A new single step hybrid block algorithm for solving fourth order ordinary differential equations directly

Raft Abdelrahim*
Department of Mathematics
College of Art and Sciences-Tabarjal
Al Jouf University
Saudi Arabia
Rafatshaab@yahoo.com

Walid Hdidi
Department of Mathematics
College of Art and Sciences-Tabarjal
Al Jouf University
Saudi Arabia
walidhadidi@gmail.com

Abstract. In this paper, a one-step hybrid block method with generalized three off-step points for solving general fourth order ordinary differential equations is developed using power series of order eight as a basis function. The technique employed for the derivation of this method are to interpolate the power series at x_n and all off-step points and to collocate the fourth derivative of the basis function at all points in the selected interval. The method derived is proven to be zero stable, consistent and then convergent. The performance of the method is tested by solving linear and non-linear fourth order initial value problems.

Keywords: one step, hybrid method, block method, fourth order differential equation, power series, three generalized off step points.

1. Introduction

In this article, we consider the numerical solution of the fourth order IVPs of the form

$$y'''' = f(x, y, y', y'', y'''),\ x \in [a, b] \tag{1}$$

with initial conditions

$$y(a) = \omega_0,\ y'(a) = \omega_1, y''(a) = \omega_2, y'''(a) = \omega_3.$$

Generally, equation (1) can be solved by converting it into system of four equations of first-order IVPs and then appropriate numerical method is applied. Another alternative approach, for solving equation (1) directly which can avoid computational burden has been discussed by Awoyemi, (1992). This approach

∗. Corresponding author

has been widely used for solving high order IVPs by many researchers. Some of these researchers are Omar, et al. (2015), Jator (2010), Fasasi et al. (2014) and Raft Abdelrahim (2021). Recently, hybrid block method for solving equation (1) directly have been proposed by Kayode et al. (2014), Adesanya (2012), Omar et al. (2004), Kayode (2008a) and Kuboye et al. (2015a). Avoiding the disadvantages in reduction method and employing the features of both block and hybrid methods which include generating numerical solutions simultaneously (Lambert, 1973) and overcoming zero stability barrier of linear multistep method are the aims of this new paper.

2. Methodology

In order to derive the new hybrid block method, the power series in equation (2) below is used.

$$y(x) = \sum_{i=0}^{d+c-1} a_i \left(\frac{x-x_n}{h}\right)^i,\ n=0,1,2,...,N-1,\ x\in[x_n,x_{n+1}], \tag{2}$$

where $c=5$ and $d=4$ represent the number of collocation and interpolation points, $h=x_n-x_{n-1}$ and $a=x_0<x_1<...<x_{N-1}<x_N=b$. Interpolating (2) at x_n, x_{n+s_1}, x_{n+s_2}, x_{n+s_3} and collocating the forth derivative of (2) at all points in the interval i.e at x_n, x_{n+s_1}, x_{n+s_2}, x_{n+s_3} and x_{n+1}. This leads to system equations as shown below:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & s_1 & s_1^2 & s_1^3 & s_1^4 & s_1^5 & s_1^6 & s_1^7 & s_1^8 \\ 1 & s_2 & s_2^2 & s_2^3 & s_2^4 & s_2^5 & s_2^6 & s_2^7 & s_2^8 \\ 1 & s_3 & s_3^2 & s_3^3 & s_3^4 & s_3^5 & s_3^6 & s_2^7 & s_3^8 \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{120s_1}{h^4} & \frac{360s_1^2}{h^4} & \frac{840s_1^3}{h^4} & \frac{1680s_1^4}{h^4} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{120s_2}{h^4} & \frac{360s_2^2}{h^4} & \frac{840s_2^3}{h^4} & \frac{1680s_2^4}{h^4} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{120s_3}{h^4} & \frac{360s_3^2}{h^4} & \frac{840s_3^3}{h^4} & \frac{1680s_3^4}{h^4} \\ 0 & 0 & 0 & 0 & \frac{24}{h^4} & \frac{120}{h^4} & \frac{360}{h^4} & \frac{840}{h^4} & \frac{1680}{h^4} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{pmatrix} = \begin{pmatrix} y_n \\ y_{n+s_1} \\ y_{n+s_2} \\ y_{n+s_3} \\ f_n \\ f_{n+s_1} \\ f_{n+s_2} \\ f_{n+s_3} \\ f_{n+1} \end{pmatrix}. \tag{3}$$

System above is solved by Gaussian Elimination Method to find $a_i's, i=0(1)8$. Secondly, substituting the values of $a_i's$ into equation (2) yields a continuous implicit scheme of the form:

$$y(x) = \alpha_0(x) + \sum_{i=1}^{3}(\alpha_{s_i}(x)y_{n+S_i} + \beta_{s_i}(x)f_{n+s_i}) + \sum_{i=0}^{1}\beta_i(x)f_{n+i}. \tag{4}$$

The first, second and third derivatives of equation (4) are

$$y'(x) = \frac{\partial}{\partial x}\alpha_0(x) + \sum_{i=1}^{3}\frac{\partial}{\partial x}(\alpha_{s_i}(x)y_{n+s_i} + \beta_{s_i}(x)f_{n+s_i}) + \sum_{i=0}^{1}\frac{\partial}{\partial x}\beta_i(x)f_{n+i},$$
$$y''(x) = \frac{\partial^2}{\partial x^2}\alpha_0(x) + \sum_{i=1}^{3}\frac{\partial^2}{\partial x^2}(\alpha_{s_i}(x)y_{n+s_i} + \beta_{s_i}(x)f_{n+s_i}) + \sum_{i=0}^{1}\frac{\partial^2}{\partial x^2}\beta_i(x)f_{n+i},$$
$$y'''(x) = \frac{\partial^3}{\partial x^3}\alpha_0(x) + \sum_{i=1}^{3}\frac{\partial^3}{\partial x^3}(\alpha_{s_i}(x)y_{n+s_i} + \beta_{s_i}(x)f_{n+s_i}) + \sum_{i=0}^{1}\frac{\partial^3}{\partial x^3}\beta_i(x)f_{n+i},$$

where,

$$\alpha_0 = \frac{(x_n - x + hs_3)(x_n - x + hs_1)(x_n - x + hs_2)}{(h^3 s_1 s_2 s_3)},$$
$$\alpha_{s_1} = \frac{(x - x_n)(x_n - x + hs_3)(x_n - x + hs_2)}{(h^3 s_1(s_1 - s_3)(s_1 - s_2)},$$
$$\alpha_{s_2} = \frac{(x - x_n)(x - x_n - hs_3)(x_n - x + hs_1)}{(h^3 s_2(s_2 - s_3)(s_1 - s_2))},$$
$$\alpha_{s_3} = \frac{(x - x_n)(x_n - x + hs_2)(x_n - x + hs_1)}{(h^3 s_3(s_2 - s_3)(s_1 - s_3)},$$
$$\begin{aligned}\beta_0 = &-\frac{(x - x_n)(x_n - x + hs_1)(x_n - x + hs_2)(x_n - x + hs_3)}{(5040h^4 s_1 s_2 s_3)}(3h^4 s_1^4 - 5h^4 s_1^3 s_2,\\
&- 5h^4 s_1^3 s_3 - 8h^4 s_1^3 - 5h^4 s_1^2 s_2^2 + 15h^4 s_1^2 s_2 s_3 + 20h^4 s_1^2 s_2 - 5h^4 s_1^2 s_3^2 + 20h^4 s_1^2 s_3\\
&- 5h^4 s_1 s_2^3 + 15h^4 s_1 s_2^2 s_3 + 20h^4 s_1 s_2^2 + 15h^4 s_1 s_2 s_3^2 - 120h^4 s_1 s_2 s_3 - 5h^4 s_1 s_3^3\\
&+ 20h^4 s_1 s_3^2 + 3h^4 s_2^4 - 5h^4 s_2^3 s_3 - 8h^4 s_2^3 - 5h^4 s_2^2 s_3^2 + 20h^4 s_2^2 s_3 - 5h^4 s_2 s_3^3\\
&+ 20h^4 s_2 s_3^2 - 8h^4 s_3^3 + 3h^3 s_1^3 x - 3h^3 s_1^3 x_n - 5h^3 s_1^2 s_2 x + 5h^3 s_1^2 s_2 x_n + 12x^3 x_n\\
&- 5h^3 s_1^2 s_3 x + 5h^3 s_1^2 s_3 x_n - 8h^3 s_1^2 x + 8h^3 s_1^2 x_n - 5h^3 s_1 s_2^2 x + 5h^3 s_1 s_2^2 x_n + 3h^4 s_3^4\\
&- 15h^3 s_1 s_2 s_3 x_n + 20h^3 s_1 s_2 x - 20h^3 s_1 s_2 x_n - 5h^3 s_1 s_3^2 x + 5h^3 s_1 s_3^2 x_n + 3hs_3 x^3\\
&- 20h^3 s_1 s_3 x_n + 3h^3 s_2^3 x - 3h^3 s_2^3 x_n - 5h^3 s_2^2 s_3 x + 5h^3 s_2^2 s_3 x_n - 8h^3 s_2^2 x - 3x_n^4\\
&- 5h^3 s_2 s_3^2 x + 5h^3 s_2 s_3^2 x_n + 20h^3 s_2 s_3 x - 20h^3 s_2 s_3 x_n + 3h^3 s_3^3 x - 3h^3 s_3^3 x_n - 3x^4\\
&+ 8h^3 s_3^2 x_n + 3h^2 s_1^2 x^2 - 6h^2 s_1^2 x x_n + 3h^2 s_1^2 x_n^2 - 5h^2 s_1 s_2 x^2 + 10h^2 s_1 s_2 x x_n\\
&- 5h^2 s_1 s_3 x^2 + 10h^2 s_1 s_3 x x_n - 5h^2 s_1 s_3 x_n^2 - 8h^2 s_1 x^2 + 16h^2 s_1 x x_n - 8h^2 s_1 x_n^2\\
&- 6h^2 s_2^2 x x_n + 3h^2 s_2^2 x_n^2 - 5h^2 s_2 s_3 x^2 + 10h^2 s_2 s_3 x x_n - 5h^2 s_2 s_3 x_n^2 - 8h^2 s_2 x^2\\
&- 8h^2 s_2 x_n^2 + 3h^2 s_3^2 x^2 - 6h^2 s_3^2 x x_n + 3h^2 s_3^2 x_n^2 - 8h^2 s_3 x^2 + 16h^2 s_3 x x_n - 8h^2 s_3 x_n^2\\
&+ 3hs_1 x^3 - 9hs_1 x^2 x_n + 9hs_1 x x_n^2 - 3hs_1 x_n^3 + 3hs_2 x^3 - 9hs_2 x^2 x_n + 9hs_2 x x_n^2\\
&+ 3h^2 s_2^2 x^2 - 9hs_3 x^2 x_n + 9hs_3 x x_n^2 - 3hs_3 x_n^3 + 6hx^3 - 18hx^2 x_n + 18hx x_n^2\\
&+ 20h^3 s_1 s_3 x - 3hs_2 x_n^3 - 5h^2 s_1 s_2 x_n^2 + 16h^2 s_2 x x_n + 15h^3 s_1 s_2 s_3 x - 8h^3 s_3^2 x\\
&+ 8h^3 s_2^2 x_n - 18x^2 x_n^2 + 12x x_n^3 - 6hx_n^3),\end{aligned}$$

$$\beta_{s_1} = -\frac{((x-x_n)(x_n-x+hs_1)(x_n-x+hs_2)(x_n-x+hs_3))}{(5040h^4s_1(s_1-s_2)(s_1-s_3)(s_1-1))}(3h^4s_1^4+3x^4$$
$$-3h^4s_1^3s_3-6h^4s_1^3-3h^4s_1^2s_2^2+5h^4s_1^2s_2s_3+8h^4s_1^2s_2-3h^4s_1^2s_3^2+8h^4s_1^2s_3$$
$$+5h^4s_1s_2^2s_3+8h^4s_1s_2^2+5h^4s_1s_2s_3^2-20h^4s_1s_2s_3-3h^4s_1s_3^3+8h^4s_1s_3^2-3h^4s_2^4$$
$$+5h^4s_2^3s_3+8h^4s_2^3+5h^4s_2^2s_3^2-20h^4s_2^2s_3+5h^4s_2s_3^3-20h^4s_2s_3^2-3h^4s_3^4$$
$$+3h^3s_1^3x-3h^3s_1^3x_n-3h^3s_1^2s_2x+3h^3s_1^2s_2x_n-3h^3s_1^2s_3x+3h^3s_1^2s_3x_n$$
$$+6h^3s_1^2x_n-3h^3s_1s_2^2x+3h^3s_1s_2^2x_n+5h^3s_1s_2s_3x-5h^3s_1s_2s_3x_n+8h^3s_1s_2x$$
$$-3h^3s_1s_3^2x+3h^3s_1s_3^2x_n+8h^3s_1s_3x-8h^3s_1s_3x_n-3h^3s_2^3x+3h^3s_2^3x_n+6hx_n^3$$
$$+5h^3s_2s_3^2x-5h^3s_2s_3^2x_n-20h^3s_2s_3x+20h^3s_2s_3x_n-3h^3s_3^3x+3h^3s_3^3x_n$$
$$-8h^3s_3^2x_n+3h^2s_1^2x^2-6h^2s_1^2xx_n+3h^2s_1^2x_n^2-3h^2s_1s_3x^2+8h^2s_2x_n^2$$
$$+6h^2s_1s_3xx_n-3h^2s_1s_3x_n^2-6h^2s_1x^2+12h^2s_1xx_n-6h^2s_1x_n^2-3h^2s_2^2x^2$$
$$-3h^2s_2^2x_n^2+5h^2s_2s_3x^2-10h^2s_2s_3xx_n+5h^2s_2s_3x_n^2+8h^2s_2x^2-16h^2s_2xx_n$$
$$+6h^2s_3^2xx_n-3h^2s_3^2x_n^2+8h^2s_3x^2-16h^2s_3xx_n+8h^2s_3x_n^2+3hs_1x^3-9hs_1x^2x_n$$
$$-3hs_2x^3+9hs_2x^2x_n-9hs_2xx_n^2+3hs_2x_n^3-3hs_3x^3+9hs_3x^2x_n$$
$$+9hs_1xx_n^2-6hx^3-3h^2s_3^2x^2-3h^2s_1s_2x^2+6h^2s_1s_2xx_n-3h^2s_1s_2x_n^2-3hs_1x_n^3$$
$$-5h^3s_2^2s_3x_n+8h^3s_2^2x-8h^3s_2^2x_n+18hx^2x_n-18hxx_n^2-12x^3x_n+18x^2x_n^2$$
$$-3h^4s_1^3s_2-3h^4s_1s_2^3+5h^3s_2^2s_3x+8h^3s_3^2x+6h^2s_2^2xx_n+3hs_3x_n^3+8h^4s_3^3$$
$$-6h^3s_1^2x-12xx_n^3-8h^3s_1s_2x_n-9hs_3xx_n^2+3x_n^4),$$

$$\beta_{s_2} = \frac{(x-x_n)(x_n-x+hs_1)(x_n-x+hs_2)(x_n-x+hs_3)}{(5040h^4s_2(s_1-s_2)(s_2-s_3)(s_2-1))}(3h^2s_1s_2x_n^2+20h^4s_1^2s_3$$
$$-5h^4s_1^3s_3-8h^4s_1^3+3h^4s_1^2s_2^2-5h^4s_1^2s_2s_3-8h^4s_1^2s_2-5h^4s_1^2s_3^2+3h^4s_3^4+3h^4s_1^3s_2$$
$$+3h^4s_1s_2^3-5h^4s_1s_2^2s_3-8h^4s_1s_2^2-5h^4s_1s_2s_3^2+20h^4s_1s_2s_3-5h^4s_1s_3^3-18hx^2x_n$$
$$+20h^4s_1s_3^2-3h^4s_2^4+3h^4s_2^3s_3+6h^4s_2^3+3h^4s_2^2s_3^2-8h^4s_2^2s_3+3h^4s_2s_3^3+18hxx_n^2$$
$$-8h^4s_2s_3^2-8h^4s_3^3+3h^3s_1^3x-3h^3s_1^3x_n+3h^3s_1^2s_2x-3h^3s_1^2s_2x_n-3x^4+12x^3x_n$$
$$+5h^3s_1^2s_3x_n-8h^3s_1^2x+8h^3s_1^2x_n+3h^3s_1s_2^2x-3h^3s_1s_2^2x_n-5h^3s_1s_2s_3x-18x^2x_n^2$$
$$+5h^3s_1s_2s_3x_n-8h^3s_1s_2x+8h^3s_1s_2x_n-5h^3s_1s_3^2x+5h^3s_1s_3^2x_n-6hx_n^3+12xx_n^3$$
$$-20h^3s_1s_3x_n-3h^3s_2^3x+3h^3s_2^3x_n+3h^3s_2^2s_3x-3h^3s_2^2s_3x_n+6h^3s_2^2x-8h^2s_3x^2$$
$$+3h^3s_2s_3^2x-3h^3s_2s_3^2x_n-8h^3s_2s_3x+8h^3s_2s_3x_n+3h^3s_3^3x-3h^3s_3^3x_n-8h^2s_1x_n^2$$
$$+8h^3s_3^2x_n+3h^2s_1^2x^2-6h^2s_1^2xx_n+3h^2s_1^2x_n^2+3h^2s_1s_2x^2-6h^2s_1s_2xx_n+3h^4s_1^4$$
$$-5h^2s_1s_3x^2+10h^2s_1s_3xx_n-5h^2s_1s_3x_n^2-8h^2s_1x^2+16h^2s_1xx_n-3x_n^4-8h^3s_3^2x$$
$$-3h^2s_2^2x^2+6h^2s_2^2xx_n-3h^2s_2^2x_n^2+3h^2s_2s_3x^2-6h^2s_2s_3xx_n+3h^2s_2s_3x_n^2-6h^3s_2^2x_n$$
$$+6h^2s_2x^2-12h^2s_2xx_n+6h^2s_2x_n^2+3h^2s_3^2x^2-6h^2s_3^2xx_n+3h^2s_3^2x_n^2+6hx^3+3hs_3x^3$$
$$-8h^2s_3x_n^2+3hs_1x^3-9hs_1x^2x_n+9hs_1xx_n^2-3hs_1x_n^3-3hs_2x^3-3hs_3x_n^3+3hs_2x_n^3$$
$$+20h^3s_1s_3x-5h^3s_1^2s_3x+9hs_2x^2x_n-9hs_2xx_n^2-9hs_3x^2x_n+9hs_3xx_n^2+16h^2s_3xx_n),$$

$$\beta_{s_3}=\frac{-(x-x_n)(x_n-x+hs_1)(x_n-x+hs_2)(x_n-x+hs_3))}{(5040h^4s_3(s_1-s_3)(s_2-s_3)(s_3-1))}(3h^4s_1^4-5h^4s_1^3s_2$$
$$+3h^4s_1^3s_3-8h^4s_1^3-5h^4s_1^2s_2^2-5h^4s_1^2s_2s_3+20h^4s_1^2s_2+3h^4s_1^2s_3^2-8h^4s_1^2s_3$$
$$-5h^4s_1s_2^3-5h^4s_1s_2^2s_3+20h^4s_1s_2^2-5h^4s_1s_2s_3^2+20h^4s_1s_2s_3+3h^4s_1s_3^3-6hx_n^3$$
$$-8h^4s_1s_3^2+3h^4s_2^4+3h^4s_2^3s_3-8h^4s_2^3+3h^4s_2^2s_3^2-8h^4s_2^2s_3+3h^4s_2s_3^3-8h^4s_2s_3^2$$
$$-3h^4s_3^4+6h^4s_3^3+3h^3s_1^3x-3h^3s_1^3x_n-5h^3s_1^2s_2x+5h^3s_1^2s_2x_n+3h^3s_1^2s_3x-3x^4$$
$$-3h^3s_1^2s_3x_n-8h^3s_1^2x+8h^3s_1^2x_n-5h^3s_1s_2^2x+5h^3s_1s_2^2x_n-5h^3s_1s_2s_3x-3x_n^4$$
$$+5h^3s_1s_2s_3x_n+20h^3s_1s_2x-20h^3s_1s_2x_n+3h^3s_1s_3^2x-3h^3s_1s_3^2x_n-8h^3s_1s_3x$$
$$+8h^3s_1s_3x_n+3h^3s_2^3x-3h^3s_2^3x_n+3h^3s_2^2s_3x-3h^3s_2^2s_3x_n-8h^3s_2^2x+8h^3s_2^2x_n$$
$$+3h^3s_2s_3^2x-3h^3s_2s_3^2x_n-8h^3s_2s_3x+8h^3s_2s_3x_n-3h^3s_3^3x+3h^3s_3^3x_n+6h^3s_3^2x$$
$$-6h^3s_3^2x_n+3h^2s_1^2x^2-6h^2s_1^2xx_n+3h^2s_1^2x_n^2-5h^2s_1s_2x^2+10h^2s_1s_2xx_n+6hx^3$$
$$-5h^2s_1s_2x_n^2+3h^2s_1s_3x^2-6h^2s_1s_3xx_n+3h^2s_1s_3x_n^2-8h^2s_1x^2+16h^2s_1xx_n$$
$$-8h^2s_1x_n^2+3h^2s_2^2x^2-6h^2s_2^2xx_n+3h^2s_2^2x_n^2+3h^2s_2s_3x^2-6h^2s_2s_3xx_n+12xx_n^3$$
$$+3h^2s_2s_3x_n^2-8h^2s_2x^2+16h^2s_2xx_n-8h^2s_2x_n^2-3h^2s_3^2x^2+6h^2s_3^2xx_n-3h^2s_3^2x_n^2$$
$$+6h^2s_3x^2-12h^2s_3xx_n+6h^2s_3x_n^2+3hs_1x^3-9hs_1x^2x_n+9hs_1xx_n^2-3hs_1x_n^3$$
$$+3hs_2x^3-9hs_2x^2x_n+9hs_2xx_n^2-3hs_2x_n^3-3hs_3x^3+9hs_3x^2x_n-9hs_3xx_n^2$$
$$+3hs_3x_n^3-18hx^2x_n+18hxx_n^2+12x^3x_n-18x^2x_n^2),$$

$$\beta_1=-\frac{(x-x_n)(x_n-x+hs_3)(x_n-x+hs_1)(x_n-x+hs_2)}{(5040h^4(s_3-1)(s_2-1)(s_1-1))}(3h^4s_1^4-5h^4s_1^3s_2$$
$$-5h^4s_1^3s_3-5h^4s_1^2s_2^2+15h^4s_1^2s_2s_3-5h^4s_1^2s_3^2-5h^4s_1s_2^3+15h^4s_1s_2^2s_3-3hs_3x_n^3$$
$$-5h^4s_1s_3^3+3h^4s_2^4-5h^4s_2^3s_3-5h^4s_2^2s_3^2-5h^4s_2s_3^3+3h^4s_3^4+3h^3s_1^3x-3h^3s_1^3x_n$$
$$-5h^3s_1^2s_2x+5h^3s_1^2s_2x_n-5h^3s_1^2s_3x+5h^3s_1^2s_3x_n-5h^3s_1s_2^2x+5h^3s_1s_2^2x_n-3x_n^4$$
$$+15h^3s_1s_2s_3x-15h^3s_1s_2s_3x_n-5h^3s_1s_3^2x+5h^3s_1s_3^2x_n+3h^3s_2^3x-3h^3s_2^3x_n$$
$$+5h^3s_2^2s_3x_n-5h^3s_2s_3^2x+5h^3s_2s_3^2x_n+3h^3s_3^3x-3h^3s_3^3x_n+3h^2s_1^2x^2-6h^2s_1^2xx_n$$
$$+3h^2s_1^2x_n^2-5h^2s_1s_2x^2+10h^2s_1s_2xx_n-5h^2s_1s_2x_n^2-5h^2s_1s_3x^2+10h^2s_1s_3xx_n$$
$$+3h^2s_2^2x^2-6h^2s_2^2xx_n+3h^2s_2^2x_n^2-5h^2s_2s_3x^2+10h^2s_2s_3xx_n-5h^2s_2s_3x_n^2$$
$$-6h^2s_3^2xx_n+3h^2s_3^2x_n^2+3hs_1x^3-9hs_1x^2x_n+9hs_1xx_n^2-3hs_1x_n^3+3hs_2x^3$$
$$-9hs_2x^2x_n+3h^2s_3^2x^2-5h^2s_1s_3x_n^2-5h^3s_2^2s_3x+9hs_2xx_n^2-3hs_2x_n^3+3hs_3x^3$$
$$+15h^4s_1s_2s_3^2-9hs_3x^2x_n+9hs_3xx_n^2-3x^4+12x^3x_n-18x^2x_n^2+12xx_n^3).$$

Equation (4) is evaluated at the non-interpolating point x_{n+1} while its derivatives are evaluated at all points in the selected interval to produce the discreet schemes. Discreet schemes and its derivatives at x_n are combined on a block of the form

$$A^{[3]_4}Y_m^{[3]_4} = \sum_{i=1}^{4} B_i^{[3]_4}R_i^{[3]_4}+h^4[D^{[3]_4}R_5^{[3]_4}+E^{[3]_4}R_6^{[3]_4}], \tag{5}$$

where

$$A^{[3]_4} = \begin{pmatrix} \frac{-(s_2-1)(s_3-1)}{s_1(s_1-s_2)(s_1-s_3)} & \frac{(s_1-1)(s_3-1)}{s_2(s_1-s_2)(s_2-s_3)} & \frac{-(s_1-1)(s_2-1)}{s_3(s_1-s_3)(s_2-s_3)} & 1 \\ \frac{-(s_2s_3)}{hs_1(s_1-s_2)(s_1-s_3)} & \frac{(s_1s_3)}{(hs_2(s_1-s_2)(s_2-s_3))} & \frac{-(s_1s_2)}{hs_3(s_1-s_3)(s_2-s_3)} & 0 \\ \frac{(2s_2+2s_3)}{h^2s_1(s_1-s_2)(s_1-s_3)} & \frac{-(2s_1+2s_3)}{h^2s_2(s_1-s_2)(s_2-s_3)} & \frac{(2s_1+2s_2)}{h^2s_3(s_1-s_3)(s_2-s_3)} & 0 \\ \frac{-6}{h^3s_1(s_1-s_2)(s_1-s_3)} & \frac{6}{h^3s_2(s_1-s_2)(s_2-s_3)} & \frac{-6}{h^3s_3(s_1-s_3)(s_2-s_3)} & 0 \end{pmatrix},$$

$$Y_m^{[3]_4} = \begin{pmatrix} y_{n+s_1} \\ y_{n+s_2} \\ y_{n+s_3} \\ y_{n+1} \end{pmatrix}, B_1^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & \frac{((s_1-1)(s_2-1)(s_3-1))}{(s_1s_2s_3)} \\ 0 & 0 & 0 & \frac{-(s_1s_2+s_1s_3+s_2s_3)}{(hs_1s_2s_3)} \\ 0 & 0 & 0 & \frac{(2(s_1+s_2+s_3))}{(h^2s_1s_2s_3)} \\ 0 & 0 & 0 & \frac{-6}{(h^3s_1s_2s_3)} \end{pmatrix}, R_1^{[3]_4} = \begin{pmatrix} y_{n-3} \\ y_{n-2} \\ y_{n-1} \\ y_n \end{pmatrix},$$

$$B_2^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, R_2^{[3]_4} = \begin{pmatrix} y'_{n-3} \\ y'_{n-2} \\ y'_{n-1} \\ y'_n \end{pmatrix}, B_3^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$R_2^{[3]_4} = \begin{pmatrix} y''_{n-3} \\ y''_{n-2} \\ y''_{n-1} \\ y''_n \end{pmatrix}, B_4^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, R_3^{[3]_4} = \begin{pmatrix} y'''_{n-3} \\ y'''_{n-2} \\ y'''_{n-1} \\ y'''_n \end{pmatrix} \; R_5^{[3]_4} = \begin{pmatrix} f_{n-3} \\ f_{n-2} \\ f_{n-1} \\ f_n \end{pmatrix},$$

$$R^{[3]_4} = \begin{pmatrix} f_{n+s_1} \\ f_{n+s_2} \\ f_{n+s_3} \\ f_{n+1} \end{pmatrix}, D^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & D_{14}^{[3]_4} \\ 0 & 0 & 0 & D_{24}^{[3]_4} \\ 0 & 0 & 0 & D_{34}^{[3]_4} \\ 0 & 0 & 0 & D_{44}^{[3]_4} \end{pmatrix},$$

$$E^{[3]_4} = \begin{pmatrix} E_{11}^{[3]_4} & E_{12}^{[3]_4} & E_{13}^{[3]_4} & E_{14}^{[3]_4} \\ E_{21}^{[3]_4} & E_{22}^{[3]_4} & E_{23}^{[3]_4} & E_{24}^{[3]_4} \\ E_{31}^{[3]_4} & E_{32}^{[3]_4} & E_{33}^{[3]_4} & E_{34}^{[3]_4} \\ E_{41}^{[3]_4} & E_{42}^{[3]_4} & E_{43}^{[3]_4} & E_{44}^{[3]_4} \end{pmatrix}.$$

The elements of $D^{[3]_4}$ and $E^{[3]_4}$ are given in Appendix A.

Multiplying equation (5) by inverse of $A^{[3]_4}$ to have a hybrid block method of the form

$$I^{[3]_4} Y_m^{[3]_4} = \sum_{i=1}^{4} \bar{B}_i^{[3]_4} R_i^{[3]_4} + h^4[\bar{D}^{[3]_4} R_5^{[3]_2} + \bar{E}^{[3]_2} R_6^{[3]_4}], \tag{6}$$

where

$$I^{[3]_4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; \bar{B}_1^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; \bar{B}_2^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & s_1h \\ 0 & 0 & 0 & s_2h \\ 0 & 0 & 0 & s_3h \\ 0 & 0 & 0 & h \end{pmatrix},$$

$$\bar{B}_3^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & \frac{s_1^2}{2}h \\ 0 & 0 & 0 & \frac{s_2^2}{2}h \\ 0 & 0 & 0 & \frac{s_3^2}{2}h \\ 0 & 0 & 0 & \frac{1}{2}h \end{pmatrix},\ \bar{B}_4^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & \frac{s_1^3}{6}h \\ 0 & 0 & 0 & \frac{s_2^3}{6}h \\ 0 & 0 & 0 & \frac{s_3^3}{6}h \\ 0 & 0 & 0 & \frac{1}{6}h \end{pmatrix},\ \bar{D}^{[3]_4} = \begin{pmatrix} 0 & 0 & 0 & \bar{D}_{14}^{[3]_4} \\ 0 & 0 & 0 & \bar{D}_{24}^{[3]_4} \\ 0 & 0 & 0 & \bar{D}_{34}^{[3]_4} \\ 0 & 0 & 0 & \bar{D}_{44}^{[3]_4} \end{pmatrix}$$

$$\bar{E}^{[3]_4} = \begin{pmatrix} \bar{E}_{11}^{[3]_4} & \bar{E}_{12}^{[3]_4} & \bar{E}_{13}^{[3]_4} & \bar{E}_{14}^{[3]_4} \\ \bar{E}_{21}^{[3]_4} & \bar{E}_{22}^{[3]_4} & \bar{E}_{23}^{[3]_4} & \bar{E}_{24}^{[3]_4} \\ \bar{E}_{31}^{[3]_4} & \bar{E}_{32}^{[3]_4} & \bar{E}_{33}^{[3]_4} & \bar{E}_{34}^{[3]_4} \\ \bar{E}_{41}^{[3]_4} & \bar{E}_{42}^{[3]_4} & \bar{E}_{43}^{[3]_4} & \bar{E}_{44}^{[3]_4} \end{pmatrix}$$

and the non-zero terms of $\bar{D}^{[3]_4}$ and $\bar{E}^{[3]_4}$ are given by

$$\bar{D}_{14}^{[3]_4} = -\frac{(s_1^4(28s_1s_2 + 28s_1s_3 - 168s_2s_3 - 8s_1^2s_2 - 8s_1^2s_3 - 8s_1^2 + 3s_1^3 + 28s_1s_2s_3))}{(5040s_2s_3)},$$

$$\bar{D}_{14}^{[3]_4} = \frac{(s_2^4(168s_1s_3 - 28s_1s_2 - 28s_2s_3 + 8s_1s_2^2 + 8s_2^2s_3 + 8s_2^2 - 3s_2^3 - 28s_1s_2s_3))}{(5040s_1s_3)},$$

$$\bar{D}_{14}^{[3]_4} = -\frac{(s_3^4(28s_1s_3 - 168s_1s_2 + 28s_2s_3 - 8s_1s_3^2 - 8s_2s_3^2 - 8s_3^2 + 3s_3^3 + 28s_1s_2s_3))}{(5040s_1s_2)},$$

$$\bar{D}_{14}^{[3]_4} = \frac{((8s_1 + 8s_2 + 8s_3 - 28s_1s_2 - 28s_1s_3 - 28s_2s_3 + 168s_1s_2s_3 - 3))}{(5040s_1s_2s_3)},$$

$$\bar{E}_{11}^{[3]_4} = \frac{(s_1^4(14s_1s_2 + 14s_1s_3 - 42s_2s_3 - 6s_1^2s_2 - 6s_1^2s_3 - 6s_1^2 + 3s_1^3 + 14s_1s_2s_3))}{(5040(s_1-1)(s_1-s_3)(s_1-s_2))},$$

$$\bar{E}_{12}^{[3]_4} = \frac{(s_1^6(28s_3 - 8s_1 - 8s_1s_3 + 3s_1^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))},$$

$$\bar{E}_{13}^{[3]_4} = -\frac{(s_1^6(28s_2 - 8s_1 - 8s_1s_2 + 3s_1^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))},$$

$$\bar{E}_{14}^{[3]_4} = \frac{(s_1^6(28s_2s_3 - 8s_1s_3 - 8s_1s_2 + 3s_1^2))}{(5040(s_3-1)(s_2-1)(s_1-1))},$$

$$\bar{E}_{21}^{[3]_4} = -\frac{(s_2^6(28s_3 - 8s_2 - 8s_2s_3 + 3s_2^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))},$$

$$\bar{E}_{22}^{[3]_4} = \frac{(s_2^4(42s_1s_3 - 14s_1s_2 - 14s_2s_3 + 6s_1s_2^2 + 6s_2^2s_3 + 6s_2^2 - 3s_2^3 - 14s_1s_2s_3))}{(5040(s_2-1)(s_2-s_3)(s_1-s_2))},$$

$$\bar{E}_{23}^{[3]_4} = \frac{(s_2^6(8s_2 - 28s_1 + 8s_1s_2 - 3s_2^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))},$$

$$\bar{E}_{24}^{[3]_4} = -\frac{(s_2^6(8s_1s_2 - 28s_1s_3 + 8s_2s_3 - 3s_2^2))}{(5040(s_3-1)(s_1-1)(s_2-1))},$$

$$\bar{E}_{31}^{[3]_4} = \frac{(s_3^6(8s_3 - 28s_2 + 8s_2s_3 - 3s_3^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))},$$

$$\bar{E}_{32}^{[3]_4} = -\frac{(h^4s_3^6(8s_3 - 28s_1 + 8s_1s_3 - 3s_3^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))},$$

$$\bar{E}_{33}^{[3]_4} = \frac{(s_3^4(14s_1s_3 - 42s_1s_2 + 14s_2s_3 - 6s_1s_3^2 - 6s_2s_3^2 - 6s_3^2 + 3s_3^3 + 14s_1s_2s_3))}{(5040(s_3-1)(s_2-s_3)(s_1-s_3))},$$
$$\bar{E}_{34}^{[3]_4} = \frac{(s_3^6(28s_1s_2 - 8s_1s_3 - 8s_2s_3 + 3s_3^2))}{(5040(s_2-1)(s_1-1)(s_3-1))},$$
$$\bar{E}_{41}^{[3]_4} = -\frac{((28s_2s_3 - 8s_3 - 8s_2 + 3))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))},$$
$$\bar{E}_{42}^{[3]_4} = \frac{((28s_1s_3 - 8s_3 - 8s_1 + 3))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))},$$
$$\bar{E}_{43}^{[3]_4} = -\frac{((28s_1s_2 - 8s_2 - 8s_1 + 3))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))},$$
$$\bar{E}_{44}^{[3]_4} = \frac{((6s_1 + 6s_2 + 6s_3 - 14s_1s_2 - 14s_1s_3 - 14s_2s_3 + 42s_1s_2s_3 - 3))}{(5040(s_3-1)(s_2-1)(s_1-1))}.$$

Equation (5) can also be written as

$$\begin{aligned}
y_{n+s_1} &= y_n + s_1hy_n' + \frac{s_1^2h^2}{2}y_n'' + \frac{s_1^3h^3}{6}y_n''' \\
&- \frac{\begin{array}{r}(h^4s_1^4(28s_1s_2 + 28s_1s_3 - 168s_2s_3 - 8s_1^2s_2 - 8s_1^2s_3 \\ - 8s_1^2 + 3s_1^3 + 28s_1s_2s_3))\end{array}}{(5040s_2s_3)}f_n \\
&+ \frac{\begin{array}{r}(h^4s_1^4(14s_1s_2 + 14s_1s_3 - 42s_2s_3 - 6s_1^2s_2 - 6s_1^2s_3 - 6s_1^2 \\ + 3s_1^3 + 14s_1s_2s_3))\end{array}}{(5040(s_1-1)(s_1-s_3)(s_1-s_2))}f_{n+s_1} \\
&+ \frac{(h^4s_1^6(28s_3 - 8s_1 - 8s_1s_3 + 3s_1^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}f_{n+s_2} \\
&- \frac{(h^4s_1^6(28s_2 - 8s_1 - 8s_1s_2 + 3s_1^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}f_{n+s_3} \\
&+ \frac{(h^4s_1^6(28s_2s_3 - 8s_1s_3 - 8s_1s_2 + 3s_1^2))}{(5040(s_3-1)(s_2-1)(s_1-1))}f_{n+1}, \\
y_{n+s_2} &= y_n + s_2hy_n' + \frac{s_2^2h^2}{2}y_n'' + \frac{s_2^3h^3}{6}y_n''' \\
&+ \frac{\begin{array}{r}(h^4s_2^4(168s_1s_3 - 28s_1s_2 - 28s_2s_3 + 8s_1s_2^2 + 8s_2^2s_3 \\ + 8s_2^2 - 3s_2^3 - 28s_1s_2s_3))\end{array}}{(5040s_1s_3)}f_n \\
&- \frac{(h^4s_2^6(28s_3 - 8s_2 - 8s_2s_3 + 3s_2^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}f_{n+s_1} \\
&+ \frac{\begin{array}{r}(h^4s_2^4(42s_1s_3 - 14s_1s_2 - 14s_2s_3 + 6s_1s_2^2 + 6s_2^2s_3 \\ + 6s_2^2 - 3s_2^3 - 14s_1s_2s_3))\end{array}}{(5040(s_2-1)(s_2-s_3)(s_1-s_2))}f_{n+s_2} \\
&+ \frac{(h^4s_2^6(8s_2 - 28s_1 + 8s_1s_2 - 3s_2^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}f_{n+s_3}
\end{aligned} \tag{7}$$

$$-\frac{(h^4 s_2^6(8s_1s_2 - 28s_1s_3 + 8s_2s_3 - 3s_2^2))}{(5040(s_3-1)(s_1-1)(s_2-1))} f_{n+1}, \tag{8}$$

$$\begin{aligned}
y_{n+s_3} = {} & y_n - s_3 h y_n' - \frac{s_3^2 h^2}{2} y_n'' - \frac{s_3^3 h^3}{6} y_n''' \\
& - \frac{\begin{array}{l}(h^4 s_3^4(28s_1s_3 - 168s_1s_2 + 28s_2s_3 - 8s_1s_3^2 - 8s_2s_3^2 \\ \qquad - 8s_3^2 + 3s_3^3 + 28s_1s_2s_3))\end{array}}{(5040s_1s_2)} f_n \\
& + \frac{(h^4 s_3^6(8s_3 - 28s_2 + 8s_2s_3 - 3s_3^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
& - \frac{(h^4 s_3^6(8s_3 - 28s_1 + 8s_1s_3 - 3s_3^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
& + \frac{\begin{array}{l}(h^4 s_3^4(14s_1s_3 - 42s_1s_2 + 14s_2s_3 - 6s_1s_3^2 - 6s_2s_3^2 - 6s_3^2 \\ \qquad + 3s_3^3 + 14s_1s_2s_3))\end{array}}{(5040(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} \\
& + \frac{(h^4 s_3^6(28s_1s_2 - 8s_1s_3 - 8s_2s_3 + 3s_3^2))}{(5040(s_2-1)(s_1-1)(s_3-1))} f_{n+1},
\end{aligned} \tag{9}$$

$$\begin{aligned}
y_{n+1} = {} & y_n + h y_n' + \frac{h^2}{2} y_n'' + \frac{h^3}{6} y_n''' \\
& + \frac{(h^4(8s_1 + 8s_2 + 8s_3 - 28s_1s_2 - 28s_1s_3 - 28s_2s_3 + 168s_1s_2s_3 - 3))}{(5040s_1s_2s_3)} f_n \\
& - \frac{(h^4(28s_2s_3 - 8s_3 - 8s_2 + 3))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))} f_{n+s_1} \\
& + \frac{(h^4(28s_1s_3 - 8s_3 - 8s_1 + 3))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))} f_{n+s_2} \\
& - \frac{(h^4(28s_1s_2 - 8s_2 - 8s_1 + 3))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))} f_{n+s_3} \\
& + \frac{(h^4(6s_1 + 6s_2 + 6s_3 - 14s_1s_2 - 14s_1s_3 - 14s_2s_3 + 42s_1s_2s_3 - 3))}{(5040(s_3-1)(s_2-1)(s_1-1))} f_{n+1}
\end{aligned} \tag{10}$$

3. Analysis of the method

3.1 Order of method

Applying the process of finding the order of a linear multistep method proposed by lambert (1973). the order of the method are found by expanding y and f-function in Taylor series

$$\begin{bmatrix}
\begin{gathered}
\sum_{j=0}^{\infty}\frac{(s_1)^j h^j}{j!}y_n^j - y_n - s_1 h y_n' - \frac{s_1^2h^2}{2}y_n'' - \frac{s_1^3h^3}{6}y_n''' \\
+\frac{(h^4 s_1^4(28s_1s_2+28s_1s_3-168s_2s_3-8s_1^2s_2-8s_1^2s_3-8s_1^2+3s_1^3+28s_1s_2s_3))}{(5040s_2s_3)}y_n^{iv} \\
-\frac{(s_1^4(14s_1s_2+14s_1s_3-42s_2s_3-6s_1^2s_2-6s_1^2s_3-6s_1^2+3s_1^3+14s_1s_2s_3))}{(5040(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^jh^{j+4}}{j!}y_n^{j+4} \\
-\frac{(s_1^6(28s_3-8s_1-8s_1s_3+3s_1^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^jh^{j+4}}{j!}y_n^{j+4} \\
+\frac{(s_1^6(28s_2-8s_1-8s_1s_2+3s_1^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^jh^{j+4}}{j!}y_n^{j+4} \\
-\frac{(s_1^6(28s_2s_3-8s_1s_3-8s_1s_2+3s_1^2))}{(5040(s_3-1)(s_2-1)(s_1-1))}\sum_{j=0}^{\infty}\frac{h^{j+4}}{j!}y_n^{j+4}
\end{gathered} \\
\begin{gathered}
\sum_{j=0}^{\infty}\frac{(s_2)^j h^j}{j!}y_n^j - y_n - s_2 h y_n' - \frac{s_2^2h^2}{2}y_n'' - \frac{s_2^3h^3}{6}y_n''' \\
-\frac{(h^4 s_2^4(168s_1s_3-28s_1s_2-28s_2s_3+8s_1s_2^2+8s_2^2s_3+8s_2^2-3s_2^3-28s_1s_2s_3))}{(5040s_1s_3)}y_n^{iv} \\
+\frac{(s_2^6(28s_3-8s_2-8s_2s_3+3s_2^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^jh^{j+4}}{j!}y_n^{j+4} \\
-\frac{(s_2^4(42s_1s_3-14s_1s_2-14s_2s_3+6s_1s_2^2+6s_2^2s_3+6s_2^2-3s_2^3-14s_1s_2s_3))}{(5040(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^jh^{j+4}}{j!}y_n^{j+4} \\
-\frac{(s_2^6(8s_2-28s_1+8s_1s_2-3s_2^2))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^jh^{j+4}}{j!}y_n^{j+4} \\
+\frac{(s_2^6(8s_1s_2-28s_1s_3+8s_2s_3-3s_2^2))}{(5040(s_3-1)(s_1-1)(s_2-1))}\sum_{j=0}^{\infty}\frac{h^{j+4}}{j!}y_n^{j+4}
\end{gathered} \\
\begin{gathered}
\sum_{j=0}^{\infty}\frac{(s_3)^j h^j}{j!}y_n^j - y_n - s_3 h y_n' - \frac{s_3^2h^2}{2}y_n'' - \frac{s_3^3h^3}{6}y_n''' \\
+\frac{(h^4 s_3^4(28s_1s_3-168s_1s_2+28s_2s_3-8s_1s_3^2-8s_2s_3^2-8s_3^2+3s_3^3+28s_1s_2s_3))}{(5040s_1s_2)}y_n^{iv} \\
-\frac{(s_3^6(8s_3-28s_2+8s_2s_3-3s_3^2))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^jh^{j+4}}{j!}y_n^{j+4} \\
+\frac{(h^4s_3^6(8s_3-28s_1+8s_1s_3-3s_3^2))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^jh^{j+4}}{j!}y_n^{j+4} \\
-\frac{(s_3^4(14s_1s_3-42s_1s_2+14s_2s_3-6s_1s_3^2-6s_2s_3^2-6s_3^2+3s_3^3+14s_1s_2s_3))}{(5040(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^jh^{j+4}}{j!}y_n^{j+4} \\
-\frac{(s_3^6(28s_1s_2-8s_1s_3-8s_2s_3+3s_3^2))}{(5040(s_2-1)(s_1-1)(s_3-1))}\sum_{j=0}^{\infty}\frac{h^{j+4}}{j!}y_n^{j+4}
\end{gathered} \\
\begin{gathered}
\sum_{j=0}^{\infty}\frac{h^j}{j!}y_n^j - y_n - h y_n' - \frac{h^2}{2}y_n'' - \frac{h^3}{6}y_n''' \\
-\frac{(h^4(8s_1+8s_2+8s_3-28s_1s_2-28s_1s_3-28s_2s_3+168s_1s_2s_3-3))}{(5040s_1s_2s_3)}y_n^{iv} \\
+\frac{((28s_2s_3-8s_3-8s_2+3))}{(5040s_1(s_1-1)(s_1-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_1)^jh^{j+4}}{j!}y_n^{j+4} \\
-\frac{((28s_1s_3-8s_3-8s_1+3))}{(5040s_2(s_2-1)(s_2-s_3)(s_1-s_2))}\sum_{j=0}^{\infty}\frac{(s_2)^jh^{j+4}}{j!}y_n^{j+4} \\
+\frac{((28s_1s_2-8s_2-8s_1+3))}{(5040s_3(s_3-1)(s_2-s_3)(s_1-s_3))}\sum_{j=0}^{\infty}\frac{(s_3)^jh^{j+4}}{j!}y_n^{j+4} \\
-\frac{((6s_1+6s_2+6s_3-14s_1s_2-14s_1s_3-14s_2s_3+42s_1s_2s_3-3))}{(5040(s_3-1)(s_2-1)(s_1-1))}\sum_{j=0}^{\infty}\frac{h^{j+4}}{j!}y_n^{j+4}
\end{gathered}
\end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Collecting like terms ($\bar{C}_i\ 's$) to h gives $\bar{C}_0 = \bar{C}_1 = \bar{C}_2 = \cdots = \bar{C}_8 = 0$, and $\bar{C}_{5+4} \neq 0$. Hence, the new method is of order$[5, 5, 5, 5]^T$ with error constant

$$\bar{C}_9 = \begin{bmatrix} -\dfrac{(s_1^6(24s_1s_2+24s_1s_3-84s_2s_3-9s_1^2s_2-9s_1^2s_3-9s_1^2+4s_1^3+24s_1s_2s_3))}{1814400} \\ \dfrac{(s_2^6(84s_1s_3-24s_1s_2-24s_2s_3+9s_1s_2^2+9s_2^2s_3+9s_2^2-4s_2^3-24s_1s_2s_3))}{1814400} \\ -\dfrac{(s_3^6(24s_1s_3-84s_1s_2+24s_2s_3-9s_1s_3^2-9s_2s_3^2-9s_3^2+4s_3^3+24s_1s_2s_3))}{1814400} \\ \dfrac{(9s_1+9s_2+9s_3-24s_1s_2-24s_1s_3-24s_2s_3+84s_1s_2s_3-4)}{1814400} \end{bmatrix}$$

which is true for all

$$\begin{aligned} & s_1, s_2, s_3 \in (0,1) \setminus \{s_2 = \frac{9s_1^2s_3+9s_1^2-4s_1^3-24s_1s_3}{24s_1-84s_3-9s_1^2+24s_1s_3}\} \\ & \cup \{s_1 = \frac{24s_2s_3-9s_2^2s_3-9s_2^2+4s_2^3}{84s_3-24s_2+9s_2^2-24s_2s_3}\} \\ & \cup \{s_2 = \frac{-24s_1s_3+9s_1s_3^2+9s_3^2-4s_3^3}{-84s_2+24s_3-9s_3^2+24s_1s_3}\} \\ & \cup \{s_1 = \frac{-9s_2-9s_3+24s_2s_3+4}{9-24s_2-24s_3+84s_2s_3}\}. \end{aligned}$$

3.2 Zero stability

In finding the zero stability of the method, definition in Fatunla (1991) is used. This is

$$\begin{aligned} \Pi(r) &= |r\ I - \bar{B}_1^{[3]_4}| \\ &= \left| z\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right| \\ &= r^3(r-1), \end{aligned}$$

which gives $r = 0,0,0,1$. According to (Fatunla, 1991), (Lambert, 1973) and (Henrici, 1962) our method is zero stable, consistent and then convergent.

3.3 Region of absolute stability

In this subsection, the locus boundary method is used to confirm the absolute stability interval. By substituting test equation $y'''' = -\lambda^4 y$ in (**??**) where $\bar{h} = \lambda^4 h^4$ and $\lambda = \frac{df}{dy}$. let $r = \cos\theta - i\sin\theta$ and considering real part yields the equation of absolute stability region.

$$\bar{h}(\theta, h) = \frac{60963840000(\cos(\theta)-1)}{(s_1^3s_2^3s_3^3(20s_1+20s_2+20s_3-10s_1s_2-10s_1s_3-10s_2s_3+4s_1s_2s_3 + s_1s_2s_3\cos(\theta)-35))}.$$

4. Numerical experimental

In this part, the following linear and non-linear IVPs available in the previous literatures were also solved to specific off step points $x_{n+\frac{1}{4}}$, $x_{n+\frac{2}{4}}$ and $x_{n+\frac{3}{4}}$ in order to compare the performance of the new method with existing ones. Computed solution (COP) , exact solution (EXT) and absolute errors (ERR) were carried out using flexible Matlab code. This is clear in Table 1 and Table 2.

Problem 1:

$$y^{iv}-(y')^2+yy''+4x^2-e^x(1-4x+x^2)=0, y(0)=1, y'(0)=1, y''(0)=1, h=\frac{1}{100}.$$

Exact solution: $y(x)=x^2+e^x$.

Problem 2:

$$y^{iv}-x=0, y(0)=0, y'(0)=1, y''(0)=0, y'''(0)=0, h=\frac{1}{320}.$$

Exact solution: $y(x)=\frac{x^5}{120}+x$.

Table 1: Comparison of the new method with (Olabode et al,2015) for solving problem 1, where $h=\frac{1}{320}$

x		New method, $P=5$	Olabode and Omole (2015), $P=6$
0.0031250	EXT	1.0031396535277390	1.003139653527739149
	CPS	1.0031396535277390	1.003139653526590265
	ERR	$0.000000e^{+00}$	$1.148884e^{-12}$
0.0062500	EXT	1.0063086345037620	1.006308634503762010
	CPS	1.0063086345037617	1.006308634484910542
	ERR	$2.220446e^{-16}$	$1.8851468e^{-11}$
0.0093750	EXT	1.0095069735890709	1.009506973589071086
	CPS	1.0095069735890709	1.009506973491318106
	ERR	$0.000000e^{+00}$	$9.7752980e^{-11}$
0.0125000	EXT	1.0127347015406345	1.012734701540634377
	CPS	1.0127347015406303	1.0127347015406341
	ERR	$4.440892e^{-16}$	$3.15759129e^{-10}$
0.0156250	EXT	1.0159918492116857	1.015991849211685747
	CPS	1.0159918492116851	1.015991848424806972
	Error	$6.661338e^{-16}$	$1.15463e^{-10}$

Table 2: Comparison of the new method with (Kayode et al, 2014) for solving problem 3, where $h = \frac{1}{10}$

x		New method, $P = 5$	(kyode et al,2014), $P = 8$
0.1	EXT	0.100000083333333340	0.100000083333334000
	CPS	0.100000083333333340	0.10000008333351720
	ERR	$0.000000e^{+00}$	$1.832e^{-13}$
0.2	EXT	0.200002666666666690	0.200002666666666900
	CPS	0.200002666666666660	0.20000266667150250
	ERR	$2.775558e^{-17}$	$4.835e^{-12}$
0.3	EXT	0.300020250000000040	0.300020250000000004
	CPS	0.300020249999999990	0.30002025000721480
	ERR	$5.551115e^{-17}$	$7.214e^{-12}$
0.4	EXT	0.400085333333333350	0.400008533333333333
	CPS	0.400085333333333350	0.40000853340160457
	ERR	$0.000000e^{+00}$	$6.832e^{-11}$
0.5	EXT	0.500260416666666650	0.500260416666666665
	CPS	0.500260416666666650	0.50026041674083458
	ERR	$0.000000e^{+00}$	$7.416e^{-11}$

5. Conclusion

A one step hybrid block method with three generalized off step points for solving linear and no-linear fourth order initial value problem has been developed in this article. The numerical properties of the new method are also established. The method competes better than its counterparts in terms of accuracy when solving fourth order initial value problems.

Appendix A:

$$\begin{aligned}D_{14}^{[3]_4} &= \frac{-(s_1-1)(s_2-1)(s_3-1)}{(5040 s_1 s_2 s_3)}(5s_1^3 s_2 + 5s_1^3 s_3 + 5s_1 - 15 s_1 s_2^2 s_3 + 5s_3^2 \\ &+ 5s_3 - 3s_1^4 + 5s_1^3 + 5s_1^2 s_2^2 - 15 s_1^2 s_2 s_3 - 15 s_1^2 s_2 + 5 s_1^2 s_3^2 - 15 s_1^2 s_3 \\ &+ 5s_1^2 + 5 s_1 s_2^3 - 3s_2^4 + 5s_2^3 - 15 s_1 s_2^2 - 15 s_1 s_2 s_3^2 - 15 s_1 s_2 \\ &+ 5 s_1 s_3^3 - 15 s_1 s_3^2 - 15 s_1 s_3 + 5 s_3^3 - 3 s_3^4 + 5 s_2 \\ &+ 105 s_1 s_2 s_3 + 5 s_2^3 s_3 + 5 s_2^2 s_3^2 - 15 s_2^2 s_3 \\ &+ 5 s_2^2 + 5 s_2 s_3^3 - 15 s_2 s_3^2 - 15 s_2 s_3 - 3),\end{aligned}$$

$$\begin{aligned}D_{24}^{[3]_4} &= \frac{1}{5040h}(3s_1^4 - 5s_1^3 s_2 + 15 s_1 s_2 s_3^2 - 120 s_1 s_2 s_3 - 5 s_1 s_3^3 + 20 s_1 s_3^2 + 3 s_2^4 \\ &- 5 s_2^3 s_3 - 8 s_1^3 - 5 s_1^2 s_2^2 + 15 s_1^2 s_2 s_3 + 20 s_1^2 s_2 - 5 s_1^2 s_3^2 + 20 s_1^2 s_3 \\ &- 5 s_1 s_2^3 + 15 s_1 s_2^2 s_3 + 20 s_1 s_2^2 - 8 s_2^3 - 5 s_1^3 s_3 \\ &- 5 s_2^2 s_3^2 + 3 s_3^4 + 20 s_2^2 s_3 - 5 s_2 s_3^3 + 20 s_2 s_3^2 - 8 s_3^3),\end{aligned}$$

$$D_{34}^{[3]_4} = \frac{-1}{(2520h^2 s_1 s_2 s_3)}(3s_1^5 s_2 + 3s_1^5 s_3 - 5s_1^4 s_2^2 - 10s_1^4 s_2 s_3 - 8s_1^4 s_2$$
$$- 5s_1^4 s_3^2 + 20s_1^3 s_3^2 - 8s_1^4 s_3 - 5s_1^3 s_2^3 + 10s_1^3 s_2^2 s_3$$
$$+ 20s_1^3 s_2^2 + 10s_1^3 s_2 s_3^2 + 40s_1^3 s_2 s_3 - 5s_1^3 s_3^3 - 5s_1^2 s_2^4$$
$$- 5s_2^2 s_3^4 + 10s_1^2 s_2 s_3^3 - 100s_1^2 s_2 s_3^2 - 5s_1^2 s_3^4 + 20s_1^2 s_3^3$$
$$+ 3s_1 s_2^5 - 10s_1 s_2^4 s_3 - 8s_1 s_2^4 + 10s_1^2 s_2^3 s_3$$
$$+ 10s_1 s_2^3 s_3^2 + 40s_1 s_2^3 s_3 + 10s_1 s_2^2 s_3^3 - 100s_1 s_2^2 s_3^2 - 10s_1 s_2 s_3^4$$
$$+ 40s_1 s_2 s_3^3 + 20s_1^2 s_2^3 + 3s_1 s_3^5 - 8s_1 s_3^4 + 3s_2^5 s_3 - 5s_2^4 s_3^2$$
$$- 8s_2^4 s_3 - 5s_2^3 s_3^3 + 20s_2^3 s_3^2 + 30s_1^2 s_2^2 s_3^2 - 100s_1^2 s_2^2 s_3$$
$$+ 20s_2^2 s_3^3 + 3s_2 s_3^5 - 8s_2 s_3^4),$$

$$D_{44}^{[3]_4} = \frac{1}{(840h^3 s_1 s_2 s_3)}(3s_1^5 - 5s_1^4 s_2 - 5s_1^4 s_3 - 8s_1^4 - 5s_1^3 s_2^2$$
$$+ 15s_1^3 s_2 s_3 + 20s_2^2 s_3^2 + 20s_1^3 s_2 - 5s_1^3 s_3^2 + 20s_1^3 s_3$$
$$- 5s_1^2 s_2^3 + 15s_1^2 s_2^2 s_3 + 20s_1^2 s_2^2 + 15s_1^2 s_2 s_3^2 - 120s_1^2 s_2 s_3$$
$$- 5s_1^2 s_3^3 + 20s_1^2 s_3^2 - 5s_1 s_2^4 + 15s_1 s_2^3 s_3 + 20s_1 s_2^3 + 15s_1 s_2^2 s_3^2$$
$$- 120s_1 s_2^2 s_3 + 15s_1 s_2 s_3^3 - 5s_2 s_3^4 - 120s_1 s_2 s_3^2 - 5s_1 s_3^4$$
$$+ 20s_1 s_3^3 - 5s_2^4 s_3 - 8s_2^4 - 5s_2^3 s_3^2 + 20s_2^3 s_3 - 5s_2^2 s_3^3$$
$$+ 20s_2 s_3^3 + 3s_3^5 - 8s_3^4 + 3s_2^5)$$

$$E_{11}^{[3]_4} = \frac{-(s_2-1)(s_3-1)}{5040s_1(s_1-s_2)(s_1-s_3)}(3s_1^4 - 3s_1^3 s_2 - 3s_1^3 s_3 - 3s_1^3$$
$$- 3s_1^2 s_2^2 + 5s_1^2 s_2 s_3 + 5s_1^2 s_2 - 3s_1^2 s_3^2 + 5s_1^2 s_3 - 3s_1^2 - 3s_1 s_2^3 + 5s_1 s_2^2 s_3$$
$$+ 5s_1 s_2^2 + 5s_1 s_2 s_3^2 - 15s_1 s_2 s_3 - 3s_1 s_3^3 + 5s_1 s_3^2$$
$$+ 5s_1 s_3 - 3s_1 - 3s_2^4 + 5s_2^3 s_3 + 5s_2^3 + 5s_2^2 s_3^2 - 15s_2^2 s_3 + 5s_2^2 + 5s_2^2$$
$$+ 5s_1 s_2 + 5s_2 s_3^3 - 15s_2 s_3^2 - 15s_2 s_3 + 5s_2 - 3s_3^4 + 5s_3^3 + 5s_3^2 + 5s_3 - 3),$$

$$E_{12}^{[3]_4} = \frac{(s_1-1)(s_3-1)}{5040s_2(s_1-s_2)(s_2-s_3)}(5s_1^3 - 3s_1^4 - 3s_1^3 s_2$$
$$+ 5s_1^3 s_3 - 3s_1^2 s_2^2 + 5s_1^2 s_2 s_3 + 5s_1^2 s_3^2 - 15s_1^2 s_3 + 5s_1^2 - 3s_1 s_2^3 + 5s_1 s_2^2 s_3$$
$$+ 5s_1 s_2^2 + 5s_1 s_2 s_3^2 - 15s_1 s_2 s_3 + 5s_1 s_2 + 5s_1 s_3^3 - 15s_1 s_3^2 - 15s_1 s_3$$
$$+ 5s_1 + 3s_2^4 - 3s_2^3 s_3 - 3s_2^3 - 3s_2^2 s_3^2 + 5s_2^2 s_3 - 3s_2^2 + 5s_3$$
$$+ 5s_1^2 s_2 - 3s_2 s_3^3 + 5s_2 s_3^2 + 5s_2 s_3 - 3s_2 - 3s_3^4 + 5s_3^3 + 5s_3^2 - 3),$$

$$E_{13}^{[3]_4} = -\frac{(s_1-1)(s_2-1)}{(5040s_3(s_1-s_3)(s_2-s_3)}(-3s_1^4 + 5s_1^3 s_2 - 3s_1^3 s_3 + 5s_1^3$$
$$+ 5s_1^2 s_2^2 + 5s_1^2 s_2 s_3 - 15s_1^2 s_2 - 3s_1^2 s_3^2 + 5s_1^2 s_3 + 5s_1^2 + 5s_1 s_2^3$$
$$+ 5s_1 s_2^2 s_3 - 15s_1 s_2^2 + 5s_1 s_2 s_3^2 - 15s_1 s_2 s_3 - 15s_1 s_2 - 3s_1 s_3^3$$
$$+ 5s_1 s_3^2 + 5s_1 s_3 + 5s_1 - 3s_2^4 - 3s_2^3 s_3 + 5s_2^3 - 3s_2^2 s_3^2 + 5s_2^2 s_3 + 5s_2^2$$
$$- 3s_2 s_3^3 + 5s_2 s_3^2 - 3s_3 + 5s_2 s_3 + 5s_2 + 3s_3^4 - 3s_3^3 - 3s_3^2 - 3),$$

$$
\begin{aligned}
E_{14}^{[3]_4} &= \frac{1}{5040}(-3s_1^4+5s_1^3s_2+5s_1^3s_3-3s_1^3+5s_1^2s_2^2-15s_1^2s_2s_3 \\
&+5s_1^2s_2+5s_1^2s_3^2-3s_1^2+5s_1s_2^3-15s_1s_2^2s_3+5s_1s_2^2 \\
&-15s_1s_2s_3^2-15s_1s_2s_3+5s_1s_2+5s_1s_3^3+5s_1s_3^2 \\
&+5s_1s_3-3s_1-3s_2^4+5s_2^3s_3-3s_2^3+5s_2^2s_3^2+5s_2^2s_3-3s_2^2+5s_2s_3^3+5s_2s_3^2 \\
&+5s_2s_3+5s_1^2s_3-3s_2-3s_3^4-3s_3^3-3s_3^2-3s_3+3) \\
E_{21}^{[3]_4} &= -\frac{s_2s_3}{5040h(s_1-s_2)(s_1-s_3)(s_1-1)}(-3s_1^3s_2-3s_1^3s_3-3s_1^2s_2^2 \\
&+8s_3^3+5s_2^2s_3^2+5s_1^2s_2s_3+8s_1^2s_2-3s_1^2s_3^2+8s_1^2s_3-3s_1s_2^3 \\
&+5s_1s_2^2s_3+8s_1s_2^2+5s_1s_2s_3^2-20s_1s_2s_3 \\
&+3s_1^4-6s_1^3-3s_1s_3^3+8s_1s_3^2-3s_2^4+5s_2^3s_3+8s_2^3-20s_2^2s_3 \\
&+5s_2s_3^3-20s_2s_3^2-3s_3^4), \\
E_{22}^{[3]_4} &= \frac{s_1s_3}{5040h(s_1-s_2)(s_2-s_3)(s_2-1)}(5s_1^3s_3-3s_1^3s_2-3s_1^2s_2^2 \\
&-3s_3^4+5s_1^2s_2s_3+8s_1^3+8s_1^2s_2+5s_1^2s_3^2-20s_1^2s_3-3s_1s_2^3+5s_1s_2^2s_3 \\
&+8s_1s_2^2+5s_1s_2s_3^2-20s_1s_2s_3-3s_1^4+5s_1s_3^3-20s_1s_3^2 \\
&+3s_2^4-3s_2^3s_3-6s_2^3-3s_2^2s_3^2+8s_2^2s_3-3s_2s_3^3+8s_2s_3^2+8s_3^3), \\
E_{23}^{[3]_4} &= -\frac{s_1s_2}{5040h(s_1-s_3)(s_2-s_3)(s_3-1)}(3s_3^4+5s_1^3s_2-3s_1^3s_3 \\
&+5s_1^2s_2^2+5s_1^2s_2s_3-20s_1^2s_2-3s_1^2s_3^2+8s_1^2s_3+5s_1s_2^3+5s_1s_2^2s_3 \\
&-20s_1s_2^2+5s_1s_2s_3^2-20s_1s_2s_3-3s_1s_3^3+8s_1^3+8s_1s_3^2 \\
&-3s_2^4-3s_2^3s_3+8s_2^3-3s_2^2s_3^2+8s_2^2s_3-3s_2s_3^3+8s_2s_3^2-6s_3^3-3s_1^4), \\
E_{24}^{[3]_4} &= \frac{s_1s_2s_3}{5040h(s_1-1)(s_2-1)(s_3-1)}(5s_2^2s_3^25s_2^3s_3-3s_1^4+5s_1^3s_2 \\
&+5s_1^3s_3+5s_1^2s_2^2-15s_1^2s_2s_3+5s_1^2s_3^2+5s_1s_2^3-15s_1s_2^2s_3-15s_1s_2s_3^2 \\
&+5s_1s_3^3-3s_2^4+5s_2s_3^3-3s_3^4), \\
E_{31}^{[3]_4} &= \frac{1}{(2520h^2s_1(s_1-s_2)(s_1-s_3)(s_1-1))}(3s_1^5s_2+3s_1^5s_3 \\
&-3s_1^4s_2^2-6s_1^4s_2s_3-6s_1^4s_2-3s_1^3s_2^3+2s_1^3s_2^2s_3+8s_1^3s_2^2 \\
&+2s_1^3s_2s_3^2+16s_1^3s_2s_3-3s_1^3s_3^3+8s_1^3s_3^2-3s_1^4s_3^2 \\
&-3s_1^2s_2^4+2s_1^2s_2^3s_3+8s_1^2s_2^3+10s_1^2s_2^2s_3^2-12s_1^2s_2^2s_3 \\
&+2s_1^2s_2s_3^3-12s_1^2s_2s_3^2-6s_1^4s_3 \\
&-3s_1^2s_3^4+8s_1^2s_3^3-3s_1s_2^5+2s_1s_2^4s_3+8s_1s_2^4+10s_1s_2^3s_3^2 \\
&-12s_1s_2^3s_3+5s_2^4s_3^2+10s_1s_2^2s_3^3-40s_1s_2^2s_3^2+2s_1s_2s_3^4 \\
&-12s_1s_2s_3^3-3s_1s_3^5+8s_1s_3^4-3s_2^5s_3+8s_2^4s_3 \\
&+5s_2^3s_3^3-20s_2^3s_3^2+5s_2^2s_3^4-20s_2^2s_3^3-3s_2s_3^5+8s_2s_3^4),
\end{aligned}
$$

$$
\begin{aligned}
E_{32}^{[3]_4} = &-\frac{1}{(2520h^2 s_2(s_1-s_2)(s_2-s_3)(s_2-1))}(-3s_1^5s_2-3s_1^5s_3\\
&-3s_1^4s_2^2+2s_1^4s_2s_3+8s_1^4s_2+5s_1^4s_3^2+8s_1^4s_3-3s_1^3s_2^3+2s_1^3s_2^2s_3\\
&+8s_1^3s_2^2+10s_1^3s_2s_3^2+2s_1s_2s_3^4+3s_1s_2^5-12s_1^3s_2s_3\\
&+5s_1^3s_3^3-20s_1^3s_3^2-3s_1^2s_2^4+2s_1^2s_2^3s_3\\
&+8s_1^2s_2^3-3s_1s_3^5+8s_1s_3^4+3s_2^5s_3+10s_1^2s_2^2s_3^2-12s_1^2s_2^2s_3\\
&+10s_1^2s_2s_3^3-40s_1^2s_2s_3^2+5s_1^2s_3^4-20s_1^2s_3^3-3s_2^4s_3^2-6s_2^4s_3\\
&-6s_1s_2^4s_3-6s_1s_2^4+2s_1s_2^3s_3^2+16s_1s_2^3s_3+2s_1s_2^2s_3^3-12s_1s_2^2s_3^2\\
&-12s_1s_2s_3^3-3s_2^3s_3^3+8s_2^3s_3^2-3s_2^2s_3^4+8s_2^2s_3^3\\
&-3s_2s_3^5+8s_2s_3^4),\\
E_{33}^{[3]_4} = &\frac{1}{(2520h^2 s_3(s_1-s_3)(s_2-s_3)(s_3-1))}(-3s_1^5s_2-3s_1^5s_3\\
&+5s_1^4s_2^2+2s_1^4s_2s_3+8s_1^4s_2-3s_1^4s_3^2+8s_1^4s_3+5s_1^3s_2^3+10s_1^3s_2^2s_3\\
&-20s_1^3s_2^2+2s_1^3s_2s_3^2+3s_1s_3^5-6s_1s_3^4-12s_1^3s_2s_3\\
&-3s_1^3s_3^3+8s_1^3s_3^2+5s_1^2s_2^4+10s_1^2s_2^3s_3\\
&-20s_1^2s_2^3+10s_1^2s_2^2s_3^2+16s_1s_2s_3^3+8s_2^4s_3+2s_1^2s_2s_3^3\\
&-12s_1^2s_2s_3^2-3s_1^2s_3^4+8s_1^2s_3^3-3s_1s_2^5\\
&+2s_1s_2^4s_3-3s_2^3s_3^3+8s_2^3s_3^2+8s_1s_2^4+2s_1s_2^3s_3^2\\
&-12s_1s_2^3s_3+2s_1s_2^2s_3^3-12s_1s_2^2s_3^2-6s_1s_2s_3^4-3s_2^5s_3-40s_1^2s_2^2s_3\\
&-3s_2^4s_3^2-3s_2^2s_3^4+8s_2^2s_3^3+3s_2s_3^5-6s_2s_3^4),\\
E_{34}^{[3]_4} = &\frac{1}{h^2(2520s_1-2520)(s_2-1)(s_3-1)}(3s_1^5s_2+3s_1^5s_3-5s_1^4s_2^2\\
&+3s_1s_3^5+3s_2^5s_3-10s_1^4s_2s_3-5s_1^4s_3^2-5s_1^3s_2^3\\
&+10s_1^3s_2^2s_3+10s_1^3s_2s_3^2-5s_1^3s_3^3-10s_1s_2^4s_3+10s_1s_2^3s_3^2\\
&-5s_1^2s_2^4+10s_1^2s_2^3s_3+30s_1^2s_2^2s_3^2+10s_1^2s_2s_3^3-5s_1^2s_3^4+3s_1s_2^5\\
&+10s_1s_2^2s_3^3-10s_1s_2s_3^4-5s_2^4s_3^2-5s_2^3s_3^3-5s_2^2s_3^4+3s_2s_3^5),\\
E_{41}^{[3]_4} = &\frac{-1}{840h^3 s_1(s_1-s_2)(s_1-s_3)(s_1-1)}(3s_1^5-3s_1^4s_2-3s_1^4s_3-6s_1^4\\
&-3s_1^3s_2^2-3s_3^5+8s_1^3s_2-3s_1^3s_3^2+8s_1^3s_3-3s_1^2s_2^3+5s_1^2s_2^2s_3\\
&+8s_1^2s_2^2+5s_1^2s_2s_3^2-20s_1^2s_2s_3-20s_2^2s_3^2\\
&-3s_1^2s_3^3+8s_1^2s_3^2-3s_1s_2^4+5s_1s_2^3s_3+8s_1s_2^3+5s_1s_2^2s_3^2\\
&-20s_1s_2^2s_3+5s_1s_2s_3^3+5s_2s_3^4-20s_1s_2s_3^2-3s_1s_3^4\\
&+8s_1s_3^3-3s_2^5+5s_2^4s_3+8s_2^4+5s_2^3s_3^2-20s_2^3s_3\\
&+5s_2^2s_3^3-20s_2s_3^3+5s_1^3s_2s_3+8s_3^4),
\end{aligned}
$$

$$
\begin{aligned}
E_{42}^{[3]_4} &= \frac{1}{840h^3 s_2(s_1-s_2)(s_2-s_3)(s_2-1)}(8s_3^4 - 3s_1^5 - 3s_1^4 s_2 + 5s_1^4 s_3 \\
&\quad - 3s_1^3 s_2^2 - 3s_3^5 + 8s_1^3 s_2 + 5s_1^3 s_3^2 - 20s_1^3 s_3 - 3s_1^2 s_2^3 + 5s_1^2 s_2^2 s_3 + 8s_1^2 s_2^2 \\
&\quad + 5s_1^2 s_2 s_3^2 - 20s_1^2 s_2 s_3 + 8s_2^2 s_3^2 + 5s_1^2 s_3^3 - 20s_1^2 s_3^2 - 3s_1 s_2^4 \\
&\quad + 5s_1 s_2^3 s_3 + 8s_1 s_2^3 + 5s_1 s_2^2 s_3^2 - 20s_1 s_2^2 s_3 + 5s_1 s_2 s_3^3 \\
&\quad - 20s_1 s_2 s_3^2 + 5s_1 s_3^4 - 20s_1 s_3^3 + 3s_2^5 - 3s_2^4 s_3 - 6s_2^4 - 3s_2^3 s_3^2 + 8s_2^3 s_3 - 3s_2^2 s_3^3 \\
&\quad + 8s_2 s_3^3 + 5s_1^3 s_2 s_3 + 8s_1^4 - 3s_2 s_3^4), \\
E_{43}^{[3]_4} &= \frac{-1}{840h^3 s_3(s_1-s_3)(s_2-s_3)(s_3-1)}(5s_1^4 s_2 - 3s_1^5 - 3s_1^4 s_3 \\
&\quad + 5s_1^3 s_2^2 + 5s_1^3 s_2 s_3 - 20s_1^3 s_2 - 3s_1^3 s_3^2 + 8s_1^3 s_3 + 5s_1^2 s_2^3 + 5s_1^2 s_2^2 s_3 \\
&\quad - 20s_1^2 s_2^2 + 5s_1^2 s_2 s_3^2 - 20s_1^2 s_2 s_3 - 3s_1^2 s_3^3 + 8s_1^2 s_3^2 + 5s_1 s_2^4 \\
&\quad + 5s_1 s_2^3 s_3 - 20s_1 s_2^3 + 5s_1 s_2^2 s_3^2 - 20s_1 s_2^2 s_3 + 5s_1 s_2 s_3^3 \\
&\quad - 20s_1 s_2 s_3^2 - 3s_1 s_3^4 + 8s_1 s_3^3 - 3s_2^5 - 3s_2^4 s_3 \\
&\quad + 8s_2^4 - 3s_2^3 s_3^2 + 8s_2^3 s_3 - 3s_2^2 s_3^3 + 8s_2^2 s_3^2 - 3s_2 s_3^4 \\
&\quad + 8s_2 s_3^3 + 3s_3^5 - 6s_3^4 + 8s_1^4), \\
E_{44}^{[3]_4} &= \frac{1}{840h^3(s_1-1)(s_2-1)(s_3-1)}(5s_1^4 s_2 - 3s_1^5 + 5s_1^4 s_3 - 3s_3^5 \\
&\quad + 5s_1^3 s_3^2 + 5s_1^3 s_2^2 + 5s_1^2 s_2^3 - 15s_1^2 s_2^2 s_3 - 15s_1^2 s_2 s_3^2 + 5s_1^2 s_3^3 + 5s_1 s_2^4 \\
&\quad - 15s_1 s_2^3 s_3 - 15s_1 s_2^2 s_3^2 - 15s_1 s_2 s_3^3 - 15s_1^3 s_2 s_3 + 5s_1 s_3^4 + 5s_2^4 s_3 \\
&\quad + 5s_2^3 s_3^2 + 5s_2^2 s_3^3 + 5s_2 s_3^4 - 3s_2^5).
\end{aligned}
$$

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Accepted: January 22, 2022