Oscillation criteria of fractional damped differential equations

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Abstract. In this paper, we prove some properties of oscillation for a class of fractional damped differential equations using generalized Riccati transformation and inequality technique, we prove some new oscillatory criteria. Recent results in the literature are generalized and significant improved. Example is shown to illustrate our main results.
Keywords: oscillatory criteria, fractional derivative, fractional damped differential equation.

1. Introduction

Consider the oscillation of the following fractional damped differential equations

$$[r(t)(D_-^\alpha y)^\eta(t)]' - p(t)(D_-^\alpha y)^\eta(t) - q(t)f(\int_t^\infty (v-t)^{-\alpha} y(v)dv) = 0, \tag{1.1}$$

where $\alpha \in (0,1)$ is a constant, and $\eta > 0$ is a quotient of odd positive integers. The differential operator $D_-^\alpha y$ is the Liouville right-sided fractional derivation

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of order α for y defined by $(D_{-}^{\alpha}y)(t) = -\frac{1}{\Gamma(1-\alpha)} \cdot \frac{d}{dt}\int_t^{\infty}(v-t)^{-\alpha}y(v)dv$ for $t \in R_+ := (0, +\infty)$. Here Γ is the gamma function $\Gamma(t) = \int_0^{\infty} v^{t-1}e^{-v}dv$ for $t \in R_+$. We assume that conditions hold:
(H_1) $r(t), p(t)$ and $q(t)$ are positive continuous functions on $[t_0, \infty)$ for a certain $t_0 > 0$. The function$f : R \to R$ is a continuous function such that $\frac{f(u)}{u^{\eta}} \geq G$ for a certain constant for $G > 0$ and for all $u \neq 0$.

By a solution of (1.1) we mean a nontrivial function $y \in C(R_+, R)$ such that $\int_t^{\infty}(v-t)^{-\alpha}y(v)dv \in C^1(R_+, R), r(t)(D_{-}^{\alpha}y)^{\eta}(t) \in C^1(R_+, R)$ and (1.1) hold for $t > 0$. We focus on those solutions of (1.1) which exist on R_+ such that $\sup\{|y(t)| : t > t_*\} > 0$ for any $t_* \geq 0$. A solution y of (1.1) is said to be called oscillatory if it is neither eventually positive nor eventually negative. Otherwise it is nonoscillatory. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

Due to its important applications on many fields, such as viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc, $(see, for example,$ $[1-12]$), in last decade, a lot of attentions has been focused on the study of the stability and properties of solutions for fractional differential equations, see, for example, [13-21].

In particular, Chen [2] studied oscillatory properties of solutions to the following fractional differential equations

$$[r(t)(D_{-}^{\alpha}y)^{\eta}(t)]' - q(t)f(\int_t^{\infty}(v-t)^{-\alpha}y(v)dv) = 0, \tag{$*$}$$

for $t > 0$, where $D_{-}^{\alpha}y$ denotes the Liouville right-sided fractional derivative of order α with the form

$$(D_{-}^{\alpha}y)(t) = \frac{1}{\Gamma(1-\alpha)} \cdot \frac{d}{dt}\int_t^{\infty}(v-t)^{-\alpha}y(v)dv,$$

for $t \in R_+ := (0, \infty)$. By using Riccati transformation technique the authors obtained some sufficient conditions, which guarantee that every solution of the equation is oscillatory.

Zhang [18] considered the oscillation of the nonlinear fractional damped fractional differential equations

$$[a(t)(D_{-}^{\alpha}x(t))^{\gamma}]' + p(t)(D_{-}^{\alpha}x(t))^{\gamma} - q(t)f(\int_t^{\infty}(\xi-t)^{-\alpha}x(\xi)d\xi) = 0, t \in [t_0, \infty),$$

where $D_{-}^{\alpha}x(t)$ denotes the Liouville right-sided fractional derivative of order α of x. By using a generalized Riccati function and the inequality technique, he established some new criteria.

Qi and Huang [19] studied the oscillation behavior of the equation:

$$(a(t)[r(t)D_{-}^{\alpha}x(t)]')' + p(t)[r(t)D_{-}^{\alpha}x(t)]' - q(t)f(\int_t^{\infty}(\xi-t)^{-\alpha}x(\xi)d\xi) = 0,$$

where $D_-^{\alpha}x(t)$ also denotes the Liouville right-sided fractional derivative and established some sufficient conditions for the oscillation of the equation.

However, as much as we know, very little is known on the oscillation of fractional damped differential equations. Only a few of papers have been published on the oscillation theory of fractional damped differential equations, such as [3,4,16-18].

In this paper, we will establish some new oscillation criteria for (1.1), by a class of new function $\Phi(t,s,l)$ and $H(t)$, generalized Riccati transformation and inequality technique.

2. Preliminaries

In this section,we present the definitions of fractional integrals, fractional derivatives and function Φ, which are used throughout this article. We also, give several lemmas, which are useful in establishing our results.

Definition 2.1 (KiLbas et al. [7])**.** The Liouville right-sided fractional integral of order $\beta > 0$ of a function $g : R_+ \to R$ on the half-axis R_+ is given by

$$(I_-^{\beta}g)(t) := \frac{1}{\Gamma(\beta)}\int_t^{\infty}(v-t)^{\beta-1}g(v)dv, \tag{2.1}$$

for $t > 0$, provided the right-hand side is pontwise defined on R_+, where Γ is the gamma function.

Definition 2.2 (Kilbas et al. [7])**.** The Liouville right-sided fractional derivative of order $\beta > 0$ of a function $g : R_+ \to R$ on the half-axis R_+ is given by

$$\begin{aligned}(D_-^{\beta}g)(t) &= (-1)^{[\beta]}\frac{d^{[\beta]}}{dt^{[\beta]}}(I_-^{[\beta]-\beta}g)(t)\\ &= (-1)^{[\beta]}\frac{1}{\Gamma([\beta]-\beta)}\cdot\frac{d^{[\beta]}}{dt^{[\beta]}}\int_t^{\infty}(v-t)^{[\beta]-\beta-1}g(v)dv,\end{aligned} \tag{2.2}$$

for $t > 0$, provided the right-hand side is pointwise defined on R_+,where$[\beta] := \min\{z \in Z : z \geq \beta\}$ is the ceilling function.

Definition 2.3 (Sun et al. [15])**.** We say that a function $\Phi = \Phi(t,s,l)$ belongs to the function class Y, denoted by $\Phi \in Y$, if $\Phi \in (E,R)$, where $E = \{(t,s,l) : t_0 \leq l \leq s \leq t < \infty\}$,which satisfies $\Phi(t,t,l) = 0, \Phi(t,l,l) = 0, \Phi(t,s,l) \neq 0$ for $l < s < t$, and has the partial derivative $\frac{\partial\Phi}{\partial s}$ on E such that $\frac{\partial\Phi}{\partial s}$ is locally integrable with respect to s in E.

Definition 2.4 (Sun et al. [15])**.** Let $\Phi \in Y, g \in C^1([t_0,+\infty),R)$, the operator $T[*;l,t]$ is defined by

$$T[g;l,t] = \int_l^t \Phi^2(t,s,l)g(s)ds, \tag{2.3}$$

for $t \geq s \geq l \geq t_0$ and $g(s) \in C[t_0, \infty)$,and the function $\varphi = \varphi(t, s, l)$ is defined by

$$\frac{\partial \Phi(t,s,l)}{\partial s} = \varphi(t,s,l)\Phi(t,s,l). \tag{2.4}$$

It is easy to verify that $T[*; l, t]$ is a linear operator and satisfies

$$T[g'; l, t] = -2T[g\varphi; l, t]. \tag{2.5}$$

Lemma 2.1. Let y be a solution of (1.1) and

$$A(t) := \int_t^\infty (v-t)^{-\alpha} y(v)dv, B(t) = e^{\int_t^\infty \frac{p(s)}{r(s)}ds}, \tag{2.6}$$

for $\alpha \in (0,1)$ and $t > 0$, then

$$[A(t)B(t)]' = -\Gamma(1-\alpha)(D_-^\alpha y)(t)B(t) - \frac{p(t)}{r(t)}A(t)B(t), \tag{2.7}$$

for $\alpha \in (0,1)$ and $t > 0$.

Proof. From(2.6) and(2.2), for $\alpha \in (0,1)$ and $t > 0$, we obtain

$$\begin{aligned}
[A(t)B(t)]' &= A'(t)B(t) + A(t)B'(t) \\
&= \Gamma(1-\alpha) \cdot \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^\infty (v-t)^{-\alpha} y(v)dv.B(t) \\
&- \int_t^\infty (v-t)^{-\alpha} y(v)dv \cdot \frac{p(t)}{r(t)}.B(t) \\
&= -\Gamma(1-\alpha)[(-1)^{[\alpha]} \frac{1}{\Gamma([\alpha]-\alpha)} \cdot \frac{d^{[\alpha]}}{dt^{[\alpha]}} \int_t^\infty (v-t)^{[\alpha]-\alpha-1} y(v)dv]B(t) \\
&- \frac{p(t)}{r(t)}A(t)B(t) \\
&= -\Gamma(1-\alpha)(D_-^\alpha y)(t)B(t) - \frac{p(t)}{r(t)}A(t)B(t).
\end{aligned}$$

The proof is complete. □

Lemma 2.2 (Hardy et al. [15])**.** If X and Y are nonnegative, then

$$mXY^{m-1} - X^m \leq (m-1)Y^m,$$

for $m > 1$, where the equality holds if and only if $X = Y$.

3. Main result

Theorem 3.1. Suppose that (H_1) and

$$\int_{t_0}^{\infty} r^{-\frac{1}{\eta}}(s)B^{1-\frac{1}{\eta}}(s)ds = \infty \tag{3.1}$$

hold. Furthermore, assume that there exists a positive function $b(t) \in C^1[t_0, \infty)$ such that

$$\limsup_{t\to\infty} \int_{t_0}^{t} [Gb(s)q(s)B(s) - \frac{r(s)B(s)[b'_+(s)]^{\eta+1}}{(\eta+1)^{(\eta+1)}[\Gamma(1-\alpha)b(s)]^{\eta}}]ds = \infty, \tag{3.2}$$

where $b'_+(s) = \max\{b'(s), 0\}$, then every solution of (1.1) is oscillatory.

Proof. Suppose that y is a non-oscillatory solution of (1.1). Without loss of generality, we may assume that $y(t)$ is an eventually positive solution of(1.1). Then, there exists $t_1 \in [t_0, \infty)$ such that

$$y(t) > 0 \text{ and } A(t)B(t) > 0, \tag{3.3}$$

for $t \in [t_1, \infty)$, where $A(t), B(t)$ is defined as in (2.6). Therefore, it follows from (1.1) that

$$\begin{aligned} [r(t)(D_-^{\alpha}y)^{\eta}(t)B(t)]' &= [r(t)(D_-^{\alpha}y)^{\eta}(t)]'B(t) - p(t)(D_-^{\alpha}y)^{\eta}(t)B(t) \\ &= q(t)f(A(t))B(t) > 0, \end{aligned} \tag{3.4}$$

for $t \in [t_1, \infty)$.

Thus, $r(t)(D_-^{\alpha}y)^{\eta}(t)B(t)$ is strictly increasing on $[t_1, \infty)$ and is eventually of one sign. Since $r(t) > 0, B(t) > 0$ for $t \in [t_1, \infty)$ and $\eta > 0$ is a quotient of odd positive integers,we see that $(D_-^{\alpha}y)(t)$ is eventually of one sign. We now claim

$$(D_-^{\alpha}y)(t) < 0,$$

for $t \in [t_1, \infty)$.

If not, then $(D_-^{\alpha}y)(t)$ is eventually positive and there exists $t_2 \in [t_1, \infty)$ such that $(D_-^{\alpha}y)(t_2) > 0$. Since $r(t)(D_-^{\alpha}y)^{\eta}(t)B(t)$ is strictly increasing on $[t_1, \infty)$, it is clear that

$$r(t)(D_-^{\alpha}y)^{\eta}(t)B(t) \geq r(t_2)(D_-^{\alpha}y)^{\eta}(t_2)B(t_2) := a_1 > 0,$$

for $t \in [t_2, \infty)$. Therefore, from (2.4), we have

$$\begin{aligned} -\frac{[A(t)B(t)]'}{\Gamma(1-\alpha)B(t)} &= -\frac{-\Gamma(1-\alpha)(D_-^{\alpha}y)(t)B(t) - \frac{p(t)}{r(t)}A(t)B(t)}{\Gamma(1-\alpha)B(t)} \\ &= (D_-^{\alpha}y)(t) + \frac{A(t)p(t)}{\Gamma(1-\alpha)r(t)} \\ &\geq (D_-^{\alpha}y)(t) \geq (\frac{a_1}{r(t)B(t)})^{\frac{1}{\eta}} = a_1^{\frac{1}{\eta}} r^{-\frac{1}{\eta}}(t)B^{-\frac{1}{\eta}}(t), \end{aligned}$$

and, then, we have

$$-\frac{[A(t)B(t)]'}{\Gamma(1-\alpha)} \geq a_1^{\frac{1}{\eta}} r^{-\frac{1}{\eta}}(t) B^{1-\frac{1}{\eta}}(t),$$

for $t \in [t_2, \infty)$.

Integrating both sides of the last inequality from t_2 to t, we have

$$\int_{t_2}^{t} r^{-\frac{1}{\eta}}(s) B^{1-\frac{1}{\eta}}(s) ds \leq -\frac{A(t)B(t) - A(t_2)B(t_2)}{a_1^{\frac{1}{\eta}} \Gamma(1-\alpha)} \leq \frac{A(t_2)B(t_2)}{a_1^{\frac{1}{\eta}} \Gamma(1-\alpha)} < \infty,$$

for $t \in [t_2, \infty)$.

Letting $t \to \infty$, we see

$$\int_{t_2}^{\infty} r^{-\frac{1}{\eta}}(s) B^{1-\frac{1}{\eta}}(s) ds \leq \frac{A(t_2)B(t_2)}{a_1^{\frac{1}{\eta}} \Gamma(1-\alpha)} < \infty.$$

This contradicts (3.1). Hence, (3.5) holds. Define the function $w(t)$ by the generalized Riccati substitution

$$w(t) = b(t) \cdot \frac{-r(t)(D_-^\alpha y)^\eta(t) B(t)}{(A(t))^\eta}, \tag{3.6}$$

for $t \in [t_1, \infty)$.

Then, we have $w(t) > 0$ for $t \in [t_1, \infty)$. From (3.6),(1.1),(3.4) and (H_1), it follows that

$$\begin{aligned}
w'(t) = {} & b'(t) \cdot \frac{-r(t)(D_-^\alpha y)^\eta(t) B(t)}{(A(t))^\eta} + b(t)[\frac{-r(t)(D_-^\alpha y)^\eta(t) B(t)}{(A(t))^\eta}]' \\
\leq {} & b_+^{'}(t) \cdot \frac{-r(t)(D_-^\alpha y)^\eta(t) B(t)}{(A(t))^\eta} \\
& + b(t) \cdot \frac{\begin{array}{c}[-r(t)(D_-^\alpha y)^\eta(t) B(t)]'(A(t))^\eta \\ + r(t)(D_-^\alpha y)^\eta(t) B(t) \eta (A(t))^{\eta-1}(-\Gamma(1-\alpha)(D_-^\alpha y)(t))\end{array}}{(A(t))^{2\eta}} \\
= {} & \frac{b_+^{'}(t)}{b(t)} w(t) + b(t)[\frac{-q(t) f(A(t)) B(t)}{(A(t))^\eta} \\
& + r(t)(D_-^\alpha y)^\eta(t) \cdot \frac{\eta B(t)[-\Gamma(1-\alpha)(D_-^\alpha y)(t)]}{(A(t))^{\eta+1}} \\
\leq {} & \frac{b_+^{'}(t)}{b(t)} w(t) - G q(t) b(t) B(t) - \eta \Gamma(1-\alpha) b(t) r(t) B(t) [\frac{w(t)}{b(t) r(t) B(t)}]^{1+\frac{1}{\eta}} \\
= {} & -G q(t) b(t) B(t) + \frac{b_+^{'}(t)}{b(t)} w(t) - \eta \Gamma(1-\alpha)[b(t) r(t) B(t)]^{-\frac{1}{\eta}} w^{1+\frac{1}{\eta}}(t), \qquad (3.7)
\end{aligned}$$

for $t \geq t_1$, where $b_+^{'}(t)$ is defined as in Theorem 3.1.

Taking

$$m = 1 + \frac{1}{\eta}, \quad X = \frac{[\eta\Gamma(1-\alpha)]^{\frac{1}{m}} w(t)}{[b(t)r(t)B(t)]^{\frac{1}{\eta+1}}}$$

and

$$Y = \frac{[\frac{b'_+(t)}{b(t)}]^{\eta}[b(t)r(t)B(t)]^{\frac{1}{m}}}{m^{\eta}[\eta\Gamma(1-\alpha)]^{\frac{\eta}{m}}},$$

from (3.7) and Lemma 2.2, we conclude that

$$w'(t) \leq -Gq(t)b(t)B(t) + \frac{r(t)B(t)[b'_+]^{\eta+1}}{(\eta+1)^{\eta+1}[\Gamma(1-\alpha)b(t)]},$$

for $t \in [t_1, \infty)$.

Integrating both sides of the last inequality from t_1 to t, we have

$$\int_{t_1}^{t} [Gb(s)q(s)B(s) - \frac{r(s)B(s)[b'_+(s)]^{\eta+1}}{(\eta+1)^{(\eta+1)}[\Gamma(1-\alpha)b(s)]^{\eta}}]ds \leq w(t_1) - w(t) < w(t_1),$$

for $t \in [t_1, \infty)$.

Letting

$$\lim_{t\to\infty} \sup \int_{t_1}^{t} [Gb(s)q(s)B(s) - \frac{r(s)B(s)[b'_+(s)]^{\eta+1}}{(\eta+1)^{(\eta+1)}[\Gamma(1-\alpha)b(s)]^{\eta}}]ds < w(t_1) < \infty,$$

which contradicts (3.2). The proof is complete. □

Remark 3.1. Theorem 3.1 in [2] is a special case of Theorem 3.1 with $p(t) = 0$, respectively. Theorem 3.1 improves and extend the results of Theorem 3.1.

Theorem 3.2. Suppose that (H_1) and (3.1) hold. Let $T_0 \geq t_0$,then there exist a and b such that $b > a > T_0$. Let

$$D(a, b) = \{U(t) \in C^1[a, b] : U(t) \neq 0, t \in (a, b), U(a) = U(b) = 0\}.$$

If there exist a function $H(t) \in D(a, b)$ such that the following condition that holds:

$$\int_a^b Gb(s)q(s)B(s)ds > \int_a^b \frac{[H^{\eta}(s)(H'(s) + \frac{b'_+(s)}{(\eta+1)b(s)})]^{\eta+1} b(s)r(s)B(s)}{[\Gamma(1-\alpha)H^{\eta+1}(s)]^{\eta}} ds. \tag{3.8}$$

Then equation (1.1) is oscillatory.

Proof. Suppose that $y(t)$ is a non-oscillatory solution of (1.1), Without loss of generality, we may suppose that $y(t)$ is an eventually positive solution of (1.1). We proceed as in proof of Theorem 3.1 to get that (3.7) holds.

Multiplying both sides of (3.7) by $H^{\eta+1}(t)$ and integrating from a to b, by $H(a)=H(b)=0$, we obtain

$$\begin{aligned}\int_a^b H^{\eta+1}(s)w'(s)ds &\le -\int_a^b Gb(s)r(s)B(s)H^{\eta+1}(s)ds\\ &+\int_a^b \frac{b'_+(s)H^{\eta+1}(s)w(s)}{b(s)}ds\\ &-\int_a^b \eta\Gamma(1-\alpha)H^{\eta+1}(s)[b(s)r(s)B(s)]^{-\frac{1}{\eta}}w^{1+\frac{1}{\eta}}(s)ds \end{aligned}\tag{3.9}$$

and then we get

$$\begin{aligned}\int_a^b Gb(s)r(s)B(s)H^{\eta+1}(s)ds &\le \int_a^b(\eta+1)H^{\eta}(s)H'(s)w(s)ds\\ &+\int_a^b \frac{b'_+(s)H^{\eta+1}(s)w(s)}{b(s)}ds\\ &-\int_a^b \eta\Gamma(1-\alpha)H^{\eta+1}(s)[b(s)r(s)B(s)]^{-\frac{1}{\eta}}w^{1+\frac{1}{\eta}}(s)ds\\ &=\int_a^b[(\eta+1)H^{\eta}(s)w(s)(H'(s)+\frac{b'_+(s)H(s)}{(\eta+1)b(s)})\\ &-\eta\Gamma(1-\alpha)H^{\eta+1}(s)[b(s)r(s)B(s)]^{-\frac{1}{\eta}}w^{1+\frac{1}{\eta}}(s)]ds. \end{aligned}\tag{3.10}$$

Taking

$$m=1+\frac{1}{\eta},\quad X=\frac{[\eta\Gamma(1-\alpha)H^{\eta+1}(s)]^{\frac{1}{m}}w(s)}{[b(s)r(s)B(s)]^{\frac{1}{\eta+1}}}$$

and

$$Y=\frac{[(\eta+1)H^{\eta}(s)(H'(s)+\frac{b'_+(s)H(s)}{(\eta+1)b(s)})]^{\eta}[b(s)r(s)B(s)]^{\frac{1}{m}}}{m^{\eta}[\eta\Gamma(1-\alpha)H^{\eta+1}(s)]^{\frac{\eta}{m}}},$$

by (3.10) and Lemma2.2, we conclude that

$$\int_a^b Gb(s)q(s)B(s)ds\le\int_a^b\frac{[H^{\eta}(s)(H'(s)+\frac{b'_+(s)H(s)}{(\eta+1)b(s)})]^{\eta+1}b(s)r(s)B(s)}{[\Gamma(1-\alpha)H^{\eta+1}(s)]^{\eta}}ds,\tag{3.11}$$

which contradicts the condition (3.8). The proof is complete. □

Remark 3.2. Theorem 3.2 is new because we introduce a new class of functions $H(t)$.

Theorem 3.3. Suppose that (H_1) and (3.1) hold. There exist a function $\Phi \in Y$. Such that

$$\limsup_{t\to\infty} \int -l^t[Gq(s)b(s)B(s) - \frac{\frac{1}{\eta}[2\varphi + \frac{b'_+(s)}{b(s)}]^{\eta+1} b(s)r(s)B(s)}{(\frac{1+\eta}{\eta})^{\eta+1}[\eta\Gamma(1-\alpha)]^\eta}]\Phi^2(t,s,l)ds > 0 \tag{3.12}$$

for each $l \geq T_0 \geq t_0$, where operator T defined by (2.3) and the function $\varphi = \varphi(t,s,l)$ is defined by (2.4). Then every solution y of (1.1) is oscillatory.

Proof. Suppose that y is a non-oscillatory solution of (1.1). Without loss of generality,we can assume that$y$is an eventually positive solution of (1.1). Similarly in the proof of Theorem 3.1 to get (3.7) hold, and then we have

$$Gq(t)b(t)B(t) \leq -w'(t) + \frac{b'_+(t)}{b(t)}w(t) - \eta\Gamma(1-\alpha)[b(t)r(t)B(t)]^{-\frac{1}{\eta}} w^{1+\frac{1}{\eta}}(t). \tag{3.13}$$

Applying $T[*;T_0,t]$ to (3.13), we have

$$\begin{aligned} &T[Gq(t)b(t)B(t);T_0,t] \\ &\leq T[-w'(t) + \frac{b'_+(t)}{b(t)}w(t) - \eta\Gamma(1-\alpha)[b(t)r(t)B(t)]^{-\frac{1}{\eta}} w^{1+\frac{1}{\eta}}(t);T_0,t] \\ &= 2T[w(t)\varphi(t,s,l);T_0,t] + T[\frac{b'_+(t)}{b(t)}w(t) \\ &- \eta\Gamma(1-\alpha)[b(t)r(t)B(t)]^{-\frac{1}{\eta}} w^{1+\frac{1}{\eta}}(t);T_0,t] \\ &= T[(2\varphi(t,s,l) + \frac{b'_+(t)}{b(t)})w(t) \\ &- \eta\Gamma(1-\alpha)[b(t)r(t)B(t)]^{-\frac{1}{\eta}} w^{1+\frac{1}{\eta}}(t);T_0,t], \end{aligned} \tag{3.14}$$

for $t \in [T_0, \infty)$.

Taking

$$m = 1 + \frac{1}{\eta}, \quad X = \frac{[\eta\Gamma(1-\alpha)]^{\frac{\eta}{1+\eta}} w(t)}{[b(t)r(t)B(t)]^{\frac{1}{\eta+1}}}$$

and

$$Y = \frac{[2\varphi(t,s,l) + \frac{b'_+(t)}{b(t)}]^\eta [b(t)r(t)B(t)]^{\frac{\eta}{\eta+1}}}{(\frac{1+\eta}{\eta})^\eta [\eta\Gamma(1-\alpha)]^{\frac{\eta^2}{1+\eta}}},$$

by (3.14) and Lemma 2.2, we conclude that

$$T[Gq(t)b(t)B(t);T_0,t] \leq T[\frac{\frac{1}{\eta}[2\varphi(t,s,l) + \frac{b'_+(t)}{b(t)}]^{\eta+1} b(t)r(t)B(t)}{(\frac{1+\eta}{\eta})^{\eta+1}[\eta\Gamma(1-\alpha)]^\eta};T_0,t].$$

Noting that (2.3) and then we have

$$T[Gq(t)b(t)B(t) - \frac{\frac{1}{\eta}[2\varphi(t,s,l) + \frac{b'_+(t)}{b(t)}]^{\eta+1} b(t)r(t)B(t)}{(\frac{1+\eta}{\eta})^{\eta+1}[\eta\Gamma(1-\alpha)]^{\eta}}; T_0, t] \leq 0.$$

Letting $t \to +\infty$, we have

$$\lim_{t\to+\infty} \sup T[Gq(t)b(t)B(t) - \frac{\frac{1}{\eta}[2\varphi(t,s,l) + \frac{b'_+(t)}{b(t)}]^{\eta+1} b(t)r(t)B(t)}{(\frac{1+\eta}{\eta})^{\eta+1}[\eta\Gamma(1-\alpha)]^{\eta}}; T_0, t] \leq 0,$$

and then

$$\lim_{t\to\infty} \sup \int_l^t [Gq(s)b(s)B(s) - \frac{\frac{1}{\eta}[2\varphi + \frac{b'_+(s)}{b(s)}]^{\eta+1} b(s)r(s)B(s)}{(\frac{1+\eta}{\eta})^{\eta+1}[\eta\Gamma(1-\alpha)]^{\eta}}]\Phi^2(t,s,l)ds \leq 0.$$

Which is a contradiction to (3.12). The proof is complete. □

If we chose $\Phi(t,s,l) = \rho(s)(t-s)^{\alpha}(s-l)^{\beta}$ for $\alpha, \beta > \frac{1}{2}$ and $\rho(t) \in C^1([t_0,\infty), (0,\infty))$, then, we have

$$\varphi(t,s,l) = \frac{\rho'(s)}{\rho(s)} + \frac{\beta t - (\alpha+\beta)s + \alpha l}{(t-s)(s-l)}.$$

Thus, by Theorem 3.3, we have the following corollary.

Corollary 3.4. Suppose that(H_1) and (3.1) hold. For each $l \geq t_0$, there exist a function $\rho(t) \in C^1([t_0,\infty), R)$ and two constants $\alpha, \beta > \frac{1}{2}$, such that

$$\lim_{t\to\infty} \sup \int_l^t \rho^2(s)(t-s)^{2\alpha}(s-l)^{2\beta}[Gq(s)b(s)B(s)$$

$$- \frac{\frac{1}{\eta}[2(\frac{\rho'(s)}{\rho(s)} + \frac{\beta t-(\alpha+\beta)s+\alpha l}{(t-s)(s-l)}) + \frac{b'_+(s)}{b(s)}]^{\eta+1} b(s)r(s)B(s)}{(\frac{1+\eta}{\eta})^{\eta+1}[\eta\Gamma(1-\alpha)]^{\eta}}]\Phi^2(t,s,l)ds] > 0. \tag{3.15}$$

All solutions of (1.1) is oscillatory.

Define

$$R(t) = \int_l^t \frac{p(s)}{r(s)} ds, t \geq l \geq t_0.$$

If we chose $\Phi(t,s,l) = \rho(s)(R(t)-R(s))^{\alpha}(R(s)-R(l))^{\beta}$ for $\alpha, \beta > \frac{1}{2}$ and $\rho(t) \in C^1([t_0,\infty), (0,\infty))$, then, we have

$$\varphi(t,s,l) = \frac{\rho'(s)}{\rho(s)} + \frac{p(t)[\beta R(t) - (\alpha+\beta)R(s) + \alpha R(l)]}{r(s)(R(t)-R(s))(R(s)-R(l))}.$$

Thus, by Theorem 3.3, we have the following Theorem.

Theorem 3.5. Suppose that(H_1) and (3.1) hold. For each $l \geq t_0$, there exist a function $\rho(t) \in C^1([t_0, \infty), R)$ and two constants $\alpha, \beta > \frac{1}{2}$, such that

$$\lim_{t\to\infty} \sup \int_l^t \rho^2(s)(R(t)-R(s))^{2\alpha}(R(s)-R(l))^{2\beta}[Gq(s)b(s)B(s) - \frac{\frac{1}{\eta}[2(\frac{\rho'(s)}{\rho(s)} + \frac{\beta R(t)-(\alpha+\beta)R(s)+\alpha R(l)}{(R(t)-R(s))(R(s)-R(l))}) + \frac{b'_+(s)}{b(s)}]^{\eta+1} b(s)r(s)B(s)}{(\frac{1+\eta}{\eta})^{\eta+1}[\eta\Gamma(1-\alpha)]^{\eta}}]\Phi^2(t,s,l)ds] > 0. \tag{3.16}$$

The every solution of (1.1) is oscillatory.

Remark 3.4. Theorems 3.3-3.5 are new because we introduce a class of kernel functions $\Phi = \Phi(t, s, l)$ which is basically a product $H(t, s)H(s, l)$ for a kernel $H(t, s)$ of Philos'type. On the other hand, when Eq. (1.1) becomes Eq. (*), conditions (3.12), (3.15), (3.16) become simpler, and they are stronger (in many case) than many exist oscillation conditions. Theorems 3.3, 3.4 improve and extend the results Theorems 3.2, 3.3 in [2].

4. Examples

Example 4.1. Consider the fractional differential equation

$$[\frac{1}{t^6}(D_-^{\frac{1}{2}}y)^{\eta}(t)]' - \frac{1}{t^7}(D_-^{\frac{1}{2}}y)^{\eta}(t) - \frac{1}{t^2}(\int_t^{\infty}(v-t)^{-\frac{1}{2}}y(v)dv) = 0, t > 0, \tag{4.1}$$

where $\alpha = \frac{1}{2}, \eta > 0$ is a quotient of odd positive integers and $(\eta+1)^{\eta+1}(\Gamma(\frac{1}{2}))^{\eta} > 1$. In (4.1),$r(t) = t^{-6}, p(t) = t^{-7}, q(t) = t^{-2}, f(u) = u$. Take $t_0 > 0, G = 1$. Since

$$B(s) = exp(-\int_{t_0}^t \frac{p(s)}{r(s)}ds) = exp(-\int_{t_0}^t \frac{1}{s}ds) = \frac{t_0}{t},$$
$$\int_{t_0}^{\infty} r^{-\frac{1}{\eta}}(s)B^{1-\frac{1}{\eta}}(s)ds = \int_{t_0}^{\infty} s^{\frac{6}{\eta}}(\frac{t_0}{s})^{1-\frac{1}{\eta}}ds = t_0^{1-\frac{1}{\eta}}\int_{t_0}^{\infty} s^{\frac{7}{\eta}-1}ds = \infty,$$

we find that(H_1) and (3.1) hold. We will apply Theorem 3.1, and it remains to satisfy the condition (3.2), taking $b(s) = s^2$, we obtain

$$\begin{aligned}
&\lim_{t\to\infty} \sup \int_{t_0}^t [Gb(s)q(s)B(s) - \frac{r(s)B(s)[b'_+(s)]^{\eta+1}}{(\eta+1)^{(\eta+1)}[\Gamma(1-\alpha)b(s)]^{\eta}}]ds \\
&= \lim_{t\to\infty} \sup \int_{t_0}^t [s^2 \cdot \frac{1}{s^2} \cdot \frac{t_0}{s} - \frac{s^{-6} \cdot \frac{t_0}{s} \cdot (2s)^{\eta+1}}{(\eta+1)^{(\eta+1)}[\Gamma(\frac{1}{2}) \cdot s^2]^{\eta}}]ds \\
&= \lim_{t\to\infty} \sup \int_{t_0}^t [\frac{t_0}{s} - \frac{s^{-\eta-6} \cdot 2^{\eta+1} \cdot t_0}{(\eta+1)^{(\eta+1)}[\Gamma(\frac{1}{2})]^{\eta}}]ds = \infty
\end{aligned}$$

which implies that (3.2) hold. Therefore, by Theorem3.1 every solution of(4.1) is oscillatory.

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References

[1] Y. Z. Wang, Z. L. Han, P. Zhao, S. Sun, *On the oscillation and asymptotic behavior for a kind of fractional differential equations*, Adv. in Diff. Equ., 50 (2014), 1-11.

[2] D. X. Chen, *Oscillatin criteria of fractional differential equations*, Adv. in Diff. Equ., 33 (2012), 1-10.

[3] B. Zhang, Q.H. Feng, *Some new oscillation criteria for a class of nonlinear fractional differential equations with damping term*, Journal of Appl. Math., 2013, 1-11.

[4] H. Qin, B.Zheng, *Oscillation of a class of fractional differential equations with damping term*, The Scientific World Journal, 115 (2013), 1-9.

[5] K. S.Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations*, John Wiley and Sons New York, 1993.

[6] I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, 1999.

[7] A. A. Kilbas, H.M. Srivastava, J.J. Trujillo, *Theory and applications of fractional differential equations*, Elsevier Science BV, Amsterdam, 2006.

[8] S. Das, *Functional fractional calculus for system identification and controls*, Springer, New York, 2008.

[9] R. L. Bagley, P.J.Torvik, *A theoretical basis for the application of fractional calculus to viscoelasticity*, J. Rheol., 27 (1983), 201-210.

[10] F. Mainardi, *Fractional calculus:Some basic problem in continuum and statistical mechanics*, In: Carpinteri, A, Mainardi, F(eds.)Fractals and Fractional Calculus in Continuum Mechanics, 291-348, Springer, New York, 1997.

[11] R. L. Magin, *Fractional calculus in bioengineering*, Crit. Rev. Biomed Eng., 32 (2004), 1-377.

[12] Y. Zhou, F. Jiao, J. Li, *Explicit and uniqueness for p-type fractional neutral differential equations*, Nonlinear Anal., 71 (2009), 2724-2733.

[13] L. Galeone, R.Garrappa, *Explicit methods for fractional differential equations and their stability properties*, J. Comput. Appl. Math., 288 (2009), 548-560.

[14] W. Deng, *Smoothness and stability of the solutions for nonlinear fracyional differential equations*, Nonlinear Anal., 72 (2010), 1768-1777.

[15] G.H. Hardy, J.E. Littlewood, G.Polya, *Ineqalities*, Cambridge University Press,Gambridge, 1988.

[16] Y. G. Sun, *New Kamenev-type oscillation criteria for second-order nonlinear differential equations with damping*, J. Math. Anal. Appl., 291 (2004), 341-351.

[17] D. X. Chen, *Oscillatory behavior of a class of fractional differential equations with damping*, Univ. Politeh. Bucharest Sci. Bull., 75 (2013), 107-118.

[18] B. Zheng, *Oscillation for a class of nonlinear fractional differential equations with damping term*, J. Adv. Math. Stud., 6 (2013), 107-115.

[19] C. Qi, S. Huang, *Interval oscillation criteria for a class of fractional differential equations with damping term*, Math Probl. Eng. 2013, Article ID 301085 (2013).

[20] Limei Feng, Yibing Sun & Zhenlai Han. *Philos-type oscillation criteria for impulsive fractional differential equations*, Journal of Applied Mathematics and Computing, 62 (2020), 361-376.

[21] Pengxian Zhu, Qiaomin Xiang. *Oscillation criteria for a class of fractional delay differential equations*, Advances in Difference Equations, 2018.

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