Congruence-free restriction semigroups

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Abstract. Restriction semigroups are common generalizations of ample semigroups and inverse semigroups. The main aim of this paper is to probe restriction semigroups with certain congruence properties. In this paper we give some characterizations of restriction semigroups each of whose proper $(2, 1, 1)$-congruences are reduced, so called H-reduced restriction semigroups. In particular, the classification of congruence-free restriction semigroups is obtained; that is, it is proved that a restriction semigroup is congruence-free if and only if it is either a simple group or an H-reduced restriction semigroup without nontrivial reduced restriction monoid $(2, 1, 1)$-congruences. These results extend and enrich the related results of inverse semigroups.

Keywords: restriction semigroup, fundamental restriction semigroup, ample semigroup, congruence.

1. Introduction

Inverse semigroups play an important role in the theory of semigroups. Many authors have tried to generalize inverse semigroups. Restriction semigroups are non-regular generalizations of inverse semigroups. They are semigroups equipped with two additional unary operators which satisfy certain identities. In particular, each inverse semigroup determines a restriction semigroup in which the unary operations assign the idempotents aa^{-1} and $a^{-1}a$, respectively, to

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any element a. The class of restriction semigroups is just the variety of algebras generated by these restriction semigroups obtained from inverse semigroups, see [8]. Restriction semigroups (formerly, called weakly E-ample semigroups) have arisen from a number of mathematical perspectives. For a detailed introduction of the history and basic properties of restricted semigroups, please refer to [13] and [18].

So far, a number of important results of the rich structure theory of inverse semigroups have been recast in the broader setting of restriction semigroups; see [11, 9, 10, 21, 25, 16]. In theory of inverse semigroups, congruences play an important role. Because restriction semigroups are generalizations of inverse semigroups, it is natural to probe the congruence theory of restriction semigroups. This is the main aim of this paper. It is an important property that any quotient of an inverse semigroup over a congruence is also inverse. This property is a key to study the congruence theory of inverse semigroups in the present ways. Unfortunately, the quotient of a restriction semigroup over a general congruence need not be still a restriction one (see [15]). So, we only consider the $(2,1,1)$-congruences on a restriction semigroup. Indeed, we are inspired by the results of El Qallali in [4] on congruences on an ample semigroup, formerly called type-A semigroups. This is because any ample semigroup is a special restriction semigroup.

We proceed as follows: after some preliminaries, in Section 3, we obtain some trace characterizations of $(2,1,1)$-congruences on a restriction semigroup. In Section 4, we consider restriction semigroups all of whose proper $(2,1,1)$-congruences are reduced, called H-reduced restriction semigroups. It is interesting that an H-reduced restriction semigroup must be an ample semigroup. Moreover, we determine when a restriction semigroup is H-reduced (Theorem 4.1). This result extends those of Tucci in [26] on inverse semigroups all of whose proper homomorphic images are groups. Section 5 is devoted to congruence-free restriction semigroups. So-called a *congruence-free restriction semigroup* is a restriction semigroup whose $(2,1,1)$-congrunces are only the identity relation and the universal relation. Such semigroups are analogue of congruence-free inverse semigroups. For congruence-free inverse semigroups, see [22, 27]. In [23], Munn further researched congruence-free regular semigroups. Indeed, any congruence-free inverse semigroup is fundamental; for fundamental inverse semigroups, see [20, 24]. It is proved that a semigroup S is a congruence-free restriction semigroup if and only if S is either a simple group, or an H-reduced restriction semigroup without nontrivial reduced $(2,1,1)$-congruences (Theorem 5.1). Our results enrich and extend the related results on inverse semigroups, or ample semigroups.

2. Preliminaries

We recall some concepts and notations, which are used in the sequel without mentions.

2.1 Restriction semigroups

A *left restriction semigroup* is defined to be an algebra of type $(2,1)$, more precisely, an algebra $S = (S,\ \cdot,\ ^+)$ where $(S,\ \cdot)$ is a semigroup and $^+$ is a unary operator such that the following identities are satisfied:

$$\begin{aligned}&(x^+)^+ = x^+,\ x^+x = x,\ x^+y^+ = y^+x^+,\\ &(x^+y)^+ = x^+y^+,\ (xy)^+ = (xy^+)^+,\ xy^+ = (xy^+)^+x.\end{aligned} \tag{2.1}$$

A *right restriction semigroup* is dually defined, that is, it is an algebra $(S,\ \cdot,\ *)$ satisfying the duals of the identities (2.1). If $S = (S,\ \cdot,\ ^+,\ *)$ is an algebra of type $(2,1,1)$ where $S = (S,\ \cdot,\ ^+)$ is a left restriction semigroup and $S = (S,\ \cdot,\ *)$ is a right restriction semigroup and the identities

$$(x^+)^* = x^+,\ (x^*)^+ = x^* \tag{2.2}$$

hold, then it is called a *restriction semigroup*. By definition, the defining properties of a restriction semigroup are left-right dual. Therefore in the sequel dual definitions and statements will not be explicitly formulated. It is well known that in a restriction semigroup, we always have

$$(xy)^+ = (xy^+)^+ \text{ and } (xy)^* = (x^*y)^* \tag{2.3}$$

(for example, see [13]).

Among restriction semigroups, the notions of subalgebra, homomorphism, congruence and factor algebra are understood in type $(2,1,1)$, which is emphasised by using the expressions $(2,1,1)$-subsemigroup, $(2,1,1)$-morphism, $(2,1,1)$-congruence and $(2,1,1)$-factor semigroup, respectively. A restriction semigroup with identity element 1 and such that $1^+ = 1 = 1^*$ is also called a *restriction monoid*.

Let S be a restriction semigroup. By (2.2), we have

$$\{x^+ : x \in S\} = \{x^* : x \in S\}.$$

This set is called the *set of projections of* S and denoted by $P(S)$. Again by (2.1) and its dual, $P(S)$ is a $(2,1,1)$-subsemigroup of S which is indeed a semilattice. We call a restriction semigroup to be *reduced* if $P(S)$ is a singleton. In this case, the unique element of $P(S)$ is the identity element of S. As in [16], we define

$$\mathfrak{C} = \{(u,v) \in S \times S : u^+ = v^+, u^* = v^*\}.$$

2.2 Ample semigroups

The relations $\mathcal{R}^*$ and $\mathcal{L}^*$ are generalizations of the usual Green's relations $\mathcal{R}$ and $\mathcal{L}$, respectively. Elements a and b of a semigroup T is related by $\mathcal{R}^*$ (respectively, $\mathcal{L}^*$) if and only if they are related by $\mathcal{R}$ (respectively, $\mathcal{L}$) in some oversemigroup of T. Equivalently, we have

$$(a,b) \in \mathcal{R}^* \text{ if and only if } xa = ya \Leftrightarrow xb = yb \text{ for any } x, y \in T^1$$

and

$$(a,b) \in \mathcal{L}^* \text{ if and only if } ax = ay \Leftrightarrow bx = by \text{ for any } x, y \in T^1.$$

A semigroup T is an *ample semigroup* if the following conditions are satisfied:

(i) for any $a \in T$, the $\mathcal{R}^*$-class R_a^* of T containing a exists uniquely one idempotent a^+;

(ii) for any $a \in T$, the $\mathcal{L}^*$-class L_a^* of T containing a exists uniquely one idempotent a^+;

(iii) the set $E(T)$ of idempotents of T becomes a commutative subsemigroup; that is, $E(T)$ is a semilattice under the multiplication of T;

(iv) for any $a \in T, e \in E(T)$, $ea = a(ea)^*$ and $ae = (ae)^+a$.

Ample semigroups are formerly called as type A semigroups. It is well known that any inverse semigroup is an ample semigroup and any ample semigroup can be viewed as a subsemigroup of some inverse semigroup. Indeed, an inverse semigroup is just an ample semigroup being regular.

For an ample semigroup T, we have that $e^+ = e = e^*$ for all $e \in E(T)$. By definition, it is easy to see that T is a restriction semigroup with unary operators:

$$^+ : T \to T;\ a \mapsto a^+$$

and

$$^* : T \to T;\ a \mapsto a^*,$$

and in this case,

(i) $P(T) = E(T)$;

(ii) $(2,1,1)$-congruences are just admissible congruences on T;

(iii) $(2,1,1)$-homomorphisms are just admissible homomorphisms on T;

(iv) any reduced $(2,1,1)$-congruence is indeed a cancellative monoid congruence;

(v) $\mathfrak{C} = \mathcal{H}^*$, where $\mathcal{H}^* = \mathcal{L}^* \sqcap \mathcal{R}^*$.

Consequently, any ample semigroup is a restriction semigroup S in which for any $a \in S$, $a^+\mathcal{R}^*a\mathcal{L}^*a^*$.

In what follows, we view an ample semigroup as a restriction semigroup with the unary operations as above.

Recall that a left (right) ideal J of a semigroup T is a *left* (*right*) $*$*-ideal* of T if $J = \sqcup_{x\in J} L_x^*$ $(J = \sqcup_{y\in J} R_y^*)$, where L_x^* (R_x^*) is the $\mathcal{L}^*$-class (the $\mathcal{R}^*$-class) of S containing a. Moreover, an ideal of T is a $*$*-ideal* of T if it is both a left $*$-ideal and a right $*$-ideal.

2.3 Unary polynomials

Given a set X of variables, by a *term* in X we mean a formal expression built up from the elements of X by means of the operational symbols— the binary operational sysmbol $\cdot$ and the unary operational symbols $^+$ and *— in finitely many steps. For example, the left and right hand sides of equalities in (2.1)-(2.3) are terms in variables x, y. If we work with an associative binary operation then we delete the unnecessary parenthesis from terms. If S is a restriction semigroup then we introduce a nullary operational symbols for every element s in S, and for simplicity, denote it also by s. By a *polynomial* of S we mean an expression obtained in a way similar to terms, but from variables and these operational symbols. A polynomial can also be interpreted in the way that such nullary operational symbols are substituted for certain variables in a term. For simplicity, later on we just say that elements of S are substituted for the variables. As it is usual for semigroups, we allow to substitute also $1 \in S^1$ for several, but not all, variables to indicate that the variables in question be deleted from the term. For example, if 1 is substituted for variable y in the terms xyz and $zy^*(x^*y)^+$ then the terms obtained are xz and $z(x^*)^+$, respectively. A *unary polynomial* of S is a polynomial with at most one variable. Their set is denoted by $\mathcal{P}_1(S)$.

If $\mathbf{t} = \mathbf{t}(x_1, x_2, \cdots, x_n)$ is a term or $\mathbf{p} = \mathbf{p}(x_1, x_2, \cdots, x_n)$ is a polynomial in the variables $x_1, x_2, \cdots, x_n$, and we substitute elements $s_1, s_2, \cdots, s_n$ of S^1 with $\{s_1, \cdots, s_n\} \cap S \neq \emptyset$ for the variables, then we can evaluate the expression so obtained in S^1. The result is an element of S which is denoted by $\mathbf{t}^S(s_1, s_2, \cdots, s_n)$ and $\mathbf{p}^S(s_1, s_2, \cdots, s_n)$, respectively. Notice that the evaluation is compatible with the interpretation of the substitution of $1 \in S^1$ for variables. The *polynomial function* of S corresponding to the polynomial $\mathbf{p}$ is the mapping

$$\mathbf{p}^S : S^n \to S, \ (s_1, s_2, \cdots, s_n) \mapsto \mathbf{p}(s_1, s_2, \cdots, s_n),$$

which is also denoted by $\mathbf{p}^S(x_1, x_2, \cdots, x_n)$.

An identity is a formal equality $\mathbf{t} = \mathbf{u}$ of two terms, considered with a common set of variables. A restriction semigroup *satisfies the identity* $\mathbf{t} = \mathbf{u}$ if

$$\mathbf{t}^S(s_1, s_2, \cdots, s_n) = \mathbf{u}^S(s_1, s_2, \cdots, s_n),$$

for any $s_1, s_2, \cdots, s_n \in S$.

Let τ be a relation on a restriction semigroup S. If $c, d \in S$ are such that

$$c = \mathbf{p}^S(a), \ d = \mathbf{p}^S(b),$$

for some $\mathbf{p} \in \mathcal{P}_1(S)$, where either (a, b) or (b, a) belongs to τ, we say that c is connected to d by a *polynomial τ-transition*, in notation, $c \overset{\mathbf{p}}{\to} d$. We denote by $\tau^\#$ the $(2, 1, 1)$-congruence on S generated by τ.

A well-known universal algebraic fact implies the following description, due to Szendrei (see [25]).

Lemma 2.1. *Let S be a restriction semigroup and τ a relation on S. Then for any $c, d \in S$, $c\tau^{\#}d$ if and only if $c = d$ or there is a sequence*

$$c = c_1 \xrightarrow{p} c_2 \xrightarrow{p} \cdots \xrightarrow{p} c_n = d$$

of polynomial τ-transitions.

3. Congruences

In this section, we need to obtain some characterizations of $(2,1,1)$-congruence on restricted semigroups. Let S be a restriction semigroup. For a $(2,1,1)$-congruence ρ on S, we have the restriction $\rho|_{P(S)}$ of ρ to $P(S)$ which is called the *projection trace* of ρ, denoted by $Ptr\rho$. It is easy to see that $Ptr\rho$ is a congruence on $P(S)$.

Definition 3.1. *A congruence τ on $P(S)$ is* projection-normal *if for any $e, f \in P(S)$ and $x \in S$, $(ex)^*\tau(fx)^*$ and $(xe)^+\tau(xf)^+$ whenever $e\tau f$.*

Corollary 3.1. *If ρ is a $(2,1,1)$-congruence on S, then $Ptr\rho$ is projection-normal.*

Proof. Let $e, f \in P(S)$ and $x \in S$. If $e\rho f$, then $ex\rho fx, xe\rho xf$, so that

$$(ex)^*\rho(fx)^*,\ (xe)^+\rho(xf)^+,$$

therefore $Ptr\rho$ is projection-normal. □

Lemma 3.1. *Let τ be a projection-normal congruence on $P(S)$ and $u, v \in S$. Then the following statements are equivalent:*

(i) *$u^*\tau v^*, ue = ve$ for some $e \in P(S), e\tau u^*$;*

(ii) *$u^+\tau v^+, fu = fv$ for some $f \in P(S), f\tau u^+$.*

Proof. (i)⇒(ii). Because S is a restriction semigroup, $ue = ve$ implies that $(ue)^+u = (ve)^+v$ and $(ue)^+ = (ve)^+$. And, by the normality of τ, $e\tau u^*$ implies that $(ue)^+\tau(uu^*)^+ = u^+$; similarly, $(ve)^+\tau v^+$. Together with the foregoing proof: $(ue)^+ = (ve)^+$, we have $u^+\tau v^+$ and (ii) holds.

(ii)⇒(i). It is similar as (i)⇒(ii). □

Proposition 3.1. *For a projection-normal congruence τ on $P(S)$, the relation*

$$\tau_{\min} = \{(u,v) \in S \times S : u^*\tau v^*, ue = ve \textit{ for some } e \in P(S), e\tau u^*\}$$

is the smallest $(2,1,1)$-congruence on S such that $Ptr\tau_{\min} = \tau$.

Proof. It is routine to check that $\tau_{\min}$ is an equivalence relation. Let $u, v, t \in S$ with $(u, v) \in \tau_{\min}$, then $u^*\tau\ v^*, ue = ve$ for some $e \in P(S)$ and $e\tau\ u^*$, so that $tue = tve$. Moreover,

$$(tu)^* = (tu)^*u^*\tau(tu)^*e = (tue)^* = (tve)^* = (tv)^*e$$

and $(tv)^* = (tv)^*v^*\tau(tv)^*e$. Therefore, $(tu)^*\tau(tv)^*$. Notice that

$$(tu)^*e = (tue)^* = (tve)^* = (tv)^*e,$$

we observe that

$$(tu)(tu)^*e = tue = tve = (tv)(tv)^*e = (tv)(tu)^*e.$$

Together with $(tu)^*e \in P(S)$, we have now proved that $(tu, tv) \in \tau_{\min}$. On the other side, we have

$$ue = ve \Rightarrow uet = vet \Rightarrow ut(et)^* = vt(et)^*.$$

By the normality of τ, $e\tau u^*$ implies that $(et)^*\tau(u^*t)^* = (ut)^*$, so that $(et)^*\tau(ut)^*$. Similarly, $(et)^*\tau(vt)^*$. Therefore $(ut)^*\tau(vt)^*$. We have now proved that $(ut, vt) \in \tau_{\min}$. Therefore, $\tau_{\min}$ is congruence.

Also, $(u^*)^* = u^*\tau v^* = (v^*)^*$, $u^*e = (ue)^* = (ve)^* = u^*e$ and $e\tau u^* = (u^*)^*$. By definition, these three formula can derive that $u^*\tau_{\min}v^*$. Similarly, by Lemma 3.1, $u^+\tau_{\min}v^+$. Consequently, $\tau_{\min}$ is indeed a $(2, 1, 1)$-congruence.

For any $e, f \in P(S)$, if $e\tau f$, then by the normality of τ, $(eu)^*\tau(fu)^*$ and $(ue)^+\tau(uf)^+$. Notice that $ef\tau e$ and $eef = fef$, we can observe that $e\tau_{\min}f$. Conversely, if $e\tau_{\min}f$ then by definition, $e\tau f$. Hence, $Ptr\tau_{\min} = \tau$.

Suppose now that ρ is a $(2, 1, 1)$-congruence on S such that $\mathrm{Ptr}\rho = \tau$, and $(u, v) \in \tau_{\min}$ for some $u, v \in S$, then $u^*\tau v^*, ue = ve$ for some $e \in P(S), e\tau u^*$. It follows that $(u^*, e), (v^*, e) \in \rho$. Therefore, $u = uu^*\rho ue = ve\rho vv^* = v$. Hence $\tau_{\min} \subseteq \rho$ and $\tau_{\min}$ is the smallest $(2, 1, 1)$-congruence on S such that $Ptr\tau_{\min} = \tau$. □

By Lemma 3.1, the following corollary is an immediate consequence of Proposition 3.1.

Corollary 3.2. *The congruence* $\tau_{\min}$ *of Proposition 3.4 has also the following from:*

$$\tau_{\min} = \{(u, v) \in S \times S : u^+\tau v^+, fu = fv \textit{ for some } f \in P(S), f\tau u^+\}.$$

By a *projection separating* $(2, 1, 1)$*-congruence* on S, we mean a $(2, 1, 1)$-congruence ρ on S in which for any $e, f \in P(S)$, if $e\rho f$, then $e = f$. Gould [11] pointed out that for a restriction semigroup S, the relation

$$\begin{aligned} \mu_S &= \{(u, v) \in S \times S : u^+ = v^+ \text{ and } (eu)^* = (ev)^* \text{ for all } e \in P(S)\} \\ &= \{(u, v) \in S \times S : u^* = v^* \text{ and } (uf)^+ = (vf)^+ \text{ for all } f \in P(S)\} \end{aligned}$$

is the greatest projection separating $(2,1,1)$-congruence on S and $\mu_S \subseteq \mathfrak{C}$. Sometime, we write also μ_S as $\mu(S)$. By definition, a $(2,1,1)$-congruence ρ on S is projection-separating if and only if $Ptr\rho = id_{P(S)}$ where $id_{P(S)}$ denotes the identity relation on $P(S)$.

For a projection-normal congruence τ on $P(S)$, we define

$$\tau_{\max} = \{(u,v) \in S \times S : (eu)^*\tau(ev)^* \text{ and } (ue)^+\tau(ve)^+ \text{ for any } e \in P(S)\}.$$

Lemma 3.2. *Let τ be a projection-normal congruence on $P(S)$. Then for any $u, v \in S$, the following statements are equivalent:*

(i) $(u,v) \in \tau_{\max}$*;*

(ii) $(eu)^*\tau(fv)^*$ *and* $(ue)^+\tau(vf)^+$*, for any* $e, f \in P(S)$ *with* $e\tau f$*;*

(iii) $(u\tau_{\min}, v\tau_{\min}) \in \mu_{S/\tau_{\min}}$.

Proof. (i)⇒(ii). For any $e, f \in P(S)$ with $e\tau f$, we have $(ev)^*\tau(fv)^*$ by normality. If $(u,v) \in \tau_{\max}$, then $(eu)^*\tau(ev)^*$ so that $(eu)^*\tau(fv)^*$; similarly, $(ue)^+\tau(vf)^+$.

(ii)⇒(i). It is clear.

(i)⇔(iii). It follows from the following implications:

$$\begin{aligned}
(u,v) \in \tau_{\max} &\Leftrightarrow (eu)^*\tau(ev)^* \text{ and } (ue)^+\tau(ve)^+ \text{ for any } e \in P(S);\\
&\Leftrightarrow (eu)^*\tau_{\min} = (ev)^*\tau_{\min} \text{ and } (ue)^+\tau_{\min} = (ve)^+\tau_{\min}\\
&\qquad \text{for any } e \in P(S);\\
&\Leftrightarrow ((eu)\tau_{\min})^* = ((ev)^{\tau}_{\min})^* \text{ and } ((ue)\tau_{\min})^+ = ((ve)\tau_{\min})^+\\
&\qquad \text{for any } e \in P(S);\\
&\Leftrightarrow (e\tau_{\min} \cdot u\tau_{\min})^* = (e\tau_{\min} \cdot b\tau_{\min})^* \text{ and}\\
&\qquad (u\tau_{\min} \cdot e\tau_{\min})^+ = (v\tau_{\min} \cdot e\tau_{\min})^+ \text{ for all } e \in P(S);\\
&\Leftrightarrow (u\tau_{\min}, v\tau_{\min}) \in \mu(S/\tau_{\min}).
\end{aligned}$$

We complete the proof. □

Proposition 3.2. *Let τ be a projection-normal congruence on $P(S)$. Then, $\tau_{\max}$ is the greatest* $(2,1,1)$*-congruence on S such that $Ptr\tau_{\max} = \tau$.*

Proof. It is routine to check that $\tau_{\max}$ is an equivalence relation. Let $u, v, t \in S$ with $(u,v) \in \tau_{\max}, e \in P(S)$. Then $(eu)^*\tau(ev)^*$ and by the normality of τ, it follows that

$$(eut)^* = ((eu)^*t)^*\tau((ev)^*t)^* = (evt)^*.$$

Notice that $(te)^\dagger \in P(S)$, we have

$$(ute)^+ = (u(te)^+)^+\tau(v(te)^+)^+ = (vte)^+.$$

Therefore, $(ut, vt) \in \tau_{\max}$; similarly, $(tu, tv) \in \tau_{\max}$. Hence, $\tau_{\max}$ is a congruence.

It is obvious that $\tau \subseteq \tau_{\max}$. Let $e, f, g \in P(S)$. If $f\tau_{\max}g$, then $ef\tau eg$. Take in turn $e = f$ and $e = g$ to get $f\tau fg, gf\tau g$. As $fg = gf$, now $f\tau g$. Thus, $Ptr\tau_{\max} = \tau$.

If $(u, v) \in \tau_{\max}$, then $(u\tau_{\min}, v\tau_{\min}) \in \mu(S/\tau_{\min})$. But, $\mu(S/\tau_{\min})$ and $\tau_{\min}$ are $(2,1,1)$-congruence, so $(u^*\tau_{\min}, v^*\tau_{\min}) \in \mu(S/\tau_{\min})$ and $(u^+\tau_{\min}, v^+\tau_{\min}) \in \mu(S/\tau_{\min})$. By Lemma 3.2, these show that $(u^*, v^*) \in \tau_{\max}$ and $(u^+, v^+) \in \tau_{\max}$. Therefore, $\tau_{\max}$ is a $(2,1,1)$-congruence.

Finally, we let ρ be a $(2,1,1)$-congruence on S such that $Ptr\rho = \tau$. If $(u, v) \in \rho$, then for any $e \in P(S)$, $(eu, ev) \in \rho$ and $(ue, ve) \in \rho$. It follows that $((eu)^*, (ev)^*), ((ue)^+, (ve)^+) \in \rho$. Thus $(eu)^*\tau(ev)^*, (ue)^+\tau(ve)^+$. Hence $\rho \subseteq \tau_{\max}$ and the proof is completed. □

In what follows, we call a $(2,1,1)$-congrunce ρ on S a *reduced* $(2,1,1)$-*congruence* if S/ρ is a reduced restriction monoid. The following proposition gives a characterization of reduced $(2,1,1)$-congruences.

Proposition 3.3. *Let ρ be a $(2,1,1)$-congruence on S. Then ρ is a reduced $(2,1,1)$-congruence on S if and only if $Ptr\rho = P(S) \times P(S)$.*

Proof. Suppose that ρ is a reduced $(2,1,1)$-congruence on S, then S/ρ is a reduced restriction monoid. This means that $|P(S/\rho)| = 1$. Obviously, for any $e, f \in P(S)$, $e\rho = f\rho$. Thus $P(S) \times P(S) \subseteq Ptr\rho$. Hence $Ptr\rho = P(S) \times P(S)$.

Conversely, suppose that $Ptr\rho = P(S) \times P(S)$, then for any $e, f \in P(S)$, $e\rho = f\rho$. This shows that $|\{e\rho : e \in P(S)\}| = 1$. On the other hand, if $a\rho$ $(a \in S)$ is a projection of S/ρ, then as ρ is a $(2,1,1)$-congruence on S, $a\rho = (a\rho)^+ = a^+\rho$. So, $P(S/\rho) = \{e\rho : e \in P(S)\}$. Therefore $|P(S/\rho)| = 1$, and so S$/\rho$ is a reduced restriction monoid. Hence ρ is a reduced $(2,1,1)$-congruence on S. □

Denote $\omega = P(S) \times P(S)$. It is obvious that ω is a normal congruence on $P(S)$. So, by Proposition 3.3, $\omega_{\min}$ and $\omega_{\max}$ are both reduced $(2,1,1)$-congruences. Again by Propositions 3.1 and 3.2, we have the following corollary.

Corollary 3.3. *Let S be a restriction semigroup. Then*

(i) $\omega_{\min}$ *is the smallest reduced* $(2,1,1)$*-congruence on* S*;*

(ii) $\omega_{\max} = S \times S$.

Evidently, the identity relation Δ on $P(S)$ is a normal congruence on $P(S)$. It is not difficult to see that for a restriction semigroup S, we have

(i) $\Delta_{\min}$ is the identity relation on S;

(ii) $\Delta_{\max} = \mu_S$.

Proposition 3.4. *Let S be a restriction semigroup. If ρ is a $(2,1,1)$-congruence on S, then $P(S/\rho) = \{e\rho : e \in P(S)\}$.*

Proof. Obviously, $\{e\rho : e \in P(S)\} \subseteq P(S/\rho)$. If $a\rho$ $(a \in S)$ is a projection of S/ρ, then $a\rho = (a\rho)^+ = a^+\rho$, so that $P(S/\rho) \subseteq \{e\rho : e \in P(S)\}$. Therefore, $P(S/\rho) = \{e\rho : e \in P(S)\}$. □

4. H-reduced restriction semigroups

In this section, we give the definition of H-reduced restricted semigroups.

Definition 4.1. *A semigroup S is an* H-reduced restriction semigroup *if*

(i) *S is not a reduced restriction monoid;*

(ii) *$|S| \geq 2$;*

(iii) *any $(2,1,1)$-congruence ρ on S is either the identical relation or a reduced $(2,1,1)$-congruence.*

Notice that a restriction semigroup is reduced if and only if its set of projections is a singleton. So, it is easy to know that for any H-reduced restriction semigroup S, we have always $|P(S)| \geq 2$.

By a 0-$\mathcal{J}^*$*-simple semigroup*, we mean a semigroup with zero element 0 and satisfying the conditions as follows:

(i) $S^2 \neq \{0\}$;

(ii) S and $\{0\}$ are the only $*$-ideals of S.

And, we call a 0-$\mathcal{J}^*$-simple semigroup having no zero element to be a $\mathcal{J}^*$*-simple semigroup*. Equivalently, a semigroup S with zero element is 0-$\mathcal{J}^*$-simple if and only if $S^2 \neq \{0\}$ and

$$\mathcal{J}^* = \{(0,0)\} \sqcup (S\backslash\{0\}) \times (S\backslash\{0\});$$

if and only if $S^2 \neq \{0\}$ and $a\mathcal{J}^*b$ for any nonzero elements a, b of S. Also, it is easy to see that a semigroup is $\mathcal{J}^*$-simple if and only if $\mathcal{J}^*$ is the universal relation on S.

Take after Gould, we call a restriction semigroup S to be *fundamental* if the maximum projection-separating $(2,1,1)$-congruence μ is the identity relation. In [11], Gould proved that any fundamental restriction semigroup is isomorphic to some full $(2,1,1)$-subsemigroup of the Munn semigroup on its projection semilattice. According to a result of Fountain in [6], any full subsemigroup of an inverse semigroup must be an ample semigroup. Because any Munn semigroup is an inverse semigroup, this shows that any fundamental restriction semigroup is always an ample semigroup.

By Definition 4.1, we have the following corollary.

Corollary 4.1. *Any H-reduced restriction semigroup is a 0-$\mathcal{J}^*$-simple ample semigroup which is fundamental.*

Proof. Let S be an H-reduced restriction semigroup. If the projection separating $(2,1,1)$-congruence μ_S is not the identity relation, then μ_S is a reduced $(2,1,1)$-congruence, and by Proposition 3.4, $|P(S/\mu_S)| = |\{e\mu_S : e \in P(S)\}|$. But μ_S is projection-separating, so $|\{e\mu_S : e \in P(S)\}| = |P(S)|$. Therefore $1 = |P(S/\mu_S)| = |P(S)|$, so that $P(S)$ is a singleton. It follows that S is a reduced restriction semigroup, contrary to Definition 4.1. Thus μ_S is the identity relation on S, so that S is a fundamental restriction semigroup. Now by the foregoing arguments before Corollary 4.1, S is an ample semigroup.

Now let U be a $*$-ideal of S and $U \neq S$. Then by [14, Lemma 2.2], the Rees congruence $R_U := U \times U \sqcup id_S$ is a $(2,1,1)$-congruence on S, where id_S is the identity relation on S.

(i) When the Rees congruence R_U is the identity relation. In this case, $U = \{0\}$.

(ii) If R_U is not the identity relation, then by hypothesis, R_U is a reduced $(2,1,1)$-congruence, and so S/R_U is a trivial semigroup, since S/R_U is a restriction semigroup with zero element and the projection set of a reduced restriction semigroup is a singleton. Therefore $U = S$.

However S has only two $*$-ideals: $\{0\}$ and S. This means that S is a 0-$\mathcal{J}^*$-simple semigroup. □

We arrive now at the main result of this section.

Theorem 4.1. *Let S be a restriction semigroup such that $|P(S)| > 1$. Then S is an H-reduced restriction semigroup if and only if the following statements hold:*

(FA) *S is a fundamental ample semigroup;*

(HR) *for any $e, f, h \in P(S)$ with $e > f$, there is a sequence*

$$e = e_1 \xrightarrow{p} e_2 \xrightarrow{p} \cdots \xrightarrow{p} e_n = h$$

of polynomial τ-transitions, where

(i) $e_1, e_2, \cdots, e_n \in P(S)$;

(ii) $\tau = \{(e, f)\}$.

Proof. Suppose that Conditions (FA) and (HR) hold. Let ρ be a $(2,1,1)$-congruence on S, and $\rho \neq S \times S$. We consider the following two cases:

(1) If $Ptr\rho = id_{P(S)}$, then ρ is a projection-separating $(2,1,1)$-congruence, so $\rho \subseteq \mu_S$. But S is fundamental, then $\mu_S = id_S$ and thus ρ is the identity congruence on S.

(2) Assume that $Ptr\rho \neq id_{P(S)}$. Then there is $e, h \in P(S)$ such that $e \neq h$ and $(e, h) \in \rho$. It follows that $(e, eh) \in \rho$.

  (a) If $e = eh$, then $e < h$. Now by Lemma 2.1, Condition (HR) implies that for any $g \in P(S)$, $(h, g) \in \tau^{\#}$ where $\tau = \{(h, e)\}$. But $\tau \subseteq \rho$, so $\tau^{\#} \subseteq \rho$. Accordingly, $(g, h) \in \rho$. This means that $P(S) \times P(S) \subseteq \rho$. Now by Proposition 3.3, ρ is a reduced $(2, 1, 1)$-congruence on S.

  (b) Assume that $e \neq eh$. We have that $eh < e$. Applying the arguments on e, h to e, eh, we can get that ρ is a reduced $(2, 1, 1)$-congruence on S.

Consequently, S is an H-reduced restriction semigroup.

Conversely, suppose that S is an H-reduced restriction semigroup. Notice that μ_S is a $(2, 1, 1)$-congruence on S. By hypothesis, $\mu_S = id_S$ or μ_S is reduced.

(A) If the first case holds, then S is a fundamental restriction semigroup. So, S is isomorphic to a full subsemigroup of the Munn semigroup on $P(S)$. But the Munn semigroup is an inverse semigroup, so any full subsemigroup of the Munn semigroup is always an ample semigroup. Therefore S is a fundamental ample semigroup.

(B) If the second case is true, then $Ptr\mu_S$ is the universal relation on $P(S)$. But μ_S is projection-separating, so $|P(S)| = 1$, contrary to the hypothesis that $|P(S)| \geq 2$.

However, S is a fundamental ample semigroup.

Let $e, f, h \in P(S)$ be such that $e > f$. Consider the relation $\tau = \{(e, f)\}$ on S, we know easily that $\tau^{\#}$ is not the identity on S. By definition, $\tau^{\#}$ is a reduced $(2, 1, 1)$-congruence on S. It follows that $(e, h) \in \tau^{\#}$. Now by Lemma 2.1, there is a sequence

$$e = c_1 \xrightarrow{p} c_2 \xrightarrow{p} \cdots \xrightarrow{p} c_n = h$$

of polynomial τ-transitions. Let $p_i \in \mathcal{P}_1(S)$ with $i = 1, 2, \cdots, n$ and such that

$$\begin{aligned} c_1 =& p_1^S(a_1), p_1^S(b_1) = c_2 = p_2^S(a_2), p_2^S(b_2) = c_3 = p_3^S(a_3), \cdots \\ & p_{n-1}^S(b_{n-1}) = c_{n-1} = p_n^S(a_n), p_n^S(b_n) = c_n, \end{aligned} \tag{4.1}$$

where either (a_i, b_i) or (b_i, a_i) belong to τ. Now let $q_i(x) = (p_i(x))^+$. Obviously, $q_i(x) \in \mathcal{P}_1(S)$. Notice that $e = e^+ = c_1^+$ and $h = h^+ = c_n^+$. By (4.1), we can obtain that

$$\begin{aligned} c_1^+ =& (p_1^S(a_1))^+, (p_1^S(b_1))^+ = c_2^+ = (p_2^S(a_2))^+, (p_2^S(b_2))^+ = c_3^+ = (p_3^S(a_3))^+, \cdots \\ & (p_{n-1}^S(b_{n-1}))^+ = c_{n-1}^+ = (p_n^S(a_n))^+, (p_n^S(b_n))^+ = c_n^+; \end{aligned}$$

that is,

$$\begin{aligned} e = c_1^+ =& q_1^S(a_1), q_1^S(b_1) = c_2^+ = q_2^S(a_2), q_2^S(b_2) = c_3^+ = q_3^S(a_3), \cdots \\ & q_{n-1}^S(b_{n-1}) = c_{n-1}^+ = q_n^S(a_n), q_n^S(b_n) = c_n^+ = h, \end{aligned} \tag{4.2}$$

where $q_k^S(x) = (p_k^+(x))^+ \in \mathcal{P}_1(S)$ for $k = 1, 2, \cdots, n$. It results Condition (HR). $\square$

By a *proper congruence* on S, we mean a congruence ρ on S with $\rho \neq S \times S$.

Let S be an inverse semigroup. It is easy to see that any congruence on S is always a $(2,1,1)$-congruence on S. Notice that for any congruence ρ on S, ρ is a group congruence on S if and only if $E(S) \times E(S) \subseteq \rho$. We can observe that ρ is a group congruence if and only if ρ is reduced. Now, the following corollary is an immediate consequence of Theorem 4.1, which is essentially the main result in [26].

Corollary 4.2. *Let S be an inverse semigroup which is not a group. Then every proper congruence of S is a group congruence if and only if S is a fundamental inverse semigroup satisfying Condition (HR).*

5. Congruence-free restriction semigroups

In this section, we shall discuss congruence-free restriction semigroups.

Definition 5.1. *A restriction semigroup S is* congruence-free *if any* $(2,1,1)$-*congruence on S is either the universal congruence or the identity congruence.*

Let S be a congruence-free restriction semigroup. Notice that the universal relation is a reduced restriction monoid. By definition, any $(2,1,1)$-congruence on a congruence-free restriction semigroup S is either the identity relation or a reduced $(2,1,1)$-congruence. So, S is either a reduced restriction semigroup or an H-reduced restriction semigroup. On the other hand, also by definition, the greatest projection separating $(2,1,1)$-congruence μ_S is the identity relation on S. So, S is a fundamental restriction semigroup. Furthermore, S is a full $(2,1,1)$-subsemigroup of the Munn semigroup on $P(S)$, so that S is an ample semigroup. Assume, in addition, that S is a reduced restriction semigroup. Obviously, S is a monoid with identity 1. Consider that an ample semigroup may be viewed as a restriction semigroup in which for any element a, $a^+\mathcal{R}^*a\mathcal{L}^*a^*$, this shows that for any $a \in S$, $a\mathcal{H}^*1$. That is, S is an $\mathcal{H}^*$-class containing an idempotent 1. By a result of Fountain in [7], S is a cancellative monoid. Therefore we have the following corollary.

Corollary 5.1. *If S is a congruence-free restriction semigroup, then S is either a cancellative monoid or an H-reduced restriction semigroup.*

Lemma 5.1. *Let S be a restriction semigroup. Then every (2, 1, 1)-congruence on S is either a projection separating* $(2,1,1)$-*congruence or a reduced* $(2,1,1)$-*congruence if and only if S satisfies Condition* (HR).

Proof. Suppose that S satisfies (HR). Indeed, in the proof of the sufficiency of Theorem 4.1, we have proved that any proper $(2,1,1)$-congruence on S is either projection-separating or reduced. It results the sufficiency.

Conversely, suppose that every $(2,1,1)$-congruence on S is either a projection-separating $(2,1,1)$-congruence or a reduce $(2,1,1)$-congruence. For $e, f, g \in P(S)$ with $e > f$, we consider the relation $\tau = \{e, f\}$. It is easy to see that $\tau^{\#}$ is not a projection-separating $(2,1,1)$-congruence on S, since the projection trace of a projection-separating congruence on S is the identity relation on $P(S)$. Furthermore, $\tau^{\#}$ is a reduced $(2,1,1)$-congruence on S. Again by the proof of the necessity of Theorem 4.1, we may obtain that S satisfies Condition (HR). The proof is finished. □

Lemma 5.2. *Let T be a cancellative monoid with identity 1. Then T is a congruence-free restriction semigroup if and only if T is a simple group.*

Proof. Suppose that T is a congruence-free restriction semigroup, and denote by $U(T)$ the set of all units of T. Then $U(T)$ is a subgroup of T, and $T \backslash U(T)$ is an ideal of T. It is easy to see that $\rho = (T \backslash U(T)) \times (T \backslash U(T)) \sqcup id_{U(T)}$ is a $(2,1,1)$-congruence on T. But T is congruence-free, so ρ is the identity relation on T. It follows that $T \backslash U(T)$ is the zero element of T. This means that $T = U(T)^0$ (the semigroup obtained from $U(T)$ by adjoining a zero). Thus $T = U(T)$ since T is cancellative. Moreover by [19, Proposition 8.2 (i), p.32], T is a simple group.

Conversely, by [19, Proposition 8.2 (i), p.32], it is clear that a simple group is a congruence-free restriction semigroup. □

We now give the main result of this section.

Theorem 5.1. *A semigroup S is a congruence-free restriction semigroup if and only if S is either a simple group or an H-reduced restriction semigroup without nontrivial reduced $(2,1,1)$-congruences.*

Proof. Suppose that S is congruence-free. By Corollary 5.1, S is either a cancellative monoid or an H-reduced restriction semigroup. If S is a cancellative monoid, then by Lemma 5.2, S is a simple group. If S is an H-reduced restriction semigroup, then any $(2,1,1)$-congruence on S is either the identity relation or a reduced $(2,1,1)$-congruence (including the universal relation), so that S has no nontrivial reduced $(2,1,1)$-congruences.

Conversely, if S is an H-reduced restriction semigroup without nontrivial reduced $(2,1,1)$-congruences, then S has only the identity relation and the universal relation. It follows that S is congruence-free. Assume that S is a simple group. By [19, Proposition 8.2 (i), p.32], any congruence on S is of the form: $\rho_N = \{(g,h) \in S \times S : gh^{-1} \in N\}$ where N is a normal subgroup of S. This shows that S is congruence-free. □

By definition, a restriction semigroup is inverse if and only if it is regular. The following corollary is an easy consequence of Theorem 5.1 and essentially the main result of Munn in [22].

Corollary 5.2. *A semigroup S is a congruence-free inverse semigroup if and only if S is either a simple group or a fundamental inverse semigroup satisfying Condition* (HR) *and without nontrivial group congruences.*

The following example is due to Tucci; for detail, see [26].

Example 5.1. Let $\mathbb{N}$ be the set of all non-negative integers. On $S = \mathbb{N} \times \mathbb{N}$, define a multiplication by

$$(m, n)(p, q) = (m - n + \max(n, p), q - p + \max(n, p)).$$

It is well known that under the above multiplication, S is an inverse semigroup. Indeed, S is the bicyclic semigroup. By [26, Corollary 7], S is a congruence-free restriction semigroup.

6. Conclusion

With the development of semigroup theory, restriction semigroups have become a hot topic in semigroup theory. This paper is based on Tucci 's inverse semigroups all of whose proper homomorphic images are groups in [26]. Moreover, EI Qallali's results in [4] on congruences on an ample semigroups give us great inspiration. In this paper, we discuss the properties of some congruences on restriction semigroups and obtain the classification of congruence-free restriction semigroups. Finally, we hope these conclusions will be helpful to the study of restriction semigroups.

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