *then* $\max(\beta) = \max(\alpha)$.
(v) *If* $z(\alpha,\max(\alpha)) = 1$ *then* $\max(\beta) = \max(\alpha) - 1$.

(vi) *If* $z(\alpha,-) = z(\alpha,\min(\alpha)) = 1$ *then* $\min(\beta) = 0$.
(vii) *If* $z(\alpha,-) = z(\alpha,\min(\alpha)) \geq 2$ *then* $min(\beta) = -1$.
(viii) $\tau(\alpha) = \tau(\beta) = 4a_j - 5 \geq -1$.

Proof. Similar to that of 3.6. □

4. The inequalities

Theorem 4.1. *Let* $n \geq 1$ *and let* $a_1, \dots, a_n$ *be integers such that* $\sum_{i=1}^n |a_i| \geq n$. *Let* $I \subseteq \{1, \dots, n\}$ *and let* z *be the number of indices* $i \in \{1, \dots, n\}$ *such that* $a_i = 0$. *Then:*

$$(1) \qquad \sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i - \sum_{i \in I}(a_i - 1) \geq \sum_{i=1}^n a_i^2 - \sum_{i=1}^n |a_i| - \sum_{i \in I}(|a_i| - 1) \geq z,$$

$$(2) \qquad \sum_{i=1}^n a_i^2 - \sum_{i=1}^n |a_i| - \sum_{i \in I}(|a_i| - 1) = z$$

if and only if $\sum_{i=1}^n |a_i| = n$, $a_i \in \{\pm 1, \pm 2\}$ *for* $i \in I$ *and* $a_i \in \{0, \pm 1\}$ *for* $i \notin I$.

Proof. (1) Since $|a_i| \geq a_i$, we can assume that all the numbers a_i are non-negative. Then $\sum_{i=1}^n a_i \geq n$. Put $b_i = a_i - 1$ for every $i = 1, \dots, n$. Evidently, $\sum_{i=1}^n b_i \geq 0$ and z is now the number of indices i such that $b_i < 0$ ($b_i = -1$ in fact). By [1, 1.4(i)] we have $\sum_{i=1}^n b_i^2 \geq 2z$. Henceforth, $\sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i - \sum_{i \in I}(a_i - 1) - z = \sum_{i=1}^n (a_i - 1)^2 + \sum_{i=1}^n (a_i - 1) - \sum_{i \in I}(a_i - 1) - z = \sum_{i=1}^n b_i^2 + \sum_{i \in I^\perp} b_i - z = (\sum_{i=1}^n b_i^2 - 2z) + (z + \sum_{i \in I^\perp} b_i) \geq z + \sum_{i \in I^\perp} b_i \geq z - z_1 \geq 0$, where $z_1 = |\{\, i \in I^\perp \mid b_i = -1 \,\}|$. Of course, $z \geq z_1 = |\{\, i \in I^\perp \mid a_i = 0 \,\}|$.

(2) If $\sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i - \sum_{i \in I}(a_i - 1) - z = 0$ then $\sum_{i=1}^n b_i^2 = 2z$, $z + \sum_{i \in I^\perp} b_i = 0$ and $z = z_1$. First, it follows from [1, 1.4(ii)] that $\sum_{i=1}^n b_i = 0$ and $b_i \in \{0, 1, -1\}$. It means that $\sum_{i=1}^n a_i = n$ and $a_i \in \{0, 1, 2\}$ for every $i = 1, \dots, n$. Since $z = z_1$, we have $a_i \neq 0$ for every $i \in I$. If $z_2 = |\{\, i \in I^\perp \mid b_i = 1 \,\}| = |\{\, i \in I^\perp \mid a_i = 2 \,\}|$ then $\sum_{i \in I^\perp} b_i = z_2 - z_1 = z_2 - z$. Since $z + \sum_{i \in I^\perp} = 0$, we have $z_2 = 0$. Thus $a_i \neq 2$ for every $i \notin I$. □

Example 4.2. (i) Put $n = 2$, $a_1 = 2$, $a_2 = 0$, $I = \{1\}$. Then $\sum_{i=1}^2 a_i^2 - \sum_{i=1}^2 a_i - (a_1 - 1) = 1 = z$ (cf. [1, 6.1(i)].

(ii) Put $n = 5$, $a_1 = a_2 = a_3 = 2$, $a_4 = a_5 = 0$, $I = \{1, 2, 3\}$. Then $\sum_{i=1}^5 a_i^2 - \sum_{i=1}^5 a_i - \sum_{i=1}^3 (a_i - 1) = 3$ and $z = 2$. We have also $\sum_{i=1}^5 b_i^2 - 2z = 1$ and $z + \sum_{i=1}^5 b_i = 0$.

Remark 4.3. Consider the situation from 4.1 (and the proof). Assume that the numbers a_i are non-negative.

(i) If $\sum_{i \in I^\perp} a_i \geq |I^\perp| = n - |I|$ then $\sum_{i \in I^\perp} b_i \geq 0$ and we conclude that $\sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i - \sum_{i \in I}(a_i - 1) \geq 2z$. For instance, if $|I| = n - 1$ then the latter inequality is true provided that $a_j \neq 0$, where $\{j\} = I^\perp$.

(ii) Assume that $a_i = 0$ for every $i \in I^\perp$. Then $\sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i - \sum_{i\in I}(a_i - 1) = \sum_{i\in I}(a_i-1)^2 = \sum_{i\in I} b_i = \sum_{i=1}^n b_i^2 - n + |I| \geq 2z - n + |I|$. For instance, if $|I| = n-1$ then $\sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i - \sum_{i\in I}(a_i - 1) \geq 2z - 1$.

Assume, moreover, that $z_1 = |\{\, i \in I \,|\, a_i = 0 \,\}| \geq n - |I|$. We have $z = z_1 + n - |I|$, so that $z \geq 2(n - |I|)$, $4z - 2n + 2|I| \geq 3z$ and $2(\sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i - \sum_{i\in I}(a_i - 1)) \geq 3z$.

Proposition 4.4. *Let $n \geq 1$ and let $a_1, \dots, a_n$ be non-negative integers. Denote $z = |\{\, i \,|\, 1 \leq i \leq n, a_i = 0 \,\}|$, $w = |\{\, i \,|\, 1 \leq i \leq n, a_i \geq 2 \,\}|$ and $a = \max(a_1, \dots, a_n)$. Assume that $a \geq 3$ and $2a^2 - 4a + 2w \geq 3z$. Then, for every subset $I \subseteq \{1, \dots, n\}$,*

$$2(\sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i - \sum_{i\in I}(a_i - 1)) \geq 3z.$$

Proof. Put $\alpha = (a_1, \dots, n)$ and $\beta = \alpha - 1 = (a_1 - 1, \dots, a_n - 1)$. Then $2r(\alpha) - 2s(\alpha) - 2\sum_{i\in I}(a_i - 1) - 3z = 2(r(\beta) + \sum_{i\in I^\perp} b_i) - 3z(\beta, -)$ (see the proof of 4.1). Furthermore, $a = \max(\beta) + 1$, $\max(\beta) \geq 2$, $w = z(\beta, +)$ and $2\max(\beta)^2 + 2z(\beta, +) - 2 - 3z(\beta, -) \geq 0$. If $(z(\beta, -) =)\ z \neq 0$ then $\min(\beta) = -1$ and $2(r(\beta) + \sum_{i\in J} b_i) - 3z(\beta, -) \geq 0$ by 3.5(v), where $J = \{\, i \,|\, b_i < 0 \,\} = \{\, i \,|\, b_i = -1 \,\} = \{\, i \,|\, a_i = 0 \,\}$. Of course, $\sum_{i\in I^\perp} b_i \geq \sum_{i\in J} b_i = -|J| = -z$, and hence $2(r(\beta) + \sum_{i\in I^\perp} b_i) - 3z(\beta, -) \geq 0$. On the other hand, if $z = 0$ then $z(\beta, -) = 0$ and $2(r(\beta) + \sum_{i\in I^\perp} b_i) - 3z(\beta, -) \geq 0$ trivially. $\square$

Remark 4.5. Let $n \geq 1$ and let $a_1, \dots, a_n$ be non-negative integers such that $\max(a_1, \dots, a_n) = 2$ and $\alpha = (a_1, \dots, a_n)$. Put $J = \{\, i \,|\, a_i = 2 \,\}$. Using 3.2(v) and proceeding similarly as in the proof of 4.4, we show that $2(\sum_{i=1}^n a_i^2 - \sum_{i=1}^n a_i - \sum_{i\in I}(a_i - 1)) \geq 3z$ if and only if $2|J| = 2z(\alpha, 2) \geq 3z$. Of course, we have $s(\alpha) = 2z(\alpha, 2) + z(\alpha, 1) = 2z(\alpha, 2) + n - z(\alpha, 2) - z = z(\alpha, 2) + n - z$. Henceforth, if $s(\alpha) \leq n$ then $1 \leq z(\alpha, 2) \leq z$.

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