Hedges in quasi-pseudo-MV algebras

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Abstract. In this paper, we introduce the notions of multiplicative interior operators (mi-operators, for short), additive closure operators (ac-operators, for short) and hedges in quasi-pseudo-MV algebras which will generalize the related contents in pseudo-MV algebras. First we discuss the relationship between mi-operators and ac-operators in a quasi-pseudo-MV algebra and investigate the properties of mi-operators in quasi-pseudo-MV algebras. Second we define and study hedges in quasi-pseudo-MV algebras. We also show that mi-operators are hedges. Finally, the properties of filters and weak filters in a quasi-pseudo-MV algebra with hedge are discussed.
Keywords: quasi-pseudo-MV algebras, Hedges, multiplicative interior operators, filters.

1. Introduction

Quasi-pseudo-MV algebras (*qp*MV-algebras, for short) were introduced in [4] as the generalizations of both pseudo-MV algebras [9] and quasi-MV algebras [11]. Considering that *qp*MV-algebras may play an important role in studying many-valued fuzzy logic and quantum computational logic, many properties of *qp*MV-algebras have been investigated in [3, 4, 5, 6, 7].

The notions of hedges were defined as operators acting on fuzzy subsets by Zadeh in order to describe linguistic hedges such as “very”, “more or less”, “much”, and so on [18]. In his paper, some examples were given to handle how to define hedges as operators. However, any sort of axiomatization was not considered. In [1], authors defined a hedge as operator on a complete lattice. The

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hedge in their definition is a mapping f which needs satisfy four axioms: (1) $f(1) = 1$, (2) $f(x) \leq x$, (3) $f(x \to y) \leq f(x) \to f(y)$ and (4) $f(f(x)) = x$. Authors also pointed out that this definition of hedge was indeed a truth function of logical connective "very true". On the other hand, the concepts of very true operators (vt-operators, for short) were introduced by Hajek on BL-algebras and the algebraic structures were called BL_{vt}-algebras [10]. A vt-operator is a mapping f which also contains four axioms: (1) $f(1) = 1$, (2) $f(x) \leq x$, (3) $f(x \to y) \leq f(x) \to f(y)$ and (4$'$) $f(x \vee y) \leq f(x) \vee f(y)$. A comparison of these two definitions indicates that they have the same axioms except the last one. Hence, a mapping which satisfies the axioms (1)(2)(3) is called a weak vt-operator [13, 15]. Moreover, on any commutative residuated lattice, Liu and Wang defined a vt-operator which is a weak vt-operator with the axiom (4$'$) and a hedge which is a weak vt-operator with the axiom (4), respectively [13]. Consequently, the concepts of vt-operators and hedges had been extended to other logical algebras such as pseudo-MV algebras [12], basic algebras [2], MTL-algebras [17], equality algebras [16], pseudo-BCK algebras [8] and so on. We need to point out that although authors named after vt-operators on some algebras, these operators are defined to satisfy the axiom (4), in other words, they are indeed "hedges" following the idea in [13]. In addition, the notions of multiplicative interior operators and additive closure operators were introduced to MV-algebras as the generalizations of topological Boolean algebras [14]. Independent of their original motivation, any multiplicative interior operator is a hedge in an MV-algebra from the purely algebraic viewpoint. Thus, it is natural to ask whether the concepts of multiplicative interior operators and hedges can be generalized to a qpMV-algebra for the more general results and new applications.

In this paper, we introduce the notions of multiplicative interior operators, additive closure operators and hedges on a qpMV-algebra and investigate the new algebraic structure. The paper is organized as follows. In Section 2, we recall some definitions and results which will be used in what follows. In Section 3, we introduce the notions of multiplicative interior operators (mi-operators, for short) and additive closure operators (ac-operators, for short) in qpMV-algebras which will generalize the related contents in pseudo-MV algebras. We discuss the relationship between mi-operators and ac-operators and investigate some properties of mi-operators in qpMV-algebras. In Section 4, we define and study hedges in qpMV-algebras. We also show that mi-operators are hedges. The properties of filters and weak filters in a qpMV-algebra with hedge are discussed.

2. Preliminary

In this section, we recall some definitions and results which will be used in the following. We list the definition and the related properties of a quasi-pseudo-MV algebra and recall the hedges on residuated lattices.

Definition 2.1 ([4])**.** *An algebra* $\boldsymbol{A} = \langle A; \oplus, ^-, ^\sim, 0\rangle$ *of type* $\langle 2, 1, 1, 0\rangle$ *is called a* quasi-pseudo-MV algebra *(qpMV-algebra, for short), if it satisfies the following axioms, for any* $x, y, z \in A$,

$(QPMV1)$ $x \oplus (y \oplus z) = (x \oplus y) \oplus z$;
$(QPMV2)$ $0^- = 0^\sim$;
$(QPMV3)$ $x \oplus 0 = 0 \oplus x$;
$(QPMV4)$ $x \oplus 0^- = 0^- = 0^- \oplus x$;
$(QPMV5)$ $(x^- \oplus y^-)^\sim = (x^\sim \oplus y^\sim)^-$;
$(QPMV6)$ $x^{-\sim} = x = x^{\sim-}$;
$(QPMV7)$ $y \oplus (x^- \oplus y)^\sim = (y \oplus x^\sim)^- \oplus y = x \oplus (y^- \oplus x)^\sim = (x \oplus y^\sim)^- \oplus x$;
$(QPMV8)$ $(x \oplus 0)^- = x^- \oplus 0$ *and* $(x \oplus 0)^\sim = x^\sim \oplus 0$;
$(QPMV9)$ $x \oplus y \oplus 0 = x \oplus y$.

A *qp*MV-algebra in which the binary operation $\oplus$ is commutative and the unary operations $^-$ and $^\sim$ coincide, is a quasi-MV algebra (*q*MV-algebra, for short). On the other hand, a *qp*MV-algebra satisfying the axiom $x \oplus 0 = x$ is a pseudo-MV algebra (*ps*MV-algebra, for short).

On any *qp*MV-algebra $\mathbf{A}$, we can define some operations, for any $x, y \in A$:

$$\begin{aligned} x \odot y &= (x^- \oplus y^-)^\sim; \\ x \vee y &= x \oplus (y^- \oplus x)^\sim; \\ x \wedge y &= (x^- \vee y^-)^\sim; \\ x \to y &= x^- \oplus y; \\ x \rightsquigarrow y &= y \oplus x^\sim. \end{aligned}$$

We can also define a relation $x \leq y$ iff $x \vee y = y \oplus 0$, or equivalently, $x \wedge y = x \oplus 0$. This is a quasi-ordering relation [4]. Moreover, if $x \leq y$ and $y \leq x$, then $x \oplus 0 = y \oplus 0$.

Below we list some properties of these operations and the relation. The proofs can be seen in [3, 4].

Proposition 2.1. *Let* $\boldsymbol{A}$ *be a qpMV-algebra. Then, for any* $x, y, z \in A$,

(P1) $0 \oplus 0 = 0$ *and* $1 \oplus 0 = 1$;
(P2) $x \wedge y = (x \to y) \odot x = x \odot (x \rightsquigarrow y)$;
(P3) $1 \to x = 1 \rightsquigarrow x$;
(P4) $x \vee y = (x \vee y) \oplus 0 = (x \oplus 0) \vee y = x \vee (y \oplus 0)$,
$x \wedge y = (x \wedge y) \oplus 0 = (x \oplus 0) \wedge y = x \wedge (y \oplus 0)$;
(P5) $x \to y = (x \to y) \oplus 0 = (x \oplus 0) \to y = x \to (y \oplus 0)$,
$x \rightsquigarrow y = (x \rightsquigarrow y) \oplus 0 = (x \oplus 0) \rightsquigarrow y = x \rightsquigarrow (y \oplus 0)$;
(P6) $x \leq y$ *iff* $x \to y = 1$ *iff* $x \rightsquigarrow y = 1$;
(P7) $0 \leq x \leq 1$;
(P8) $x \leq 1 \to x$ *and* $1 \to x \leq x$;
(P9) $x \leq y$ *iff* $y^- \leq x^-$ *iff* $y^\sim \leq x^\sim$;
(P10) $x \to y \leq (y \to z) \rightsquigarrow (x \to z)$ *and* $x \rightsquigarrow y \leq (y \rightsquigarrow z) \to (x \rightsquigarrow z)$;
(P11) $x \to y \leq (z \to x) \to (z \to y)$ *and* $x \rightsquigarrow y \leq (z \rightsquigarrow x) \rightsquigarrow (z \rightsquigarrow y)$;

(P12) $x \le y \to x$ *and* $x \le y \rightsquigarrow x$;
(P13) $x \odot y \le z$ *iff* $x \le y \to z$ *iff* $y \le x \rightsquigarrow z$;
(P14) *If* $x \le y$, *then* $y \to z \le x \to z$ *and* $y \rightsquigarrow z \le x \rightsquigarrow z$;
(P15) *If* $x \le y$, *then* $z \to x \le z \to y$ *and* $z \rightsquigarrow x \le z \rightsquigarrow y$;
(P16) $(x \vee y)^- = x^- \wedge y^-$ *and* $(x \vee y)^\sim = x^\sim \wedge y^\sim$,
$(x \wedge y)^- = x^- \vee y^-$ *and* $(x \wedge y)^\sim = x^\sim \vee y^\sim$;
(P17) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$ *and* $(x \vee y) \rightsquigarrow z = (x \rightsquigarrow z) \wedge (y \rightsquigarrow z)$,
$z \to (x \vee y) = (z \to x) \vee (z \to y)$ *and* $z \rightsquigarrow (x \vee y) = (z \rightsquigarrow x) \vee (z \rightsquigarrow y)$;
(P18) $z \to (x \wedge y) = (z \to x) \wedge (z \to y)$ *and* $z \rightsquigarrow (x \wedge y) = (z \rightsquigarrow x) \wedge (z \rightsquigarrow y)$,
$(x \wedge y) \to z = (x \to z) \vee (y \to z)$ *and* $(x \wedge y) \rightsquigarrow z = (x \rightsquigarrow z) \vee (y \rightsquigarrow z)$.

Given that $\mathbf{A}$ is a *qp*MV-algebra and consider the set $R(A) = \{x \in A | x \oplus 0 = x\}$. Then, we have that $R(A)$ is a non-empty subset of A by Proposition 2.1 and elements in $R(A)$ are called *regular*. Moreover, $\mathbf{R_A} = \langle R(A); \oplus, ^-, ^\sim, 0\rangle$ is a pseudo-MV subalgebra of $\mathbf{A}$. We recall that a *qp*MV-algebra in which $0 = 1$ is called *flat*. Then, we can show the following result.

Theorem 2.1 ([4])**.** *For any qpMV-algebra $\boldsymbol{A}$, there exist a pseudo-MV algebra $\boldsymbol{M}$ and a flat qpMV-algebra $\boldsymbol{F}$ such that $\boldsymbol{A}$ can be embedded into the direct product $\boldsymbol{M} \times \boldsymbol{F}$.*

Let $\mathbf{A}$ be a *qp*MV-algebra. A non-empty subset F of A is called a *filter* of $\mathbf{A}$, if, for any $x, y \in A$, the following conditions are satisfied (F1) $1 \in F$; (F2) if $x, y \in F$, then $x \odot y \in F$; (F3) if $x \in F$ and $y \in A$ with $x \le y$, then $y \in F$. A non-empty subset F of A is called a *weak filter* of $\mathbf{A}$, if, for any $x, y \in A$, the following conditions are satisfied (WF1) $1 \in F$; (WF2) if $x, y \in F$, then $x \odot y \in F$; (WF3) if $x \in F$ and $y \in A$, then $y \oplus x \in F$ and $x \oplus y \in F$. Moreover, a (weak) filter F is called *normal*, if $x \to y \in F$ iff $x \rightsquigarrow y \in F$, for any $x, y \in A$. Finally, we recall that a congruence θ on $\mathbf{A}$ is called *filter congruence*, if $\langle x \odot 1, y \odot 1\rangle \in \theta$ can imply $\langle x, y\rangle \in \theta$, for any $x, y \in A$.

The filter is the dual notion of an ideal in any *qp*MV-algebra. In [6], we have proved that there exists a bijective correspondence between normal ideals and ideal congruences on a *qp*MV-algebra. On the basis of the proof, we can get the following result.

Theorem 2.2. *Let $\boldsymbol{A}$ be a qpMV-algebra, F be a normal filter of $\boldsymbol{A}$ and θ be a filter congruence on $\boldsymbol{A}$. Then,*

(1) $f(F) = \{\langle x, y\rangle \in A^2 | x \to y \in F \text{ and } y \to x \in F\}$ *is a filter congruence on $\boldsymbol{A}$;*

(2) $g(\theta) = \{x \in A | \langle x, 1\rangle \in \theta\}$ *is a normal filter of $\boldsymbol{A}$;*

(3) $g(f(F)) = F$;

(4) $f(g(\theta)) = \theta$.

3. Interior and closed operators

MV-algebras with multiplicative interior operators (interior MV-algebras) or additive closure operators (closure MV-algebras) were introduced in [14]. In fact, a multiplicative interior operator (or an additive closure operator) on an MV-algebra generalizes that of a topological interior operator (or closure operator) on a Boolean algebra. In this section, we generalize these notions to *qp*MV-algebras.

Definition 3.1. *Let* $\boldsymbol{A}$ *be a qpMV-algebra and* $f : A \to A$ *be a mapping. Then,* f *is called a* multiplicative interior operator *(*mi-operator*, for short) on* $\boldsymbol{A}$*, if the following conditions are satisfied, for any* $x, y \in A$*,*

$(MI1)$ $f(1) = 1$*;*

$(MI2)$ $f(x) \leq x$*;*

$(MI3)$ $f(x \odot y) = f(x) \odot f(y)$*;*

$(MI4)$ $f(f(x)) = f(x)$*.*

The pair $(\boldsymbol{A}, f)$ *is called an* interior *qp*MV-algebra*. For any* $x \in A$*, the element* $f(x)$ *is called the* interior *of* x*. An element* $x \in A$ *is called* open*, if* $f(x) = x$*.*

Similarly, we have the following definition.

Definition 3.2. *Let* $\boldsymbol{A}$ *be a qpMV-algebra and* $g : A \to A$ *be a mapping. Then,* g *is called an* additive closure operator *(*ac-operator*, for short) on* $\boldsymbol{A}$*, if the following conditions are satisfied, for any* $x, y \in A$*,*

$(AC1)$ $g(0) = 0$*;*

$(AC2)$ $x \leq g(x)$*;*

$(AC3)$ $g(x \oplus y) = g(x) \oplus g(y)$*;*

$(AC4)$ $g(g(x)) = g(x)$*.*

The pair $(\boldsymbol{A}, g)$ *is called a* closure *qp*MV-algebra*. For any* $x \in A$*, the element* $g(x)$ *is called the* closure *of* x*. An element* $x \in A$ *is called* closed*, if* $g(x) = x$*.*

Proposition 3.1. *Let* $\boldsymbol{A}$ *be a qpMV-algebra and* f *be an mi-operator on* $\boldsymbol{A}$*. Then, the mappings* $f_{\sim}^{-}$ *defined by* $f_{\sim}^{-}(x) = (f(x^{\sim}))^{-}$ *and* $f_{-}^{\sim}$ *defined by* $f_{-}^{\sim}(x) = (f(x^{-}))^{\sim}$*, for any* $x \in A$*, are ac-operators on* $\boldsymbol{A}$*.*

Proof. We only check the case of $f_{\sim}^{-}$. The other can be proved similarly.

(AC1) We have $f_{\sim}^{-}(0) = (f(0^{\sim}))^{-} = (f(1))^{-} = 1^{-} = 0$.

(AC2) Since $f_{\sim}^{-}(x) = (f(x^{\sim}))^{-}$ and $f(x^{\sim}) \leq x^{\sim}$ by (MI2), we have $x \leq f_{\sim}^{-}(x)$ by (P10).

(AC3) We have $f_{\sim}^{-}(x \oplus y) = (f((x \oplus y)^{\sim}))^{-} = (f(x^{\sim} \odot y^{\sim}))^{-} = (f(x^{\sim}) \odot f(y^{\sim}))^{-} = (f(x^{\sim}))^{-} \oplus (f(y^{\sim}))^{-} = f_{\sim}^{-}(x) \oplus f_{\sim}^{-}(y)$.

(AC4) We have $f_{\sim}^{-}(f_{\sim}^{-}(x)) = f_{\sim}^{-}((f(x^{\sim}))^{-}) = (f(f(x^{\sim})))^{-} = (f(x^{\sim}))^{-} = f_{\sim}^{-}(x)$. □

Dually, we get the following result.

Proposition 3.2. *Let* $\boldsymbol{A}$ *be a qpMV-algebra and* g *be an ac-operator on* $\boldsymbol{A}$*. Then, the mappings* $g_{\sim}^{-}$ *defined by* $g_{\sim}^{-}(x) = (g(x^{\sim}))^{-}$ *and* $g_{-}^{\sim}$ *defined by* $g_{-}^{\sim}(x) = (g(x^{-}))^{\sim}$*, for any* $x \in A$*, are mi-operators on* $\boldsymbol{A}$*.*

As shown above, there exist the corresponding relations between mi-operators and ac-operators on a qpMV-algebra, thus we will only discuss mi-operators in the rest.

Proposition 3.3. *Let $\boldsymbol{A}$ be a qpMV-algebra and f be an mi-operator on $\boldsymbol{A}$. Then, for any $x, y \in A$,*

(1) *f keeps regular elements, i.e., if $x = x \oplus 0$, then $f(x) = f(x) \oplus 0$;*

(2) *$f(0) = 0$;*

(3) *$f(x^-) \leq (f(x))^-$ and $f(x^\sim) \leq (f(x))^\sim$;*

(4) *If $x \leq y$, then $f(x) \leq f(y)$;*

(5) *$f(x \to y) \leq f(x) \to f(y)$ and $f(x \rightsquigarrow y) \leq f(x) \rightsquigarrow f(y)$.*

Proof. (1) Since $x = x \oplus 0$ iff $x = x \odot 1$, for any $x \in A$, we get the result by (MI3) and (MI1).

(2) Since $0 \leq f(0) \leq 0$ by (MI2), we have $f(0) \oplus 0 = 0 \oplus 0$. Note that 0 is a regular element, it turns out that $f(0) = 0$ by (1).

(3) By (MI2), we have $f(x) \leq x$, so $x^- \leq (f(x))^-$. Using (MI2) again, we have $f(x^-) \leq x^-$, it turns out that $f(x^-) \leq (f(x))^-$ by the transitivity. The other can be proved similarly.

(4) If $x \leq y$, then $x \odot 1 = x \wedge y = y \odot (y \rightsquigarrow x)$. On the one hand, we have $f(x \odot 1) = f(x) \odot f(1) = f(x) \odot 1 \geq f(x)$ by (MI3) and (MI1). On the other hand, we have $f(x \wedge y) = f(y \odot (y \rightsquigarrow x)) = f(y) \odot f(y \rightsquigarrow x) \leq f(y) \odot 1 \leq f(y)$. Hence, $f(x) \leq f(y)$.

(5) Since $(x \to y) \odot x = x \wedge y \leq y$, we have $f((x \to y) \odot x) = f(x \to y) \odot f(x) \leq f(y)$ by (MI3) and (4), so $f(x \to y) \leq f(x) \to f(y)$. The other can be proved similarly. $\square$

Proposition 3.4. *Let $\boldsymbol{A}$ be a qpMV-algebra and f be an mi-operator on $\boldsymbol{R_A}$. Then, f can be extended to an mi-operator on $\boldsymbol{A}$.*

Proof. For any $x \in A$, define $\bar{f}(x) = \begin{cases} f(x), & x \in R(A), \\ f(x \oplus 0), & x \in A \setminus R(A). \end{cases}$ Then, $\bar{f}$ is an mi-operator on $\mathbf{A}$. Indeed, $\bar{f}(1) = f(1) = 1$, so the condition (MI1) is true. Now, we check the conditions (MI2)-(MI4).

(MI2) If $x \in R(A)$, then $\bar{f}(x) = f(x) \leq x$. If $x \notin R(A)$, then $\bar{f}(x) = f(x \oplus 0) \leq x \oplus 0 \leq x$.

(MI3) If $x, y \in R(A)$, then $\bar{f}(x \odot y) = f(x \odot y) = f(x) \odot f(y) = \bar{f}(x) \odot \bar{f}(y)$. If $x \in R(A)$ and $y \notin R(A)$, then $\bar{f}(x \odot y) = f(x \odot y) = f(x \odot (y \oplus 0)) = f(x) \odot f(y \oplus 0) = \bar{f}(x) \odot \bar{f}(y)$. If $x \notin R(A)$ and $y \in R(A)$, the proof is similar as above. If $x, y \notin R(A)$, then $\bar{f}(x \odot y) = f(x \odot y) = f((x \oplus 0) \odot (y \oplus 0)) = f(x \oplus 0) \odot f(y \oplus 0) = \bar{f}(x) \odot \bar{f}(y)$.

(MI4) If $x \in R(A)$, then $\bar{f}(\bar{f}(x)) = f(f(x)) = f(x) = \bar{f}(x)$. If $x \notin R(A)$, then $\bar{f}(x) = f(x \oplus 0) = f(f(x \oplus 0))$ and $\bar{f}(\bar{f}(x)) = \bar{f}(f(x \oplus 0)) = f(f(x \oplus 0) \oplus 0) = f(f(x \oplus 0))$, so $\bar{f}(\bar{f}(x)) = \bar{f}(x)$. $\square$

In [14], authors showed that for a complete MV-algebra, every topological closure operator on the Boolean algebra of additively idempotent elements can be extended to a closure operator on the whole MV-algebra. Since the set of additively idempotent elements in a pseudo-MV algebra is also a Boolean algebra [9], we can extend the result to a complete pseudo-MV algebra. Suppose that $\mathbf{M} = \langle M; \oplus, ^-, ^\sim, 0, 1\rangle$ is a pseudo-MV algebra and denote $B(M) =$ the set of additive idempotent elements in M. Then, $\mathbb{B}(\mathbf{M}) = \langle B(M); \vee, \wedge, 0, 1\rangle$ is a Boolean algebra, where $x \vee y = x \oplus (y^- \oplus x)^\sim$ and $x \wedge y = (x^- \vee y^-)^\sim$, for any $x, y \in B(M)$.

Lemma 3.1. *Let $\boldsymbol{M}$ be a interior complete pseudo-MV algebra and f be a topological interior operator on $\mathbb{B}(\boldsymbol{M})$. Then, there is an mi-operator on $\boldsymbol{M}$ such that its restriction on $B(M)$ is equal to f.*

Proposition 3.5. *Let $\boldsymbol{A}$ be a qpMV-algebra and $\boldsymbol{R_A}$ be its interior complete pseudo-subalgebra of $\boldsymbol{A}$. If f is a topological interior operator on the Boolean algebra $\mathbb{B}(\boldsymbol{R_A})$, then there is an mi-operator on $\boldsymbol{A}$ such that its restriction on $B(R(A))$ is equal to f.*

Proof. Follows from Proposition 3.4 and Lemma 3.1. □

Proposition 3.6. *Let $\boldsymbol{A}$ be a qpMV-algebra and f_1, f_2 be mi-operators on $\boldsymbol{A}$. Then, $f_1 \le f_2$ iff $f_1 f_2|_{R(A)} = f_1|_{R(A)}$.*

Proof. Suppose that $f_1 \le f_2$. Then, for any $x \in A$, we have $f_1(x) \le f_2(x)$. By (MI4) and Proposition 3.3(4), $f_1(x) = f_1(f_1(x)) \le f_1(f_2(x)) = (f_1 f_2)(x)$. Meanwhile, since $f_2(x) \le x$, it follows that $(f_1 f_2)(x) = f_1(f_2(x)) \le f_1(x)$ using Proposition 3.3(4) again. Thus, $f_1(x) \oplus 0 = f_1 f_2(x) \oplus 0$. By Proposition 2.1(1), if $x \in R(A)$, then $(f_1 f_2)(x) = f_1(x)$, i.e., $f_1 f_2|_{R(A)} = f_1|_{R(A)}$. Conversely, if $f_1 f_2|_{R(A)} = f_1|_{R(A)}$, then, for any $x \in A$, we have $f_1(x) \le f_1(x \oplus 0) = (f_1 f_2)(x \oplus 0) \le f_2(x \oplus 0) \le f_2(x)$, so $f_1 \le f_2$. □

Following the proof of Proposition 3.6, we can get the result.

Proposition 3.7. *Let $\boldsymbol{A}$ be a qpMV-algebra and f_1, f_2 be isotone mappings on $\boldsymbol{A}$. If f_1 and f_2 restricted on $\boldsymbol{R_A}$ are mi-operators, then $f_1 \le f_2$ iff $f_1 f_2|_{R(A)} = f_1|_{R(A)}$.*

Proposition 3.8. *Let $\boldsymbol{A}$ be a qpMV-algebra and f_1, f_2 be isotone mappings on $\boldsymbol{A}$. If f_1 and f_2 restricted on $\boldsymbol{R_A}$ are mi-operators, then the following conditions are equivalent:*

(1) $f_1 f_2|_{R(A)} = f_2 f_1|_{R(A)}$;
(2) $f_1 f_2|_{R(A)}$ *and* $f_2 f_1|_{R(A)}$ *are mi-operators;*
(3) $f_1 f_2 f_1 f_2|_{R(A)} = f_1 f_2|_{R(A)}$ *and* $f_2 f_1 f_2 f_1|_{R(A)} = f_2 f_1|_{R(A)}$.

Proof. (1)⇒(2) For any $x \in R(A)$, we have $(f_1 f_2)(1) = 1$ and $(f_1 f_2)(x) \le f_2(x) \le x$. Moreover, $(f_1 f_2)(x \odot y) = f_1(f_2(x) \odot f_2(y)) = (f_1 f_2)(x) \odot (f_1 f_2)(y)$

and $(f_1f_2)((f_1f_2)(x)) = (f_1f_2f_1f_2)(x)) = (f_1f_1)(f_2f_2)(x) = (f_1f_2)(x)$. Hence, $f_1f_2|_{R(A)}$ is an mi-operator. The case of $f_2f_1|_{R(A)}$ can be proved similarly.

(2)⇒(3) Since $f_1f_2 \leq f_1f_2$ and $f_2f_1 \leq f_2f_1$, we have the result by Proposition 3.7.

(3)⇒(1) On the one hand, for any $x \in R(A)$, we have $(f_1f_2)(x) = (f_1f_2f_1f_2)(x) \leq (f_2f_1f_2)(x) \leq (f_2f_1)(x)$. On the other hand, for any $x \in R(A)$, we have $(f_2f_1)(x) = (f_2f_1f_2f_1)(x) \leq (f_1f_2f_1)(x) \leq (f_1f_2)(x)$. Hence, we get $f_1f_2|_{R(A)} = f_2f_1|_{R(A)}$. □

Let $\mathbf{A}$ be a qpMV-algebra and f be any mi-operator on $\mathbf{A}$. We denote the set of all open elements of A by $O_f(\mathbf{A}) = \{x \in A | f(x) = x\}$.

Theorem 3.1. *Let $\boldsymbol{A}$ be a qpMV-algebra and f_1, f_2 be mi-operators on $\boldsymbol{A}$. If $O_{f_1}(\boldsymbol{A}) = O_{f_2}(\boldsymbol{A})$, then $f_1|_{R(A)} = f_2|_{R(A)}$.*

Proof. For any $x \in A$, since $f_1(f_1(x)) = f_1(x)$, we have $f_1(x) \in O_{f_1}(\mathbf{A}) = O_{f_2}(\mathbf{A})$, it follows that $f_2(f_1(x)) = f_1(x)$. Similarly, we have $f_1(f_2(x)) = f_2(x)$. Since $f_1(x) \leq x$, we get $f_2(f_1(x)) \leq f_2(x)$, it turns out that $f_1(x) \leq f_2(x)$. Meanwhile, since $f_2(x) \leq x$, we get $f_1(f_2(x)) \leq f_1(x)$, it turns out that $f_2(x) \leq f_1(x)$. Hence, $f_1(x) \oplus 0 = f_2(x) \oplus 0$ which means that $f_1(x \oplus 0) = f_2(x \oplus 0)$ and then we get $f_1|_{R(A)} = f_2|_{R(A)}$. □

4. Hedges in quasi-pseudo-MV algebras

In this section, we introduce the notion of hedge in a qpMV-algebra and show some basic properties of it. We also investigate some properties of (weak) filters in qpMV-algebras with hedges and discuss the relationship between normal filters and filter congruences on qpMV-algebras with hedges.

Definition 4.1. *Let $\boldsymbol{A}$ be a qpMV-algebra and $h : A \to A$ be a mapping. Then, h is called a* weak hedge *in $\boldsymbol{A}$, if the following conditions are satisfied, for any $x, y \in A$,*

$(H1)$ $h(1) = 1$;

$(H2)$ $h(x) \leq x$;

$(H3)$ $h(x \to y) \leq h(x) \to h(y)$ and $h(x \rightsquigarrow y) \leq h(x) \rightsquigarrow h(y)$.

If a weak hedge h satisfies (H4) $h(h(x)) = h(x)$, then it is called a hedge *in $\boldsymbol{A}$. The pair $(\boldsymbol{A}, h)$ is called a* qpMV-algebra with hedge*. Moreover, if a hedge h keeps regular elements, then it is called a* strong hedge *in $\boldsymbol{A}$ and the pair $(\boldsymbol{A}, h)$ is called a* qpMV-algebra with strong hedge*.*

Example 4.1. Let $\mathbf{F}$ be a flat qpMV-algebra and $h : F \to F$ be a mapping satisfying $h(1) = 1$ and $h(x) \leq x$, for any $x \in F$. Then, h is a weak hedge in $\mathbf{F}$. In fact, since $1 = 0$ and $x \oplus y = x \oplus y \oplus 0 = x \oplus y \oplus 1 = 1$, we have that $x \to y$, $x \rightsquigarrow y$, $h(x) \to h(y)$ and $h(x) \rightsquigarrow h(y)$ are equal to 1. Hence, the condition (H3) is satisfied. Moreover, if the condition (H4) is also satisfied, then it is a hedge in $\mathbf{F}$ and also a strong hedge in $\mathbf{F}$.

Example 4.2. Let $\mathbf{A}$ be a qpMV-algebra. It is easy to see that the identity mapping $\mathbf{Id}_A$ is a hedge in $\mathbf{A}$. That is to say that any qpMV-algebra can be regarded as a qpMV-algebra with hedge.

Example 4.3. Let $\mathbf{A}$ be a qpMV-algebra and satisfy $x \leq y$ or $y \leq x$, for any $x, y \in A$. We define a mapping $h : A \to A$ by $h(1) = 1$ and $h(x) = 0$, for any $x < 1$. Then, h is a hedge in $\mathbf{A}$.

Example 4.4. Let $\mathbf{A}$ be a qpMV-algebra in which the operations are defined as follows:

$\oplus$	0	a	b	1
0	0	b	b	1
a	b	1	1	1
b	b	1	1	1
1	1	1	1	1

and

$'$	
0	1
a	a
b	b
1	0

. In fact, it is a quasi-MV algebra [11]. Define $h(1) = 1$, $h(0) = 0$, $h(a) = h(b) = a$. Then, h is a hedge in $\mathbf{A}$.

Example 4.5. Let $(\mathbf{M}, h_1)$ be a pseudo-MV algebra with hedge and $(\mathbf{F}, h_2)$ be a flat qpMV-algebra with hedge. Then, $\mathbf{M} \times \mathbf{F}$ is a qpMV-algebra. If we define $h(\langle x, y\rangle) = (h_1(x), h_2(y))$, for any $\langle x, y\rangle \in M \times F$, then h is a hedge in $\mathbf{M} \times \mathbf{F}$.

Remark 4.1. Following from Proposition 3.3, it is immediate to see that any mi-operator on a qpMV-algebra is a hedge. However, the converse is not true in general. In Example 4.4, we calculate $h(b \odot 1) = h(b \oplus 0) = a$ and $h(b) \odot h(1) = a \odot 1 = a \oplus 0 = b$, which imply that $h(b \odot 1) \neq h(b) \odot h(1)$, so h is not an mi-operator on $\mathbf{A}$.

Proposition 4.1. *Let $\boldsymbol{A}$ be a qpMV-algebra and h be a weak hedge in $\boldsymbol{A}$. Then, for any $x, y \in A$,*

(1) $h(0) \oplus 0 = 0$;
(2) *If* $h(x) = 1$, *then* $x \oplus 0 = 1$;
(3) *If* $x \leq y$, *then* $h(x) \leq h(y)$;
(4) *If* $h(x) \leq h(y)$, *then* $h(x) \leq y$;
(5) $h(x^-) \leq (h(x))^-$ *and* $h(x^\sim) \leq (h(x))^\sim$;
(6) $h(x) \odot h(y) \leq h(x \odot y)$;
(7) $h(x \oplus 0) \oplus 0 = h(x) \oplus 0$.
If h is a hedge in $\boldsymbol{A}$, then
(8) $h(x) \leq h(y)$ *iff* $h(x) \leq y$;
(9) $Im(h) = \mathrm{Fix}_h(A) = \{x \in A | h(x) = x\}$;
(10) *If h is surjective, then* $h = \mathrm{Id}_A$.

Proof. (1) By (H2), we have $h(0) \leq 0$. And $0 \leq h(0)$, it turns out that $h(0) \oplus 0 = 0$.

(2) Since $1 = h(x) \leq x$ and $x \leq 1$, we have $x \oplus 0 = 1$.

(3) Since $x \leq y$, we have $x \to y = 1$. By (H1) and (H3), we get $1 = h(1) = h(x \to y) \leq h(x) \to h(y)$. Note that, $h(x) \to h(y)$ is a regular element and $h(x) \to h(y) \leq 1$, we have $h(x) \to h(y) = 1$, so $h(x) \leq h(y)$.

(4) Since $h(x) \leq h(y)$ and $h(y) \leq y$ by (H2), we have $h(x) \leq y$.

(5) Since $x^- \leq (1 \to x)^- = (1 \to x) \to 0$, we have $h(x^-) \leq h((1 \to x) \to 0) \leq h(1 \to x) \to h(0) \leq h(x) \to h(0) = h(x) \to (h(0) \oplus 0) = h(x) \to 0 \leq (h(x))^-$ using (H3) and (1). The other can be proved similarly.

(6) Since $x \odot y \leq x \odot y$, we have $x \leq y \to (x \odot y)$, it turns out that $h(x) \leq h(y \to (x \odot y)) \leq h(y) \to h(x \odot y)$ by (3) and (H3), so $h(x) \odot h(y) \leq h(x \odot y)$.

(7) Since $x \leq x \oplus 0$ and $x \oplus 0 \leq x$, we have $h(x) \leq h(x \oplus 0)$ and $h(x \oplus 0) \leq h(x)$ by (3), so $h(x) \oplus 0 = h(x \oplus 0) \oplus 0$.

(8) If $h(x) \leq y$, then $h(x) = h(h(x)) \leq h(y)$ by (3). The converse follows from (4).

(9) For any $x \in Im(h)$, then there exists $a \in A$ such that $x = h(a)$, it follows that $h(x) = h(h(a)) = h(a) = x$, so $x \in \mathrm{Fix}_h(A)$. Conversely, for any $x \in \mathrm{Fix}_h(A)$, then $h(x) = x$, we have $x \in Im(h)$.

(10) If h is surjective, then $Im(h) = A = \mathrm{Fix}_h(A)$ by (9), it follows that $h(x) = x = \mathrm{Id}_A(x)$, for any $x \in A$, so $h = \mathrm{Id}_A$. $\square$

Proposition 4.2. *Let $\boldsymbol{A}$ be a qpMV-algebra and h be a hedge in $\boldsymbol{R_A}$. Then, h can be extended to a hedge in $\boldsymbol{A}$.*

Proof. For any $x \in A$, define $\bar{h}(x) = \begin{cases} h(x), & x \in R(A); \\ h(x \oplus 0), & x \in A \setminus R(A). \end{cases}$ Then, $\bar{h}$ is a hedge in $\mathbf{A}$. Indeed, $\bar{h}(1) = h(1) = 1$, so the condition (H1) is true. Now, we check the conditions (H2)-(H4).

(H2) For any $x \in A$, if $x \in R(A)$, then $\bar{h}(x) = h(x) \leq x$. If $x \notin R(A)$, then $\bar{h}(x) = h(x \oplus 0) \leq x \oplus 0 \leq x$. Hence, $\bar{h}(x) \leq x$.

(H3) We only prove the first one. The other can be proved similarly. If $x, y \in R(A)$, then $\bar{h}(x \to y) = h(x \to y) \leq h(x) \to h(y) = \bar{h}(x) \to \bar{h}(y)$. If $x \in R(A)$ and $y \notin R(A)$, then $\bar{h}(x \to y) = h(x \to y) = h(x \to (y \oplus 0)) \leq h(x) \to h(y \oplus 0) = \bar{h}(x) \to \bar{h}(y)$. If $x \notin R(A)$ and $y \in R(A)$, the proof is similar as above. If $x, y \notin R(A)$, then $\bar{h}(x \to y) = h(x \to y) = h((x \oplus 0) \to (y \oplus 0)) \leq h(x \oplus 0) \to h(y \oplus 0) = \bar{h}(x) \to \bar{h}(y)$.

(H4) If $x \in R(A)$, then $\bar{h}(\bar{h}(x)) = h(h(x)) = h(x) = \bar{h}(x)$. If $x \notin R(A)$, then $\bar{h}(x) = h(x \oplus 0) = h(h(x \oplus 0))$ and $\bar{h}\bar{h}(x) = \bar{h}(h(x \oplus 0)) = h(h(x \oplus 0) \oplus 0) = h(h(x \oplus 0))$, so $\bar{h}(\bar{h}(x)) = \bar{h}(x)$. $\square$

Definition 4.2. *Let $(\boldsymbol{A}, h)$ be a qpMV-algebra with hedge and F be a (weak) filter of $\boldsymbol{A}$. Then, F is called an* (weak) h-filter *of $(\boldsymbol{A}, h)$, if $h(F) \subseteq F$. In addition, if F is a (weak) h-filter of $(\boldsymbol{A}, h)$ and satisfies $x \to y \in F$ iff $x \rightsquigarrow y \in F$, for any $x, y \in A$, then it is called a* normal (weak) h-filter *of $(\boldsymbol{A}, h)$.*

Definition 4.3. *Let $(\boldsymbol{A}, h)$ be a qpMV-algebra with hedge and θ be a congruence on $\boldsymbol{A}$. Then, θ is called a* congruence *on $(\boldsymbol{A}, h)$, if $\langle x, y \rangle \in \theta$ implies $\langle h(x), h(y) \rangle \in \theta$, for any $x, y \in A$. In addition, if θ is a congruence on $(\boldsymbol{A}, h)$ and $\langle x \odot 1, y \odot 1 \rangle \in \theta$ can imply $\langle x, y \rangle \in \theta$, for any $x, y \in A$, then it is called a* filter congruence *on $(\boldsymbol{A}, h)$.*

Theorem 4.1. *Let* $(\boldsymbol{A}, h)$ *be a qpMV-algebra with hedge. Then, there exists a bijection between normal h-filters and filter congruences on* $(\boldsymbol{A}, h)$.

Proof. Let F be a normal h-filter of $(\mathbf{A}, h)$. Then, $\theta_F = \{\langle x, y\rangle \in A^2 | x \to y \in F \text{ and } y \to x \in F\}$ is a filter congruence on $\mathbf{A}$ by Proposition 2.2. Moreover, since F is a h-filter of $(\mathbf{A}, h)$, we have $h(x \to y) \in F$ and $h(y \to x) \in F$. By (H3), we have $h(x \to y) \leq h(x) \to h(y)$ and $h(y \to x) \leq h(y) \to h(x)$, it follows that $h(x) \to h(y) \in F$ and $h(y) \to h(x) \in F$, so $\langle h(x), h(y)\rangle \in \theta_F$. Conversely, let θ be a filter congruence on $(\mathbf{A}, h)$. Then, $F_\theta = \{x \in A | \langle x, 1\rangle \in \theta\}$ is a normal filter of $\mathbf{A}$ using Proposition 2.2 again. Moreover, for any $x \in F_\theta$, we have $\langle h(x), 1\rangle = \langle h(x), h(1)\rangle \in \theta$, so $h(x) \in F_\theta$. The left is obtained by Proposition 2.2. □

Let $(\mathbf{A}, h)$ be a qpMV-algebra with hedge and F be a normal h-filter of $(\mathbf{A}, h)$. Then, $A/F = \{x/F | x \in A\}$ where $x/F = \{y \in A | x \to y \in F \text{ and } y \to x \in F\}$ is a *quotient set* with respect to F. We define some operations as follows: $(x/F) \oplus (y/F) = (x \oplus y)/F$, $(x/F)^- = x^-/F$ and $(x/F)^\sim = x^\sim/F$. Then, $\mathbf{A}/F = \langle A/F; \oplus, ^-, ^\sim, 0/F, 1/F\rangle$ is a pseudo-MV algebra by [6].

Theorem 4.2. *Let* $(\boldsymbol{A}, h)$ *be a qpMV-algebra with hedge and F be a normal h-filter of* $(\boldsymbol{A}, h)$. *Define* $\bar{h} : A/F \to A/F$ *by* $\bar{h}(x/F) = h(x)/F$, *for any* $x \in A$. *Then,* $(\boldsymbol{A}/F, \bar{h})$ *is a pseudo-MV algebra with hedge.*

Proof. It is easy to see that $\bar{h}$ is well-defined. Now, we check that the conditions (H1-H4) are satisfied. Obviously, $\bar{h}(1/F) = h(1)/F = 1/F$ and $\bar{h}(x/F) = h(x)/F \leq x/F$. For any $x/F, y/F \in A/F$, we have $\bar{h}(x/F \to y/F) = \bar{h}((x \to y)/F) = h(x \to y)/F \leq (h(x) \to h(y))/F = h(x)/F \to h(y)/F = \bar{h}(x/F) \to \bar{h}(y/F)$. Similarly, we have $\bar{h}(x \rightsquigarrow y) \leq \bar{h}(x) \rightsquigarrow \bar{h}(y)$. Finally, we have $\bar{h}(\bar{h}(x/F)) = h(h(x))/F = h(x)/F = \bar{h}(x/F)$. □

Proposition 4.3. *Let* $(\boldsymbol{A}, h)$ *be a qpMV-algebra with strong hedge. Then,* $\ker(h)$ *is a weak h-filter of* $(\boldsymbol{A}, h)$.

Proof. Obviously, $1 \in \ker(h)$. For any $x, y \in \ker(h)$, then $h(x) = h(y) = 1$, we have $h(x \odot y) \geq h(x) \odot h(y) = 1 \odot 1 = 1$, so $h(x \odot y) = 1$ and then $x \odot y \in \ker(h)$. Let $x \in \ker(h)$ and $y \in A$. Then, $1 = h(x) \leq h(x \oplus y)$, we have $h(x \oplus y) = 1$, so $x \oplus y \in \ker(h)$. Similarly, we have $y \oplus x \in \ker(h)$. Hence, $\ker(h)$ is a weak filter of $(\mathbf{A}, h)$. Moreover, for any $x \in \ker(h)$, we have $h(h(x)) = h(1) = 1$, so $h(x) \in \ker(h)$. Hence, $\ker(h)$ is a weak h-filter of $(\mathbf{A}, h)$. □

Since any mi-operator is a strong hedge in a qpMV-algebra, we have the following result.

Corollary 4.1. *Let* $(\boldsymbol{A}, f)$ *be an interior qpMV-algebra. Then,* $\ker(f)$ *is a weak f-filter of* $(\boldsymbol{A}, f)$.

Definition 4.4. *Let $(\boldsymbol{A}, h_1)$ and $(\boldsymbol{B}, h_2)$ be qpMV-algebras with hedges and $\varphi : A \to B$ be a mapping. Then, φ is called a qp*MV-algebra with hedge homomorphism, *if it satisfies the following conditions, for any $x, y \in A$,*

$(HH1)$ $\varphi(1) = 1$;
$(HH2)$ $\varphi(x \oplus y) = \varphi(x) \oplus \varphi(y)$;
$(HH3)$ $\varphi(x^-) = (\varphi(x))^-$;
$(HH4)$ $\varphi(x^\sim) = (\varphi(x))^\sim$;
$(HH5)$ $\varphi(h_1(x)) = h_2(\varphi(x))$.

Proposition 4.4. *Let $(\boldsymbol{A}, h_1)$ and $(\boldsymbol{B}, h_2)$ be qpMV-algebras with hedges and φ be a homomorphism from $(\boldsymbol{A}, h_1)$ to $(\boldsymbol{B}, h_2)$. Then, the following conditions are equivalent:*

(1) $\varphi(x \oplus y) = \varphi(x) \oplus \varphi(y)$;
(2) $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$;
(3) $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$;
(4) $\varphi(x \odot y) = \varphi(x) \odot \varphi(y)$;
(5) $\varphi(x \to y) = \varphi(x) \to \varphi(y)$;
(6) $\varphi(x \rightsquigarrow y) = \varphi(x) \rightsquigarrow \varphi(y)$.

Proof. (1) $\Rightarrow$ (2) We have $\varphi(x \vee y) = \varphi(y \oplus (x^- \oplus y)^\sim) = \varphi(y) \oplus \varphi((x^- \oplus y)^\sim) = \varphi(y) \oplus (\varphi(x^- \oplus y))^\sim = \varphi(y) \oplus (\varphi(x)^- \oplus \varphi(y))^\sim = \varphi(x) \vee \varphi(y)$.

(2) $\Rightarrow$ (3) We have $\varphi(x \wedge y) = \varphi((x^- \vee y^-)^\sim) = (\varphi(x^- \vee y^-))^\sim = (\varphi(x)^- \vee \varphi(y)^-)^\sim = \varphi(x) \wedge \varphi(y)$.

(3) $\Rightarrow$ (1) Since $x \oplus y = x \oplus (y \wedge x^\sim)$, we have $\varphi(x \oplus y) = \varphi(x) \oplus \varphi(y \wedge x^\sim) = \varphi(x) \oplus (\varphi(y) \wedge \varphi(x)^\sim) = (\varphi(x) \oplus \varphi(y)) \wedge (\varphi(x) \oplus \varphi(x)^\sim) = \varphi(x) \oplus \varphi(y)$.

(1) $\Leftrightarrow$ (4) Since $x \odot y = (x^- \oplus y^-)^\sim$ and $x \oplus y = (x^- \odot y^-)^\sim$, we get the result.

(1) $\Leftrightarrow$ (5) Since $x \to y = x^- \oplus y$ and $x \oplus y = x^\sim \to y$, we get the result.

(1) $\Leftrightarrow$ (6) Analogously. $\square$

Recall that a *h-subalgebra* $(\mathbf{S}, h)$ of a *qp*MV-algebra with hedge $(\mathbf{A}, h)$, if $\mathbf{S}$ is a subalgebra of $\mathbf{A}$ and $h(S) \subseteq S$.

Theorem 4.3. *Let $(\boldsymbol{A}, h_1)$ and $(\boldsymbol{B}, h_2)$ be qpMV-algebras with hedges and φ be a homomorphism from $(\boldsymbol{A}, h_1)$ to $(\boldsymbol{B}, h_2)$. Then,*

(1) *If $(\boldsymbol{S}, h_1)$ is a h_1-subalgebra of $(\boldsymbol{A}, h_1)$, then $(\varphi(\boldsymbol{S}), h_2)$ is a h_2-subalgebra of $(\boldsymbol{B}, h_2)$;*

(2) *If φ is surjective and F is a weak h_1-filter of $(\boldsymbol{A}, h_1)$, then $\varphi(F)$ is a weak h_2-filter of $(\boldsymbol{B}, h_2)$;*

(3) *If F is a (weak) h_2-filter of $(\boldsymbol{B}, h_2)$, then $\varphi^{-1}(F)$ is a (weak) h_1-filter of $(\boldsymbol{A}, h_1)$;*

(4) $\ker(\varphi) = \{x \in A | \varphi(x) = 1\}$ *is a normal weak h_1-filter of $(\boldsymbol{A}, h_1)$.*

Proof. (1) It is easy to show that $\varphi(\mathbf{S})$ is a subalgebra of $\mathbf{B}$. Moreover, for any $\varphi(x) \in \varphi(S)$ where $x \in S$, we have $h_2(\varphi(x)) = \varphi(h_1(x)) \in \varphi(S)$.

(2) Since $1 \in F$, we have $1 = \varphi(1) \in \varphi(F)$. Let $x, y \in \varphi(F)$. Then, there exist $m, n \in F$ such that $\varphi(m) = x$ and $\varphi(n) = y$, it turns out that $x \odot y = \varphi(m) \odot \varphi(n) = \varphi(m \odot n) \in \varphi(F)$. Now, let $x \in \varphi(F)$ and $y \in B$. Because φ is surjective, there exist $m \in F$ and $n \in A$ such that $x = \varphi(m)$ and $y = \varphi(n)$. We have $x \oplus y = \varphi(m) \oplus \varphi(n) = \varphi(m \oplus n) \in \varphi(F)$. Similarly, $y \oplus x = \varphi(n \oplus m) \in \varphi(F)$. Hence, $\varphi(F)$ is a weak filter of $(\mathbf{B}, h_2)$. For any $\varphi(x) \in \varphi(F)$ where $x \in F$, we have $h_2(\varphi(x)) = \varphi(h_1(x)) \in \varphi(F)$, so $\varphi(F)$ is a weak h_2-filter of $(\mathbf{B}, h_2)$.

(3) We only prove the case of filters. The case of weak filters can be proved similarly. Obviously, $\varphi(1) = 1 \in F$, so $1 \in \varphi^{-1}(F)$. For any $x, y \in \varphi^{-1}(F)$, then there exist $a, b \in F$ such that $\varphi(x) = a$ and $\varphi(y) = b$, it follows that $\varphi(x \odot y) = \varphi(x) \odot \varphi(y) = a \odot b \in F$, so $x \odot y \in \varphi^{-1}(F)$. Let $x \in \varphi^{-1}(F)$ and $y \in A$ with $x \leq y$. Then, there exists $a \in F$ such that $\varphi(x) = a$ and $a = \varphi(x) \leq \varphi(y)$. Because $a \in F$, we have $\varphi(y) \in F$, so $y \in \varphi^{-1}(F)$. For any $x \in \varphi^{-1}(F)$, there exists $a \in F$ such that $\varphi(x) = a$, then we have $\varphi(h_1(x)) = h_2(\varphi(x)) = h_2(a) \in F$, so $h_1(x) \in \varphi^{-1}(F)$. Hence, $\varphi^{-1}(F)$ is a h_1-filter of $(\mathbf{A}, h_1)$.

(4) Obviously, $1 \in \ker(\varphi)$. For any $x, y \in \ker(\varphi)$, we have $\varphi(x \odot y) = \varphi(x) \odot \varphi(y) = 1 \odot 1 = 1$, so $x \odot y \in \ker(\varphi)$. Let $x \in \ker(\varphi)$ and $y \in A$. Then, $\varphi(x \oplus y) = \varphi(x) \oplus \varphi(y) = 1 \oplus \varphi(y) = 1$ and $\varphi(y \oplus x) = \varphi(y) \oplus \varphi(x) = \varphi(y) \oplus 1 = 1$, it follows that $x \oplus y \in \ker(\varphi)$ and $y \oplus x \in \ker(\varphi)$. For $x \in \ker(\varphi)$, we have $\varphi(h_1(x)) = h_2(\varphi(x)) = h_2(1) = 1$, so $h_1(x) \in \ker(\varphi)$. Hence, $\ker(\varphi)$ is a weak h_1-filter of $(\mathbf{A}, h_1)$. Finally, for any $x, y \in A$, we have $x \to y \in \ker(\varphi)$ iff $\varphi(x \to y) = 1$ iff $\varphi(x) \to \varphi(y) = 1$ iff $\varphi(x) \leq \varphi(y)$ iff $\varphi(x) \rightsquigarrow \varphi(y) = 1$ iff $\varphi(x \rightsquigarrow y) = 1$ iff $x \rightsquigarrow y \in \ker(\varphi)$. So $\ker(\varphi)$ is normal. $\square$

Corollary 4.2. *Let $(\boldsymbol{A}, h_1)$ and $(\boldsymbol{B}, h_2)$ be qpMV-algebras with strong hedges and φ be a homomorphism from $(\boldsymbol{A}, h_1)$ to $(\boldsymbol{B}, h_2)$. Then,*

(1) *$\varphi^{-1}(\ker(h_2))$ is a weak h_1-filter of $(\boldsymbol{A}, h_1)$;*

(2) *If φ is surjective, $\varphi(\ker h_1)$ is a weak h_2-filter of $(\boldsymbol{B}, h_2)$.*

Let $(\mathbf{A}, h)$ be a *qp*MV-algebra with hedge and F be a normal h-filter of $(\mathbf{A}, h)$. Then, $(\mathbf{A}/F, \bar{h})$ is a pseudo-MV algebra with hedge by Theorem 4.2. Define $\pi : A \to A/F$ by $x \mapsto x/F$, for any $x \in A$. Then, we have the following result.

Proposition 4.5. *Let $(\boldsymbol{A}, h)$ be a qpMV-algebra with hedge and F be a normal h-filter of $(\boldsymbol{A}, h)$. Then,*

(1) *π is a homomorphism from $(\boldsymbol{A}, h)$ to $(\boldsymbol{A}/F, \bar{h})$ and* $\ker \pi = F$*;*

(2) *$\pi^{-1}(\ker \bar{h}) \subseteq h^{-1}(F)$;*

(3) *$\pi(\ker h) \subseteq \ker(\bar{h})$.*

Proof. (1) It is easy to check that π is a homomorphism $(\mathbf{A}, h)$ to $(\mathbf{A}/F, \bar{h})$. For any $x \in \ker(\pi)$, then $\pi(x) = x/F = 1/F$, it turns out that $1 \to x \in F$. Since $1 \to x \leq x$, we have $x \in F$, so $\ker(\pi) \subseteq F$. For any $x \in F$, then $1 \to x \in F$

and $x \to 1 = 1 \in F$, we have $1 \in x/F$, so $1/F \subseteq x/F$. Conversely, for any $y \in x/F$, then $y \to x \in F$ and $x \to y \in F$. Because $x \in F$, we have $y \in F$, it turns out that $y \to 1 = 1 \in F$ and $1 \to y \in F$, so $y \in 1/F$ and $x/F \subseteq 1/F$. Thus, $1/F = x/F$ which means that $x \in \ker(\pi)$, we have $F \subseteq \ker(\pi)$. Hence, $\ker(\pi) = F$.

(2) For any $x \in \pi^{-1}(\ker \bar{h})$, then $\pi(x) \in \ker(\bar{h})$, so $1/F = \bar{h}(\pi(x)) = \pi(h(x)) = h(x)/F$ and then $1 \to h(x) \in F$. Since $1 \to h(x) \leq h(x)$, we have $h(x) \in F$, so $x \in h^{-1}(F)$. Hence, $\pi^{-1}(\ker \bar{h}) \subseteq h^{-1}(F)$.

(3) For any $x \in \pi(\ker(h))$, there exists $m \in \ker(h)$ such that $\pi(m) = x$, then we have $\bar{h}(x) = \bar{h}(\pi(m)) = \pi(h(m)) = \pi(1) = 1/F$, so $x \in \ker(\bar{h})$ and then $\pi(\ker h) \subseteq \ker(\bar{h})$. □

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