A unified generalization and refinement of Hermite-Hadamard-type and Simpson-type inequalities via s-convex functions

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Abstract. In this paper, by introducing the incomplete beta function, we establish a multi-parameter integral inequality via s-convex functions, which provides a unified generalization and refinement of Hermite-Hadamard-type and Simpson-type inequalities. As applications, we illustrate that a number of Hermite-Hadamard-type and Simpson-type inequalities can be derived from the special cases of the main result.

Keywords: Hermite-Hadamard-type inequalities, Simpson-type inequalities, s-convex function, generalization, refinement, incomplete beta function.

1. Introduction

The theory of inequalities has been greatly developed since Jensen introduced the concept of convex functions 100 years ago. There are a large number of inequalities which are established by the convexity of functions (see, [1, 2, 3, 4]). Among these results, the Hermite-Hadamard inequality is one of the best known results in the literature, which is stated as follows:

Let $f : [a, b] \to \mathbb{R}$ be a convex function with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

Nowadays, the Hermite-Hadamard inequality has been studied extensively both in theory and in practical applications, see, e.g., [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17] and the references cited therein.

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Recently, it has attracted our attention that an extraordinary generalization of Hermite-Hadamard-type inequality was posted by Deng and Wu in [18], in which the Hermite-Hadamard type inequality was generalized by the way of n-time differentiable functions, as follows:

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|\\ &\le \frac{(|n-2|+1)(b-a)^n}{2(|n-2|+3)n!}\\ &\times\left[\frac{((n+1)\,|n-2|+n)\,\left|f^{(n)}(a)\right|^q+(|n-2|+2)\,\left|f^{(n)}(b)\right|^q}{(n+2)(\,|n-2|+1\,)}\right]^{\frac{1}{q}},\end{aligned}\tag{2}$$

where $f^{(n)}$ is integrable and $|f^{(n)}|^q$ is convex on $[a,b]$, $n \ge 1$, $q \ge 1$.

The main role of above-mentioned inequality is to provides an estimation to the difference between the middle and rightmost terms in the Hermite-Hadamard inequality (1). This result also leads us to pay attention to another famous inequality, called Simpson's inequality, which gives the estimate of the error term in the quadrature formula [19], i.e.,

$$\left|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{(b-a)^4}{2880}\|f^{(4)}\|_\infty,\tag{3}$$

where $f:[a,b]\to\mathbb{R}$ is a four times continuously differentiable function on (a,b) and $\|f^{(4)}\|_\infty=\sup_{x\in(a,b)}|f^{(4)}(x)|<\infty$.

Motivated by the above-mentioned results, in this paper, by introducing more parameters, we establish a multi-parameter integral inequality via s-convex functions, which provides a unified generalization and refinement of inequalities (2) and (3). The methods used are mainly based on the representation of integral using the incomplete beta function and the extension of convexity via the s-convex functions.

The remaining parts of this paper are organized as follows. In Section 2, we present some definitions and lemmas which are essential in the proof of the main results. In Sections 3, we establish our main result, in which a unified generalization and refinement of inequalities (2) and (3) is proved. In Sections 3 and 4, we explain the applications of our main result with two aspects corresponding to the two types of integral inequalities, we show that a lot of Hermite-Hadamard-type and Simpson-type inequalities can be derived respectively when some suitable values are assigned to the parameters.

2. Definitions and lemmas

We begin with introducing some essential definitions and lemmas in preparation for the proof of our main result.

Definition 2.1 ([5]). *Let $I \subseteq \mathbb{R}$ be an interval. Then a real-valued function $f : I \to \mathbb{R}$ is said to be convex on I if the inequality*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) \tag{4}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

In [20], Hudzik and Maligranda introduced the class of functions which are s-convex in the second sense, as follows:

Definition 2.2. *A real-valued function $f : [0, \infty) \to \mathbb{R}$ is said to be s-convex in the second sense if*

$$f(\lambda x + (1-\lambda)y) \leq \lambda^s f(x) + (1-\lambda)^s f(y) \tag{5}$$

holds for all $x, y \in [0, \infty)$, $\lambda \in [0, 1]$ and for some fixed $s \in (0, 1]$.

It can be easily observed that for $s = 1$ s-convexity reduces to ordinary convexity of functions defined on $[0, \infty)$.

Below are two lemmas, we will give a representation of integral via the incomplete beta function and establish an integral identity.

Lemma 2.1. *Let $\sigma > 0$, $\upsilon > 0$, $\zeta \geq 0$, $0 < \tau \leq 1$. Then we have*

$$\begin{aligned} \mathcal{N}(\tau, \sigma, \upsilon, \zeta) &:= \int_0^\tau x^{\sigma-1}(1-x)^{\upsilon-1}|\zeta - x|dx \\ &= \begin{cases} \zeta B_\tau(\sigma, \upsilon) - B_\tau(\sigma+1, \upsilon), & \zeta \geq \tau, \\ 2\zeta B_\zeta(\sigma, \upsilon) - 2B_\zeta(\sigma+1, \upsilon) & \\ \qquad -\zeta B_\tau(\sigma, \upsilon) + B_\tau(\sigma+1, \upsilon), & 0 \leq \zeta < \tau \end{cases} \end{aligned} \tag{6}$$

where $B_t(\kappa, \iota)$ $(0 < t < 1)$ and $B_1(\kappa, \iota)$ denote respectively the incomplete beta function and the beta function [21], i.e.,

$$B_t(\kappa, \iota) = \int_0^t x^{\kappa-1}(1-x)^{\iota-1}dx, \quad \kappa, \iota > 0,$$
$$B_1(\kappa, \iota) = \int_0^1 x^{\kappa-1}(1-x)^{\iota-1}dx, \quad \kappa, \iota > 0.$$

Proof. We compute the integral $\mathcal{N}(\tau,\sigma,\upsilon,\zeta)$ by discussing separately two cases of $\zeta \geq \tau$ and $0 \leq \zeta < \tau$, it follows that

$$\begin{aligned}
\mathcal{N}(\tau,\sigma,\upsilon,\zeta) &= \int_0^\tau x^{\sigma-1}(1-x)^{\upsilon-1}|\zeta - x|dx \\
&= \begin{cases} \int_0^\tau x^{\sigma-1}(1-x)^{\upsilon-1}(\zeta - x)dx, & \zeta \geq \tau, \\ \int_0^\zeta x^{\sigma-1}(1-x)^{\upsilon-1}(\zeta - x)dx \\ \qquad + \int_\zeta^\tau x^{\sigma-1}(1-x)^{\upsilon-1}(x-\zeta)dx, & 0 \leq \zeta < \tau. \end{cases} \\
&= \begin{cases} \int_0^\tau x^{\sigma-1}(1-x)^{\upsilon-1}(\zeta - x)dx, & \zeta \geq \tau, \\ 2\int_0^\zeta x^{\sigma-1}(1-x)^{\upsilon-1}(\zeta - x)dx \\ \qquad - \int_0^\tau x^{\sigma-1}(1-x)^{\upsilon-1}(\zeta-x)dx, & 0 \leq \zeta < \tau \end{cases} \\
&= \begin{cases} \int_0^\tau \zeta x^{\sigma-1}(1-x)^{\upsilon-1}dx \\ \qquad - \int_0^\tau x^{\sigma}(1-x)^{\upsilon-1}dx, & \zeta \geq \tau, \\ 2\int_0^\zeta \zeta x^{\sigma-1}(1-x)^{\upsilon-1}dx \\ \qquad -2\int_0^\zeta x^{\sigma}(1-x)^{\upsilon-1}dx \\ \qquad - \int_0^\tau \zeta x^{\sigma-1}(1-x)^{\upsilon-1}dx \\ \qquad + \int_0^\tau x^{\sigma}(1-x)^{\upsilon-1}dx, & 0 \leq \zeta < \tau \end{cases} \\
&= \begin{cases} \zeta B_\tau(\sigma,\upsilon) - B_\tau(\sigma+1,\upsilon), & \zeta \geq \tau, \\ 2\zeta B_\zeta(\sigma,\upsilon) - 2B_\zeta(\sigma+1,\upsilon) \\ \qquad -\zeta B_\tau(\sigma,\upsilon) + B_\tau(\sigma+1,\upsilon), & 0 \leq \zeta < \tau. \end{cases}
\end{aligned}$$

The proof of Lemma 2.1 is complete. □

Lemma 2.2. *Let $f : [a,b] \to \mathbb{R}$, and let $f^{(n)}$ $(n \geq 1)$ be integrable on $[a,b]$. Then for $\mu, \lambda \in \mathbb{R}$ and $\theta \in [0,1]$, we have the following identity:*

$$\frac{(b-a)^n}{n!}\left[\int_0^\theta x^{n-1}(n\lambda - x)f^{(n)}(xa+(1-x)b)dx \right.$$
$$\left. - \int_\theta^1 (x-1)^{n-1}(n\mu + x - 1)f^{(n)}(xa+(1-x)b)dx\right]$$

$$
\begin{aligned}
(7)\qquad & = \mu f(a) + \lambda f(b) - (\lambda + \mu - 1) f(\theta a + (1-\theta) b) - \frac{1}{b-a}\int_a^b f(x)dx \\
& - \sum_{k=1}^{n-1} \frac{(b-a)^k}{(k+1)!}\Big[\lambda(k+1)\theta^k - \theta^{k+1} + \mu(k+1)(\theta-1)^k + (\theta-1)^{k+1}\Big] \\
& \times f^{(k)}(\theta a + (1-\theta) b).
\end{aligned}
$$

Proof. Using integration by parts for $n-1$ times, we obtain

$$
\begin{aligned}
I = & \int_0^\theta x^{n-1}(n\lambda - x) f^{(n)}(xa + (1-x)b)dx \\
& - \int_\theta^1 (x-1)^{n-1}(n\mu + x - 1) f^{(n)}(xa + (1-x)b)dx \\
& = \sum_{j=1}^{n-1} \frac{f^{(n-j)}(\theta a + (1-\theta)b)}{(b-a)^j} \frac{n!}{(n+1-j)!}\Big[(\theta^{n+1-j} - (n+1-j)\lambda\theta^{n-j}) \\
& - ((\theta-1)^{n+1-j} + (n+1-j)\mu(\theta-1)^{n-j})\Big] + \frac{n!}{(b-a)^{n-1}} \\
& \times \left[\int_0^\theta (\lambda - x) f'(xa + (1-x)b)dx - \int_\theta^1 (\mu + x - 1) f'(xa + (1-x)b)dx\right].
\end{aligned}
$$

Again, using integration by parts again yields

$$
\begin{aligned}
I = & \sum_{j=1}^{n-1} \frac{f^{(n-j)}(\theta a + (1-\theta)b)}{(b-a)^j} \frac{n!}{(n+1-j)!}\Big[(\theta^{n+1-j} - (n+1-j)\lambda\theta^{n-j}) \\
& - ((\theta-1)^{n+1-j} + (n+1-j)\mu(\theta-1)^{n-j})\Big] - \frac{n!}{(b-a)^{n+1}}\int_a^b f(x)dx \\
& + \frac{n!\Big(\mu f(a) + \lambda f(b) - (\lambda + \mu - 1) f(\theta a + (1-\theta)b\Big)}{(b-a)^n}.
\end{aligned}
$$

Performing a substitution $j \to n-k$ gives

$$
\begin{aligned}
I = & \sum_{k=1}^{n-1} \frac{f^{(k)}(\theta a + (1-\theta)b)}{(b-a)^{n-k}} \frac{n!}{(k+1)!}\Big[(\theta^{k+1} - (k+1)\lambda\theta^k) \\
& - ((\theta-1)^{k+1} + (k+1)\mu(\theta-1)^k)\Big] - \frac{n!}{(b-a)^{n+1}}\int_a^b f(x)dx \\
& + \frac{n!\Big(\mu f(a) + \lambda f(b) - (\lambda + \mu - 1) f(\theta a + (1-\theta)b\Big)}{(b-a)^n}.
\end{aligned}
$$

Multiplying both side of the above equation by $(b-a)^n/n!$ leads to the desired identity (7). This completes the proof of Lemma 2.2. □

3. Main result

Our main result is stated in the following theorem, which provides a unified generalization and refinement of inequalities (2) and (3).

Theorem 3.1. *Let $f:[0,\infty)\to\mathbb{R}$ be n-time differentiable function and $a,b\in[0,\infty)$ with $a<b$. If $f^{(n)}$ is integrable and $|f^{(n)}|^q$ is s-convex on $[a,b]$, $n\geq 1$, $q\geq 1$, $0<s\leq 1$ and $\theta,\lambda,\mu\in[0,1]$, then*

$$\begin{aligned}
&\Bigg|\mu f(a)+\lambda f(b)-(\lambda+\mu-1)f(\theta a+(1-\theta)b)-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=1}^{n-1}\frac{(b-a)^k}{(k+1)!}\\
&\times\Big[\lambda(k+1)\theta^k-\theta^{k+1}+\mu(k+1)(\theta-1)^k+(\theta-1)^{k+1}\Big]f^{(k)}(\theta a+(1-\theta)b)\Bigg|\\
&\leq\frac{(b-a)^n}{n!}\Bigg[(\mathcal{N}(\theta,n,1,n\lambda))^{1-\frac{1}{q}}\Big(\mathcal{N}(\theta,n+s,1,n\lambda)|f^{(n)}(a)|^q \qquad (8)\\
&+\mathcal{N}(\theta,n,s+1,n\lambda)|f^{(n)}(b)|^q\Big)^{\frac{1}{q}}+(\mathcal{N}(1-\theta,n,1,n\mu))^{1-\frac{1}{q}}\\
&\times\Big(\mathcal{N}(1-\theta,n,s+1,n\mu)|f^{(n)}(a)|^q+\mathcal{N}(1-\theta,n+s,1,n\mu)|f^{(n)}(b)|^q\Big)^{\frac{1}{q}}\Bigg]\\
&\leq\frac{(b-a)^n}{n!}(\mathcal{N}(\theta,n,1,n\lambda)+\mathcal{N}(1-\theta,n,1,n\mu))^{1-\frac{1}{q}} \qquad (9)\\
&\times\Bigg[\Big(\mathcal{N}(\theta,n+s,1,n\lambda)+\mathcal{N}(1-\theta,n,s+1,n\mu)\Big)|f^{(n)}(a)|^q\\
&+\Big(\mathcal{N}(\theta,n,s+1,n\lambda)+\mathcal{N}(1-\theta,n+s,1,n\mu)\Big)|f^{(n)}(b)|^q\Bigg]^{\frac{1}{q}},
\end{aligned}$$

where $\mathcal{N}(\theta,n,1,n\lambda)$, $\mathcal{N}(\theta,n+s,1,n\lambda)$, $\mathcal{N}(\theta,n,s+1,n\lambda)$, $\mathcal{N}(1-\theta,n,1,n\mu)$, $\mathcal{N}(1-\theta,n,s+1,n\mu)$, $\mathcal{N}(1-\theta,n+s,1,n\mu)$ are given by the formula (6).

Proof. Let

$$\begin{aligned}
&\mathcal{H}(\theta,n,\lambda,\mu):=\mu f(a)+\lambda f(b)-(\lambda+\mu-1)f(\theta a+(1-\theta)b)\\
&-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=1}^{n-1}\frac{(b-a)^k}{(k+1)!}\\
&\times\Big[\lambda(k+1)\theta^k-\theta^{k+1}+\mu(k+1)(\theta-1)^k+(\theta-1)^{k+1}\Big]f^{(k)}(\theta a+(1-\theta)b).
\end{aligned}$$

Then, form Lemma 2.2, one has

$$|\mathcal{H}(\theta,n,\lambda,\mu)|=\frac{(b-a)^n}{n!}\left|\int_0^\theta x^{n-1}(n\lambda-x)f^{(n)}(xa+(1-x)b)dx\right.$$

$$- \int_\theta^1 (x-1)^{n-1}(n\mu + x - 1) f^{(n)}(xa + (1-x)b)dx\Big|$$
$$\le \frac{(b-a)^n}{n!}\left[\int_0^\theta \Big|x^{n-1}(n\lambda - x) f^{(n)}(xa+(1-x)b)\Big| dx \right.$$
$$\left. + \int_\theta^1 \Big|(x-1)^{n-1}(n\mu + x - 1) f^{(n)}(xa+(1-x)b)\Big| dx\right].$$

Using the Hölder integral inequality, we obtain

$$|\mathcal{H}(\theta, n, \lambda, \mu))| \le \frac{(b-a)^n}{n!}\left[\Big(\int_0^\theta |x^{n-1}(n\lambda - x)|dx\Big)^{1-\frac{1}{q}}\right.$$
$$\times \Big(\int_0^\theta |x^{n-1}(n\lambda - x)||f^{(n)}(xa+(1-x)b)|^q dx\Big)^{\frac{1}{q}}$$
$$+ \Big(\int_\theta^1 |(x-1)^{n-1}(n\mu + x - 1)|dx\Big)^{1-\frac{1}{q}}$$
$$\left.\times \Big(\int_\theta^1 |(x-1)^{n-1}(n\mu + x - 1)||f^{(n)}(xa+(1-x)b)|^q dx\Big)^{\frac{1}{q}}\right].$$

Further, utilizing the s-convexity of $|f^{(n)}(x)|^q$, we deduce that

$$|\mathcal{H}(\theta, n, \lambda, \mu))| \le \frac{(b-a)^n}{n!}\left[\Big(\int_0^\theta |x^{n-1}(n\lambda - x)|dx\Big)^{1-\frac{1}{q}}\right.$$
$$\times \Big(\int_0^\theta \Big(x^{n+s-1}|n\lambda - x||f^{(n)}(a)|^q + x^{n-1}(1-x)^s|n\lambda - x||f^{(n)}(b)|^q\Big)dx\Big)^{\frac{1}{q}}$$
$$+ \Big(\int_\theta^1 |(x-1)^{n-1}(n\mu + x - 1)|dx\Big)^{1-\frac{1}{q}}$$
$$\times \Big(\int_\theta^1 \Big((1-x)^{n-1}x^s|(n\mu + x - 1)||f^{(n)}(a)|^q$$
$$\left. + (1-x)^{n+s-1}|(n\mu + x - 1)||f^{(n)}(b)|^q\Big)dx\Big)^{\frac{1}{q}}\right]$$
$$= \frac{(b-a)^n}{n!}\left[\Big(\int_0^\theta |x^{n-1}(n\lambda - x)|dx\Big)^{1-\frac{1}{q}}\right.$$
$$\times \Big(\int_0^\theta x^{n+s-1}|n\lambda - x||f^{(n)}(a)|^q dx + \int_0^\theta x^{n-1}(1-x)^s|n\lambda - x||f^{(n)}(b)|^q dx\Big)^{\frac{1}{q}}$$
$$+ \Big(\int_0^{1-\theta} x^{n-1}|(n\mu - x)|dx\Big)^{1-\frac{1}{q}}\Big(\int_0^{1-\theta} x^{n-1}(1-x)^s|(n\mu - x)||f^{(n)}(a)|^q dx$$
$$\left. + \int_0^{1-\theta} x^{n+s-1}|(n\mu - x)||f^{(n)}(b)|^q dx\Big)^{\frac{1}{q}}\right].$$

Now, by means of the integral representation via the incomplete beta function described in Lemma 2.1, we obtain

$$|\mathcal{H}(\theta,n,\lambda,\mu))| \leq \frac{(b-a)^n}{n!}\Bigg[(\mathcal{N}(\theta,n,1,n\lambda))^{1-\frac{1}{q}}\Big(\mathcal{N}(\theta,n+s,1,n\lambda)|f^{(n)}(a)|^q$$

$$+\mathcal{N}(\theta,n,s+1,n\lambda)|f^{(n)}(b)|^q\Big)^{\frac{1}{q}}+(\mathcal{N}(1-\theta,n,1,n\mu))^{1-\frac{1}{q}}$$

$$\times\Big(\mathcal{N}(1-\theta,n,s+1,n\mu)|f^{(n)}(a)|^q+\mathcal{N}(1-\theta,n+s,1,n\mu)|f^{(n)}(b)|^q\Big)^{\frac{1}{q}}\Bigg],$$

which is the desired inequality (8).

Now, let us turn to the proof of inequality (9). In fact, the inequality (9) can be derived directly from the following discrete Hölder inequality

$$x_1^{1-\frac{1}{q}}x_2^{\frac{1}{q}}+y_1^{1-\frac{1}{q}}y_2^{\frac{1}{q}} \leq (x_1+y_1)^{1-\frac{1}{q}}(x_2+y_2)^{\frac{1}{q}} \quad (q\geq 1)$$

with a choice of

$$x_1=\mathcal{N}(\theta,n,1,n\lambda),$$
$$x_2=\mathcal{N}(1-\theta,n,1,n\mu),$$
$$y_1=\mathcal{N}(\theta,n+s,1,n\lambda)|f^{(n)}(a)|^q+\mathcal{N}(\theta,n,s+1,n\lambda)|f^{(n)}(b)|^q,$$
$$y_2=\mathcal{N}(1-\theta,n,s+1,n\mu)|f^{(n)}(a)|^q+\mathcal{N}(1-\theta,n+s,1,n\mu)|f^{(n)}(b)|^q.$$

The proof of Theorem 3.1 is complete. □

4. Applications to the establishing of Hermite-Hadamard-type inequalities

In this section, we illustrate that some Hermite-Hadamard-type inequalities can be derived from the special cases of Theorem 3.1.

Putting $\lambda=\mu=\frac{1}{2}$ and $\theta=1$ in the inequalities of Theorem 3.1, we obtain a generalization of Hermite-Hadamard-type inequality (2), as follows:

Corollary 4.1. *Let $f:[0,\infty)\to\mathbb{R}$ be n-time differentiable function and $a,b\in[0,\infty)$ with $a<b$. If $f^{(n)}$ is integrable and $|f^{(n)}|^q$ is s-convex on $[a,b]$, $n\geq 1$, $q\geq 1$, $0<s\leq 1$, then*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$

$$\leq \frac{(b-a)^n}{n!}\Big(\mathcal{N}\Big(1,n,1,\frac{n}{2}\Big)\Big)^{1-\frac{1}{q}} \tag{10}$$

$$\times\left[\Big(\mathcal{N}\Big(1,n+s,1,\frac{n}{2}\Big)\Big)|f^{(n)}(a)|^q+\Big(\mathcal{N}\Big(1,n,s+1,\frac{n}{2}\Big)\Big)|f^{(n)}(b)|^q\right]^{\frac{1}{q}},$$

where $\mathcal{N}(1,n,1,\frac{n}{2})$, $\mathcal{N}(1,n+s,1,\frac{n}{2})$, $\mathcal{N}(1,n,s+1,\frac{n}{2})$ *are given by the formula* (6).

If we take $n=1$ and $n=2$ in inequality (10) respectively, we obtain

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{(b-a)}{4\times(2s+2^{1-s})^{-\frac{1}{q}}}\left[\frac{|f'(a)|^q+|f'(b)|^q}{(s+1)(s+2)}\right]^{\frac{1}{q}}, \tag{11}$$

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{(b-a)^2}{2\times 6^{1-\frac{1}{q}}}\left[\frac{|f''(a)|^q+|f''(b)|^q}{(s+2)(s+3)}\right]^{\frac{1}{q}}. \tag{12}$$

Setting $s=1$ in (10), we get

Corollary 4.2. *Let* $f:[0,\infty)\to\mathbb{R}$ *be* n*-time differentiable function and* $a,b\in[0,\infty)$ *with* $a<b$. *If* $f^{(n)}$ *is integrable and* $|f^{(n)}|^q$ *is convex on* $[a,b]$, $n\ge 1$, $q\ge 1$, *then*

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|\\ &\le\frac{(b-a)^n}{n!}\left(\mathcal{N}\left(1,n,1,\frac{n}{2}\right)\right)^{1-\frac{1}{q}}\\ &\times\left[\left(\mathcal{N}\left(1,n+1,1,\frac{n}{2}\right)\right)|f^{(n)}(a)|^q+\left(\mathcal{N}\left(1,n,2,\frac{n}{2}\right)\right)|f^{(n)}(b)|^q\right]^{\frac{1}{q}}.\end{aligned} \tag{13}$$

For $n=1$, inequality (13) becomes

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{(b-a)}{4}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}. \tag{14}$$

For $n=2$, inequality (13) reduces to

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right|\le\frac{(b-a)^2}{12}\left[\frac{|f''(a)|^q+|f''(b)|^q}{2}\right]^{\frac{1}{q}}. \tag{15}$$

Especially, for $n\ge 2$, a simple computation gives

$$\mathcal{N}\left(1,n,1,\frac{n}{2}\right)=\frac{n-1}{2(n+1)},$$

$$\mathcal{N}\left(1,n+1,1,\frac{n}{2}\right)=\frac{n^2-2}{2(n+1)(n+2)},$$

$$\mathcal{N}\left(1,n,2,\frac{n}{2}\right)=\frac{n}{2(n+1)(n+2)},$$

and then substituting them into inequality (13), we get

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx-\sum_{k=2}^{n-1}\frac{(k-1)(b-a)^k}{2(k+1)!}f^{(k)}(a)\right|$$

$$\leq \frac{(n-1)(b-a)^n}{2(n+1)!}\left[\frac{(n^2-2)\left|f^{(n)}(a)\right|^q+n\left|f^{(n)}(b)\right|^q}{(n-1)(n+2)}\right]^{\frac{1}{q}}, \tag{16}$$

which is equivalent to the inequality (2) ($n \geq 2$) that we have mentioned at the beginning section.

5. Applications to the establishing of Simpson-type inequalities

In this section, we show that some Simpson-type inequalities can be derived from the special cases of Theorem 3.1.

Putting $\lambda = \mu = \frac{1}{6}$ and $\theta = \frac{1}{2}$ in the inequalities of Theorem 3.1, we obtain a generalization of Simpson's inequality (3), as follows:

Corollary 5.1. *Let $f : [0,\infty) \to \mathbb{R}$ be n-time differentiable function and $a, b \in [0,\infty)$ with $a < b$. If $f^{(n)}$ is integrable and $|f^{(n)}|^q$ is s-convex on $[a,b]$, $n \geq 1$, $q \geq 1$, $0 < s \leq 1$, then*

$$\left|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\right.$$

$$\left.-\sum_{k=1}^{n-1}\frac{k-2}{6}\Big[\Big(-\frac{1}{2}\Big)^k+\Big(\frac{1}{2}\Big)^k\Big]\frac{(b-a)^k}{(k+1)!}f^{(k)}\Big(\frac{a+b}{2}\Big)\right|$$

$$\leq \frac{(b-a)^n}{n!}\Big(2\mathcal{N}\Big(\frac{1}{2},n,1,\frac{n}{6}\Big)\Big)^{1-\frac{1}{q}} \tag{17}$$

$$\times\Big(\mathcal{N}\Big(\frac{1}{2},n+s,1,\frac{n}{6}\Big)+\mathcal{N}\Big(\frac{1}{2},n,s+1,\frac{n}{6}\Big)\Big)^{\frac{1}{q}}\Big[|f^{(n)}(a)|^q+|f^{(n)}(b)|^q\Big]^{\frac{1}{q}},$$

where $\mathcal{N}(\frac{1}{2},n,1,\frac{n}{6})$, $\mathcal{N}(\frac{1}{2},n+s,1,\frac{n}{6})$, $\mathcal{N}(\frac{1}{2},n,s+1,\frac{n}{6})$ are given by the formula (6).

If we take $n = 1$ in inequality (17), we obtain

$$\left|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\right| \tag{18}$$

$$\leq \frac{5(b-a)}{36}\left[\frac{(s-4)6^{s+1}+2\times 5^{s+2}-2\times 3^{s+2}+2}{5\times 6^s(s+1)(s+2)}\right]^{\frac{1}{q}}\Big[|f'(a)|^q+|f'(b)|^q\Big]^{\frac{1}{q}}.$$

Setting $s = 1$ in (17), we get

Corollary 5.2. *Let $f:[0,\infty)\to\mathbb{R}$ be n-time differentiable function and $a,b\in[0,\infty)$ with $a<b$. If $f^{(n)}$ is integrable and $|f^{(n)}|^q$ is convex on $[a,b]$, $n\geq 1$, $q\geq 1$, then*

$$\begin{aligned}
&\bigg|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\\
&-\sum_{k=1}^{n-1}\frac{k-2}{6}\Big[\big(-\frac{1}{2}\big)^k+\big(\frac{1}{2}\big)^k\Big]\frac{(b-a)^k}{(k+1)!}f^{(k)}\Big(\frac{a+b}{2}\Big)\bigg|\\
&\leq\frac{(b-a)^n}{n!}\Big(2\mathcal{N}\Big(\frac{1}{2},n,1,\frac{n}{6}\Big)\Big)^{1-\frac{1}{q}}\\
&\times\Big(\mathcal{N}\Big(\frac{1}{2},n+1,1,\frac{n}{6}\Big)+\mathcal{N}\Big(\frac{1}{2},n,2,\frac{n}{6}\Big)\Big)^{\frac{1}{q}}\Big[|f^{(n)}(a)|^q+|f^{(n)}(b)|^q\Big]^{\frac{1}{q}}.
\end{aligned}\tag{19}$$

Choosing $n=1$ in inequality (19) yields

$$\begin{aligned}
&\bigg|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\bigg|\\
&\leq\frac{5(b-a)}{36}\left[\frac{|f'(a)|^q+|f'(b)|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}\tag{20}$$

Choosing $n=2$ in inequality (19), we obtain

$$\begin{aligned}
&\bigg|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\bigg|\\
&\leq\frac{(b-a)^2}{81}\left[\frac{|f''(a)|^q+|f''(b)|^q}{2}\right]^{\frac{1}{q}}.
\end{aligned}\tag{21}$$

Especially, for $n\geq 3$, a simple computation gives

$$\begin{aligned}
\mathcal{N}\Big(\frac{1}{2},n,1,\frac{n}{6}\Big)&=\frac{n-2}{6(n+1)}\left(\frac{1}{2}\right)^n,\\
\mathcal{N}\Big(\frac{1}{2},n+1,1,\frac{n}{6}\Big)&=\frac{n^2-n-3}{3\,(n+2)\,(n+1)}\left(\frac{1}{2}\right)^{n+2},\\
\mathcal{N}\Big(\frac{1}{2},n,2,\frac{n}{6}\Big)&=\frac{n^2+n-5}{3\,(n+2)\,(n+1)}\left(\frac{1}{2}\right)^{n+2},
\end{aligned}$$

and then substituting them into inequality (19), we get

$$\begin{aligned}
&\Bigg|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx \\
&-\sum_{k=1}^{n-1}\frac{k-2}{6}\Big[\Big(-\frac{1}{2}\Big)^k+\Big(\frac{1}{2}\Big)^k\Big]\frac{(b-a)^k}{(k+1)!}f^{(k)}\Big(\frac{a+b}{2}\Big)\Bigg| \\
&\le \frac{(n-2)(b-a)^n}{3\times 2^n(n+1)!}\left[\frac{|f^{(n)}(a)|^q+|f^{(n)}(b)|^q}{2}\right]^{\frac{1}{q}}. \qquad (22)
\end{aligned}$$

For $n=3$, inequality (22) reduces to

$$\begin{aligned}
&\Bigg|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\Bigg| \\
&\le \frac{(b-a)^3}{576}\left[\frac{|f'''(a)|^q+|f'''(b)|^q}{2}\right]^{\frac{1}{q}}. \qquad (23)
\end{aligned}$$

For $n=4$, inequality (22) becomes

$$\begin{aligned}
&\Bigg|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\Bigg| \\
&\le \frac{(b-a)^4}{2880}\left[\frac{|f^{(4)}(a)|^q+|f^{(4)}(b)|^q}{2}\right]^{\frac{1}{q}}. \qquad (24)
\end{aligned}$$

Inequality (24) is just the Simpson's inequality (3) that we have mentioned in the introduction section.

6. Concluding remarks

In Sections 3 and 4, we have shown the applications of our main result in establishing some generalizations of Hermite-Hadamard-type and Simpson-type inequalities, respectively. Here, we demonstrate that our main result given by Theorem 3.1 can also generate some refined inequalities of Hermite-Hadamard and Simpson type when some suitable values are assigned to the parameters. For example, if we take $s=1, \theta=\frac{1}{2}$ and $\lambda=\mu=\frac{1}{6}$ in Theorem 3.1, then, under the assumptions of Corollary 5.2, we have the following refinement of Simpson's inequalities:

$$\begin{aligned}
&\Bigg|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx \\
&-\sum_{k=1}^{n-1}\frac{k-2}{6}\Big[\Big(-\frac{1}{2}\Big)^k+\Big(\frac{1}{2}\Big)^k\Big]\frac{(b-a)^k}{(k+1)!}f^{(k)}\Big(\frac{a+b}{2}\Big)\Bigg|
\end{aligned}$$

$$\leq \frac{(b-a)^n}{n!}\Big(\mathcal{N}\Big(\frac{1}{2},n,1,\frac{n}{6}\Big)\Big)^{1-\frac{1}{q}} \tag{25}$$
$$\times\left[\Big(\mathcal{N}\Big(\frac{1}{2},n+1,1,\frac{n}{6}\Big)|f^{(n)}(a)|^q+\mathcal{N}\Big(\frac{1}{2},n,2,\frac{n}{6}\Big)|f^{(n)}(b)|^q\Big)^{\frac{1}{q}}\right.$$
$$\left.+\Big(\mathcal{N}\Big(\frac{1}{2},n,2,\frac{n}{6}\Big)|f^{(n)}(a)|^q+\mathcal{N}\Big(\frac{1}{2},n+1,1,\frac{n}{6}\Big)|f^{(n)}(b)|^q\Big)^{\frac{1}{q}}\right]$$
$$\leq \frac{(b-a)^n}{n!}\Big(2\mathcal{N}\Big(\frac{1}{2},n,1,\frac{n}{6}\Big)\Big)^{1-\frac{1}{q}} \tag{26}$$
$$\times\Big(\mathcal{N}\Big(\frac{1}{2},n+1,1,\frac{n}{6}\Big)+\mathcal{N}\Big(\frac{1}{2},n,2,\frac{n}{6}\Big)\Big)^{\frac{1}{q}}\Big[|f^{(n)}(a)|^q+|f^{(n)}(b)|^q\Big]^{\frac{1}{q}}.$$

Especially, when $n\geq 3$, the above inequalities reduce to the following refined inequalities of Simpson type.

$$\Bigg|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx$$
$$-\sum_{k=1}^{n-1}\frac{k-2}{6}\Big[\big(-\frac{1}{2}\big)^k+\big(\frac{1}{2}\big)^k\Big]\frac{(b-a)^k}{(k+1)!}f^{(k)}\Big(\frac{a+b}{2}\Big)\Bigg|$$
$$\leq \frac{(n-2)(b-a)^n}{3\times 2^{n+1}(n+1)!}\times\left[\Big(\frac{n^2-n-3}{2n^2-8}|f^{(n)}(a)|^q+\frac{n^2+n-5}{2n^2-8}|f^{(n)}(b)|^q\Big)^{\frac{1}{q}}\right. \tag{27}$$
$$\left.+\Big(\frac{n^2+n-5}{2n^2-8}|f^{(n)}(a)|^q+\frac{n^2-n-3}{2n^2-8}|f^{(n)}(b)|^q\Big)^{\frac{1}{q}}\right]$$
$$\leq \frac{(n-2)(b-a)^n}{3\times 2^n(n+1)!}\left[\frac{|f^{(n)}(a)|^q+|f^{(n)}(b)|^q}{2}\right]^{\frac{1}{q}}. \tag{28}$$

As a direct consequence, a refinement of Simpson's inequality can be derived by taking $n=4$ in the above inequalities, i.e.,

$$\Bigg|\frac{1}{6}\Big[f(a)+f(b)+4f\Big(\frac{a+b}{2}\Big)\Big]-\frac{1}{b-a}\int_a^b f(x)dx\Bigg|$$
$$\leq \frac{(b-a)^4}{5760}\left[\left(\frac{3}{8}|f^{(4)}(a)|^q+\frac{5}{8}|f^{(4)}(b)|^q\right)^{\frac{1}{q}}\right. \tag{29}$$
$$\left.+\left(\frac{5}{8}|f^{(4)}(a)|^q+\frac{3}{8}|f^{(4)}(b)|^q\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{(b-a)^4}{2880}\left[\frac{|f^{(4)}(a)|^q+|f^{(4)}(b)|^q}{2}\right]^{\frac{1}{q}}. \tag{30}$$

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