Separation coordinates in a Hamiltonian quartic system

Nicola Sottocornola
Department of Mathematics and Statistics
Zayed University
Abu Dhabi
UAE
nicola.spinelli@zu.ac.ae

Abstract. The separability of Hamiltonian integrable systems has been the object of a considerable amount of attention in the last decades. Over the years several techniques have been proposed to deal with this difficult problem. In this paper we make use of the method of the Kowalewski's Conditions. To illustrate the effectiveness of the method we consider the Hénon-Heiles system known as HH4 1:6:8. This system is integrable in two cases. For one of them, separated only in some particular cases, we provide the separation coordinates in the generic form. The other case remains unsolved.
Keywords: integrable systems, separation of coordinates, integration in quadratures.

1. Introduction

Hénon-Heiles (HH) systems are Hamiltonian systems in $\mathbb{R}^4$ endowed with the standard symplectic form $dp_1 \wedge dx + dp_2 \wedge dy$. The Hamiltonian function has the form

$$H = \frac{1}{2}({p_1}^2 + {p_2}^2) + V(x, y),$$

where V is a polynomial function. There are four nontrivial integral cases with quartic potential that can be "generalized" adding inverse terms without destroying the integrability in the Liouville sense. This means that every one of these systems possesses an integral of the motion called K. The most general forms of H and K, for all the integrable HH systems, have been given by Hietarinta [3].

Once proved the Liouville integrability of these systems, the question arises of an explicit integration of the equations of motion. The most efficient way to bring the systems to quadratures is to find coordinates that separate the Hamilton-Jacobi equation. This is such a difficult task that, after decades of efforts, only one of the four quartic systems has been separated in the generic form [2]. In this paper we will deal with the so called HH4 1:6:8 system only (1, 6, 8 are the coefficients of the quartic monomials). The Hamiltonian function is

$$(1)\quad H = \frac{{p_1}^2}{2} + \frac{{p_2}^2}{2} + x^4 + 6\,x^2y^2 + 8\,y^4 + \omega\,\left(x^2 + 4\,y^2\right) + \frac{a}{y^2} + \frac{b^2}{x^2} - \frac{c^2}{2\,x^6} + ey,$$

where $\omega,\ a,\ b,\ c$ and e are arbitrary constants.

The function K for this system is quite complicated [3]:

$$\begin{aligned} K = {} & {p_1}^4 + 2\,{p_1}^2\left(2x^4 + 12\,x^2y^2 + 2\omega\,x^2 + 2\frac{b^2}{x^2} - \frac{c^2}{x^6} + 2ey\right) \\ & - 4xp_1p_2\left(4\,x^2y + e\right) + 4\,x^4{p_2}^2 \\ & + 4\,\frac{b^4}{x^4} + 8\,b^2x^2 + 16\,b^2y^2 + 4\,\omega^2x^4 + 8\,\omega\,x^6 + 16\,\omega\,x^4y^2 + 4\,x^8 \\ & + 16\,x^6y^2 + 16\,x^4y^4 + 8\,\frac{ax^4}{y^2} - c^2\left(4\,\frac{b^2}{x^8} - \frac{c^2}{x^{12}} + 4\frac{\omega}{x^4} + 4\,\frac{1}{x^2} + 24\,\frac{y^2}{x^4}\right) \\ & + 2e\left(4\,\frac{b^2y}{x^2} - ex^2 - 4\,x^4y - 8\,x^2y^3 - 4\,\omega\,x^2y\right). \end{aligned} \tag{2}$$

The reader can easily check that the Poisson bracket of H and K is

$$\{H,K\} = -\frac{4e\left(2ax^8p_1 - 3\,c^2xy^3p_2 + 6\,c^2y^4p_1\right)}{x^7y^3}$$

and this lets us with two cases of integrability:

- Case I: $a = c = 0$
- Case II: $e = 0$.

The first case has been solved only under the additional hypothesis $be = 0$ [13] and $e = 2\sqrt{2}\,b$ [12]; the separation coordinates for the generic case remain unknown.

Case II has been studied in the particular case $e = c = ab = 0$ [8]. The authors wrote, about adding the term in x^{-6} or the linear term: "it would be interesting to extend our approach to these cases although we anticipate serious technical difficulties". The aim of this paper is to show that these difficulties can be bypassed looking at the problem from a different perspective. Using the method of the Kowalewski Conditions (KC) we will be able to provide the separation coordinates, for Case II, in the generic form.

2. The method of the vector field Z

Let's introduce quickly the method adopted in the following calculations. A comprehensive presentation, with all the necessary proofs that are omitted here, can be found in [7] and [10].

Separable Hamiltonian systems come equipped with a torsionless recursive tensor N (Nijenhuis tensor), compatible with the Poisson tensor P, i.e. forming a so called PN manifold. If the manifold is 4-dimensional and N has two functionally independent eigenvalues, then they are the separation coordinates of the system (under suitable hypotheses, see below and [5]).

The explicit calculation of the tensor N can be quite cumbersome except in some simple cases [11]. Nevertheless, the essential remark is that N acts on the vector fields tangent to the Lagrangian foliation given by $H = c_1$ and $K = c_2$, so that one can simply calculate the eigenvalues of the restriction of N to the bi-dimensional foliation. This restricted tensor, given a basis on the leaves, reduces to a 2x2 matrix M called the Control Matrix. In the basis associated with the flows of the Hamiltonian vector fields X_H and X_K, this matrix has the form $M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$. The Kowalewski Conditions (KC), introduced by F. Magri in [6], characterize the entries of the matrix M. These functions verify four differential constraints if and only if[1] the eigenvalues of M are separation coordinates:

$$
\begin{aligned}
X_H(m_3) &= X_K(m_1) \\
X_H(m_4) &= X_K(m_2) \\
X_H(m_1 m_3 + m_3 m_4) &= X_K(m_1^2 + m_2 m_3) \\
X_H(m_2 m_3 + m_4^2) &= X_K(m_1 m_2 + m_2 m_4)
\end{aligned}
\tag{3}
$$

and the involutivity of the trace and the determinant if we want the eigenvalues of M to be canonical coordinates:

$$\{m_1 + m_4, m_1 m_4 - m_2 m_3\} = 0. \tag{4}$$

This is a system of 5 differential equations in 4 unknown functions and it is, in general, difficult to solve. A possible strategy to attack this problem is outlined in the following steps:

1. We start looking for two "Fundamental Functions" F and G verifying

   $$X_H(G) = X_K(F) \tag{5}$$

   and

   $$dF \wedge dG \wedge dH \wedge dK \neq 0.$$

   We can see (F, G, H, K) as non-canonical coordinates associated to the Lagrangian foliation. We use these coordinates to write the Control Matrix in the simplified form:

   $$M = \begin{pmatrix} AF + B & 1 \\ AG + C & D \end{pmatrix} \tag{6}$$

   where A, B, C and D are constant of the motion. In this way the first two equations in (3) are automatically satisfied and the constants A, B, C, D have to be chosen in such a way that the other equations are verified too. One could say that the method of the Fundamental Functions reduces the problem to the search of two functions only: F and G. An example of application of this method can be found in [9].

1. There are some additional technical conditions that are clearly verified in the present case. The complete Theorem can be found in [6] or [9].

2. The next step consists in introducing a "potential function" V and the canonical vector field Z associated to V : $Z = X_V$. The functions F and G can be generated by V in the following way:

$$F = Z(H) \qquad G = Z(K) \tag{7}$$

and equation (5) is still verified for any choice of V [10].

3. Unfortunately the method of the potential function seems excessively restrictive and many interesting problems don't fall under this scheme (several examples are given in [10]). The set of all possible fields Z must be enlarged. The idea is to use the constants a, b and c present in the Hamiltonian functions as variables, and turn the symplectic system into a Poisson one in $\mathbb{R}^7$ with coordinates $(p_1, p_2, x, y, a, b, c)$. This is easily obtained adding three lines and columns of zeros to the matrix representing the standard Poisson tensor and extending the canonical vector field X_f, associated to a function f, to $\widetilde{X}_f$:

$$\widetilde{X}_f = \left(-\frac{\partial f}{\partial x}, -\frac{\partial f}{\partial y}, \frac{\partial f}{\partial p_1}, \frac{\partial f}{\partial p_2}, 0, 0, 0\right)^T.$$

In this framework, the vector field Z can be extended with extra terms in this way

$$Z = X_V + w_1 \frac{\partial}{\partial a} + w_2 \frac{\partial}{\partial b} + w_3 \frac{\partial}{\partial c} \tag{8}$$

where w_1, w_2 and w_3 are constants.

At the end of the process, the problem is reduced to the determination of a single function V and, eventually, a few constants w_1, w_2 and w_3.
We are now ready to solve the generic Case II.

3. The separation coordinates for Case II

According to the discussion in the Introduction we replace $e = 0$ in (1) and (2). Our problem is to calculate the separation coordinates of this system without imposing any additional restriction to the remaining constants a, b and c.

Writing Z in the extended form (8), we can calculate the Fundamental Functions with (7) and finally obtain the Control Matrix in the simplified form (6).

Now, we have to replace $m_1, \dots, m_4$ into the KC (3). The first two equations are verified for any choice of the potential function and constants [10]. The second couple of KC are verified with $V = c/(2x^2)$ and the constant $w_1 = 0$, $w_2 = b/2$ and $w_3 = c$. Therefore, the vector field Z has the simple form

$$Z = \frac{c}{x^3} \frac{\partial}{\partial p_1} + \frac{b}{2} \frac{\partial}{\partial b} + c \frac{\partial}{\partial c}. \tag{9}$$

This vector field contains all the essential information needed to separate the system. Finally we still have to choose the constants of motion A, B, C and D in order to verify (4). The results can be summarized in the following

Theorem 3.1. *Consider the integrable Hamiltonian system* (1)-(2) *with* $e = 0$. *Let* Z *be the vector field in* (9) *and* F *and* G *the functions in* (7). *Then, the Control Matrix of the system takes the form*

$$M = \begin{pmatrix} -16F + 8H & 1 \\ -16G + 16K & 8H \end{pmatrix} \tag{10}$$

i.e. the eigenvalues of (10) *are canonical separation coordinates for both* H *and* K.

Proof. The functions F and G can be calculated directly with (7):

$$F = \frac{b^2 x^4 + p_1 c x^3 - c^2}{x^6}$$

and

$$\begin{aligned} G = \frac{1}{x^{12}} \Big[& 8\, b^2 x^{14} + 8\, c x^{13} p_1 + \left(16\, b^2 y^2 - 16\, c y p_2\right) x^{12} + 8\, c \left(6 y^2 + \omega\right) p_1 x^{11} \\ & + \left(4\, b^2 {p_1}^2 - 8\, c^2\right) x^{10} + 4\, c x^9 {p_1}^3 + \left(-48\, c^2 y^2 - 8\, \omega c^2 + 8\, b^4\right) x^8 \\ & + 8\, b^2 c x^7 p_1 - 4\, c^2 x^6 {p_1}^2 - 12\, c^2 b^2 x^4 - 4\, c^3 x^3 p_1 + 4\, c^4 \Big]. \end{aligned}$$

Replacing these functions in (10) one can find the explicit form of $m_1, \ldots, m_4$. According to the results in [6], it is enough to prove that these functions verify the KC (3), as well as the condition of canonicity (4). All these conditions can be easily checked with a software like Maple. □

Remark 3.1. Different Control Matrices can be obtained using more complicated entries, for instance quadratic functions in F and G:

$$M' = \begin{pmatrix} -16F^2 + G & F \\ -16FG + 16KF & G \end{pmatrix}.$$

The eigenvalues of M' provide a different set of separation coordinates. These coordinates reduce to the ones found by Ravoson et al. [8] in the case $a = c = 0$.

A similar method can be applied to Case I too and provides an alternative way to calculate the separation coordinates for the degenerate cases $be = 0$ [10]. In [12] we find the separation coordinates under the particular condition $e = 2\sqrt{2}\, b$. The idea was to guess the form of the potential function V taking example from these particular cases. However the application of the theory to the general case presents some difficulties: it seems that neither linear nor quadratic functions in F and G verify all conditions (3) and (4). Finding separation coordinates for Case I in the generic form remains an open problem.

On the other hand, separation coordinates in Case II could be found without any additional condition on the coefficients and the potential function is as simple as $V = c/(2x^2)$. This system represents, in our opinion, one of the most convincing examples of the effectiveness of the method of the KC. The complete separation of the system goes beyond the scopes of the paper and requires additional work. Nevertheless this paper could be considered as a first step in that direction.

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Accepted: December 14, 2021