Ruled hypersurfaces in nonflat complex space forms satisfying Fischer-Marsden equation

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Abstract. In this paper, we prove that there exist no ruled hypersurfaces in a nonflat complex space form satisfying the Fischer-Marsden equation. This answers partially an open question posed by Venkatesha et al. in (Ann. Univ. Ferrara, 67 (2021), 203–216).
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1. Introduction

It is well known that there exist no Einstein real hypersurfaces in a nonflat complex space form $M^n(c)$ (cf. [2, 12]). Here by a nonflat complex space form $M^n(c)$ we refer to a complete and connected Kähler manifold with constant holomorphic sectional curvature $c \neq 0$ of complex dimension $n \geq 2$. It is complex analytically isometric to a complex projective space $\mathbb{C}P^n(c)$ if $c > 0$ or a complex hyperbolic space $\mathbb{C}H^n(c)$ if $c < 0$. In geometry of real hypersurface, it has been an active and interesting problem for a long time to research the existence and classification of some geometric conditions which generalize Einstein condition. For example, in 1979, Kon in [10] introduced pseudo-Einstein hypersurfaces and later they became an important research subject (see many references related with these hypersurfaces in [2, 12]). In 2009, Cho and Kimura in [4] first initiated the study of Ricci soliton on real hypersurfaces. Here by a Ricci soliton defined on a Riemannian manifold (M, g), we mean a triple (g, V, λ) (or shortly, a metric g) satisfying

$$\frac{1}{2}\mathcal{L}_V g + \mathrm{Ric} = \lambda g, \tag{1}$$

where V is a non-zero vector field, $\mathcal{L}$ is the Lie derivative and λ is a constant. When V is a Killing vector field, then a Ricci soliton becomes an Einstein metric. In particular, if V is the gradient of a smooth function f, then (1) becomes

$$\mathrm{Hess} f + \mathrm{Ric} = \lambda g, \tag{2}$$

and it is called a gradient Ricci soliton, where Hess denotes the Hessian operator. Ricci solitons are fixed points of the Ricci flow and play very important roles in modern differential geometry.

It was proved by Cho and Kimura in [5] that there exist no Hopf hypersurfaces which admits a gradient Ricci soliton in a nonflat complex space form. Some other studies involving Ricci solitons on real hypersurfaces can be seen in [1, 8, 11]. These results motivate many other research in which some other extensions of Einstein metrics were discussed. Next we exhibit some of them. A Riemannian manifold (M, g) is said to admit a Miao-Tam critical metric if on M there exists a smooth function f such that

$$\text{Hess}f - (\triangle f)g - f\text{Ric} = g. \tag{3}$$

Note that (3) reduces to an Einstein metric when f is a nonzero constant, just like that case in a gradient Ricci soliton. Applying Cho and Kimura's techniques in [5], Chen in [3] proved that there exist no Hopf real hypersurfaces with Miao-Tam critical metric in a nonflat complex space form. Similarly, a Riemannian manifold (M, g) is said to admit an m-quasi-Einstein metric if on M there exists a smooth function f such that

$$\text{Hess}f - \frac{1}{m}\text{d}f \otimes \text{d}f + \text{Ric} = \lambda g, \tag{4}$$

where m denotes a positive constant. Note that (4) reduces to still an Einstein metric if f is a constant. Applying those techniques in [5], Cui and Chen in [6] proved that there exist no Hopf real hypersurfaces with m-quasi Einstein metric in nonflat complex space forms. A Riemannian manifold (M, g) is said to admit Fischer-Marsden metric if on M there exists a smooth function f such that

$$\text{Hess}f - f\text{Ric} = (\Delta f)g, \tag{5}$$

The well known Fischer-Marsden conjecture states that a compact Riemannian manifold is Einstein if it admits a non-trivial solution to equation (5) (cf. [7]). In view of this, Fischer-Marsden equation (5) is also a nice extension of Einstein metrics. Applying those techniques in [5], Venkatesha et al. in [13] proved that there exist no complete Hopf hypersurfaces satisfying Fischer-Marsden equation in a nonflat complex space form. In addition, Venkatesha et al. in [13] proposed an open question:

Are there real hypersurfaces in nonflat complex space forms satisfying Fischer-Marsden equation?

The present paper aims to investigate the above problem on a special hypersurface. We prove that there exist no ruled hypersurfaces in a nonflat complex space form satisfying Fischer-Marsden equation. The proof of this result is given in the last section of the paper.

2. Preliminaries

Let M be a real hypersurface immersed in a complex space form $M^n(c)$ and N be a unit normal vector field of M. We denote by $\overline{\nabla}$ the Levi-Civita connection of the metric $\overline{g}$ of $M^n(c)$ and J the complex structure. Let g and ∇ be the induced metric from the ambient space and the Levi-Civita connection of g respectively. Then the Gauss and Weingarten formulas are given respectively as the following:

$$\overline{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \ \overline{\nabla}_X N = -AX, \tag{6}$$

for any $X, Y \in \mathfrak{X}(M)$, where A denotes the shape operator of M in $M^n(c)$. For any vector field X tangent to M, we put

$$JX = \phi X + \eta(X)N, \ JN = -\xi. \tag{7}$$

We can define on M an almost contact metric structure (ϕ, ξ, η, g) satisfying

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \ \eta(\xi) = 1, \ \phi\xi = 0, \tag{8}$$

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \ \eta(X) = g(X, \xi), \tag{9}$$

for any $X, Y \in \mathfrak{X}(M)$. If the structure vector field ξ is principal, that is, $A\xi = \alpha\xi$ at each point, where $\alpha = \eta(A\xi)$, then M is called a Hopf hypersurface and α is called Hopf principal curvature.

Moreover, applying the parallelism of the complex structure (i.e., $\overline{\nabla} J = 0$) of $M^n(c)$ and using (6), (7) we have

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{10}$$

$$\nabla_X \xi = \phi AX, \tag{11}$$

for any $X, Y \in \mathfrak{X}(M)$. Let R be the Riemannian curvature tensor of M. Because $M^n(c)$ is of constant holomorphic sectional curvature c, the Gauss and Codazzi equations of M in $M^n(c)$ are given respectively as the following:

$$\begin{aligned} R(X, Y)Z =& \frac{c}{4}\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &- 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY, \end{aligned} \tag{12}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = \frac{c}{4}\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}, \tag{13}$$

for any $X, Y \in \mathfrak{X}(M)$. From the Gauss equation, the Ricci operator is given by

$$Q = \frac{c}{4}((2n+1)\mathrm{id} - 3\eta \otimes \xi) + (\mathrm{trace}A)A - A^2. \tag{14}$$

3. Main results

Taking a regular curve γ in a nonflat complex space form $M^n(c)$ with tangent vector field X. There is a unique complex projective or hyperbolic hyperplane at each point of γ such that it cuts γ so as to be orthogonal to both X and JX.

The union of these hyperplanes is said to be a ruled real hypersurface ([9, 12]). A ruled hypersurface cannot be Hopf and has some interesting characterizations. For example, a real hypersurface in a nonflat complex space form is ruled if and only if $g(AX, Y) = 0$, for any vector fields X and Y orthogonal to ξ (cf. [9]). It follows that

$$\begin{aligned} A\xi &= \alpha\xi + \beta U, \\ AU &= \beta\xi, \\ AZ &= 0,\ \forall\, Z \in \{\xi, U\}^{\perp}, \end{aligned} \tag{15}$$

where $\alpha = g(A\xi, \xi)$, β is a smooth nowhere vanishing function and U is a unit vector field parallel to $\phi\nabla_{\xi}\xi$. Putting (15) into (14) we have

$$\begin{aligned} Q\xi &= \left(\frac{1}{2}(n-1)c - \beta^2\right)\xi, \\ QU &= \left(\frac{1}{4}(2n+1)c - \beta^2\right)U, \\ QZ &= \frac{1}{4}(2n+1)cZ,\ \forall\, Z \in \{\xi, U\}^{\perp}. \end{aligned} \tag{16}$$

It follows directly that the scalar curvature is $r = (n^2-1)c - 2\beta^2$. We collect some necessary properties of ruled hypersurfaces (cf. [9]) in the following lemma.

Lemma 3.1. *On a ruled hypersurface the following relations are valid:*

$$\begin{gathered} \nabla_U\phi U = \left(\frac{c}{4\beta} - \beta\right)U,\ \nabla_{\phi U}U = 0, \\ U(\beta) = 0,\ \phi U(\beta) = \beta^2 + \frac{c}{4},\ W(\beta) = 0,\ \forall\, W \in \{\xi, U, \phi U\}^{\perp}. \end{gathered} \tag{17}$$

Lemma 3.2. *On a real hypersurface in a noflat complex space form satisfying Fischer-Marsden equation, the following relation is valid:*

$$\begin{aligned} &\left(\frac{1}{2(n-1)}Y(fr) - \frac{c}{4}Y(f)\right)X - \left(\frac{1}{2(n-1)}X(fr) - \frac{c}{4}X(f)\right)Y \\ &+ (X(f)QY - Y(f)QX) + f((\nabla_X Q)Y - (\nabla_Y Q)X) - \frac{c}{2}g(X, \phi Y)\phi Df \\ &+ \frac{c}{4}(\phi X(f)\phi Y - \phi Y(f)\phi X) + AX(f)AY - AY(f)AX = 0, \end{aligned} \tag{18}$$

for any vector field X, Y, where Df denotes the gradient of function f.

Proof. Note that the Fischer-Marsden equation (5) can be transformed into the following

$$\nabla_X Df = (\Delta f)X + fQX,$$

for any vector field X. Contracting the above equality over X gives that $\Delta f = -\frac{fr}{2(n-1)}$. Putting this into the above equality gives

$$\nabla_X Df = -\frac{fr}{2(n-1)}X + fQX.$$

Taking the derivative of this equality we obtain

$$\nabla_Y\nabla_X Df = -\frac{1}{2(n-1)}Y(fr)X - \frac{fr}{2(n-1)}\nabla_Y X + Y(f)QX + f\nabla_Y(QX),$$

for any vector fields X, Y. Applying this equality and previous one in definition of the curvature tensor we have

$$\begin{aligned} R(X,Y)Df =& \frac{1}{2(n-1)}(Y(fr)X - X(fr)Y) \\ &+X(f)QY - Y(f)QX + f(\nabla_X Q)Y - f(\nabla_Y Q)X. \end{aligned} \tag{19}$$

On the other hand, replacing Z by Df in (12) we get

$$\begin{aligned} R(X,Y)Df =& \frac{c}{4}(Y(f)X - X(f)Y + \phi Y(f)\phi X - \phi X(f)\phi Y) \\ &+\frac{c}{2}g(X,\phi Y)\phi Df + AY(f)AX - AX(f)AY. \end{aligned}$$

Comparing the above equality with (19) gives (18). □

With the help of (15), (16) and Lemma 3.1, by a direct calculation we have

$$(\nabla_\xi Q)U - (\nabla_U Q)\xi = -2\beta\xi(\beta)U - \beta^2\nabla_\xi U.$$

Note that we have applied $\nabla_\xi U \in \{\xi, U\}^\perp$ due to $g(\nabla_\xi U, \xi) = 0$ and $g(\nabla_\xi U, U) = 0$. Form now on, suppose that a real hypersurface in a nonflat complex space form satisfies Fischer-Marsden equation. In (18), replacing X and Y by ξ and U, respectively, we obtain an equality. Taking the ξ-component of this equality gives

$$\frac{1}{2(n-1)}U(fr) - \frac{c}{4}U(f) - \left(\frac{n-1}{2}c - \beta^2\right)U(f) + \beta^2 U(f) = 0.$$

Substituting the scalar curvature $r = (n^2-1)c - 2\beta^2$ into the above equality and applying Lemma 3.1, we get

$$\left(\frac{2n-3}{n-1}\beta^2 + \frac{3}{4}c\right)U(f) = 0.$$

Suppose that there exists a point p on the hypersurface such that $U(f) \neq 0$ at p and hence on an open neighborhood Ω around p. Thus, working on Ω we obtain $\frac{2n-3}{n-1}\beta^2 + \frac{3}{4}c = 0$. Then β is a constant. Applying (17) again we obtain $\beta^2 + \frac{c}{4} = 0$. Putting this into the previous one reduces to either $n = 0$ or $c = 0$, a contradiction. Therefore, $U(f) = 0$ holds on the whole of the hypersurface.

With the help of (15), (16) and Lemma 3.1, by a direct calculation we have

$$(\nabla_\xi Q)\phi U - (\nabla_{\phi U} Q)\xi = \frac{2n+1}{4}c\nabla_\xi\phi U - Q\nabla_\xi\phi U + 2\beta\left(\beta^2 + \frac{c}{4}\right)\xi.$$

In (18), replacing X and Y by ξ and ϕU, respectively, we obtain an equality. Taking the ξ-component of this equality gives

$$\frac{1}{2(n-1)}\phi U(fr) - \frac{c}{4}\phi U(f) - \left(\frac{n-1}{2}c - \beta^2\right)\phi U(f) + \frac{2n+1}{4}cfg(\nabla_\xi \phi U, \xi) - fg(Q\nabla_\xi \phi U, \xi) + 2f\beta(\beta^2 + \frac{c}{4}) = 0.$$

Substituting the scalar curvature $r = (n^2-1)c - 2\beta^2$ into the above equality and applying Lemma 3.1, we get

$$\left(\frac{n-2}{n-1}\beta^2 + \frac{3}{4}c\right)\phi U(f) + \frac{n-3}{n-1}f\beta^3 - \frac{n+1}{4(n-1)}cf\beta = 0. \tag{20}$$

With the help of (15), (16) and Lemma 3.1, by a direct calculation we have

$$(\nabla_U Q)\phi U - (\nabla_{\phi U} Q)U = \beta\left(\beta^2 + \frac{c}{2}\right)U.$$

In (18), replacing X and Y by U and ϕU, respectively, we obtain an equality. Applying the fact $U(f) = 0$ and taking the U-component of this equality gives

$$\frac{1}{2(n-1)}\phi U(fr) - \frac{c}{4}\phi U(f) - \left(\frac{2n+1}{4}c - \beta^2\right)\phi U(f) + f\beta\left(\beta^2 + \frac{c}{2}\right) - \frac{c}{4}\phi U(f) - \frac{c}{2}\phi U(f) = 0.$$

Substituting the scalar curvature $r = (n^2-1)c - 2\beta^2$ into the above equality and applying Lemma 3.1, we get

$$\left(\frac{n-2}{n-1}\beta^2 - \frac{3}{4}c\right)\phi U(f) + \frac{n-3}{n-1}f\beta^3 + \frac{n+1}{2(n-1)}cf\beta = 0. \tag{21}$$

Subtracting (20) from (21) we obtain $\phi U(f) = \frac{1}{2}f\beta$ because of $c \neq 0$. Substituting this into (20) we get

$$\frac{3n-8}{2(n-1)}\beta^2 + \frac{3}{8}c - \frac{n+1}{4(n-1)}c = 0.$$

This means that β is a constant, and hence from Lemma 3.1 we have $\beta^2 + \frac{c}{4} = 0$. Putting this into the above equality we arrive at a contradiction. Therefore, we obtain the following result.

Theorem 3.1. *There are no ruled hypersurfaces in nonflat complex space forms satisfying Fischer-Marsden equation.*

Remark 3.1. Hopf and ruled hypersurfaces are ones of the most classical real hypersurfaces in a nonflat complex space form. Except for this two types of real hypersurfaces, the existence and classification problems of general non-Hopf real hypersurfaces satisfying Fischer-Marsden equation are still open questions.

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