A characterization of PSL$(4, p^2)$ by some character degree

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Abstract. Let G be a finite group and cd(G) be the set of irreducible character degree of G. In this paper we prove that if p is a prime number, then the simple group PSL(4,p^2) are uniquely determined by its order and some its character degrees.

Keywords: character degrees, order, projective special linear group.

1. Introduction

All groups considered are finite and all characters are complex characters. Let G be a group. Denote by Irr(G) the set of all irreducible characters of G. Let cd(G) be the set of all irreducible character degree of G.

Many authors were recently concerned with the following question:

What can be said about the structures of a finite group G , if some information is known about the arithmetical structure of the degree of the irreducible characters of G (see, [17,18]). A finite group G is called a K_3-group if $|G|$ has exactly three distinct prime divisors.

Yan et al. [17] and [18] proved that all simple k_3-group and the Mathieu groups are uniquely determined by their orders and some its character degrees.

Also, Khosravi et al. in [9] and [10] proved that the simple groups PSL$(2, p)$ and PSL$(2, p^2)$ are uniquely determined by its order and its largest and second largest irreducible character degrees, where p is an odd prime. Also, Hung

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and Thomson in [13] proved that the simple group PSL$(4, q)$ whit $q \geq 13$ are determined by the set of their character degrees.

Let p be an odd prime number. In [14] the authors proved that the simple group $PSL(4, p)$ is uniquely by its order and some character degrees.

The goal of this paper is to introduce a new characterization for the finite group PSL$(4, p^2)$, where p is prime, by its order and some its character degrees. In fact we prove the following theorem.

Theorem 1.1 (Main Theorem)**.** *Let $p > 7$ be a prime. If G is a finite group such that the following statements hold, then G is isomorphic to $PSL(4, p^2)$.*
(i) $|G| = |PSL(4, p^2)|$.
(ii) $kp^{12} \in cd(G)$ if only if $k = 1$, where k is an integer number.
(iii) $p^2(p^4 + p^2 + 1)$ is the smallest nonlinear character degree of G.
(iv) $\{p^2(p^2 + 1)^2(p^4 + 1), (p^2 + 1)(p^4 + 1)\} \subset cd(PSL(4, p^2))$.

2. Notation and preliminary

We know that if p is an odd prime, then

$$|PSL(4, p^2)| = \frac{p^{12}(p^4 - 1)(p^6 - 1)(p^8 - 1)}{(4, p^2 - 1)}$$

and let Φ_k denote the kth cyclotomic polynomial evaluated at p^2. In particular,

$$\Phi_1 = p^2 - 1, \Phi_2 = p^2 + 1, \Phi_3 = p^4 + p^2 + 1, \Phi_4 = p^4 + 1.$$

The data in [18] gives the character degree of $PSL(4, q)$. From there, we are able to extract the character degree of $PSL(4, p^2)$.These degrees are given in Table 1. The word "possible" in the second column means that the condition for the existence of corresponding degree in fairly complicated

$$\{p^{12}, p^2\Phi_3, p^2\Phi_2^2\Phi_4, \Phi_2\Phi_4\} \subset cd(PSL(4,^2 p))$$

and the smallest nonlinear character degrees of $PSL(4, p^2)$ is $p^2\Phi_3$.

If n is an integer and r is a prime number, then we write $r^\alpha || n$, when $r^\alpha | n$ but $r^{\alpha+1} \mid n$. All other notations are standard and we refer to [1].

If $N \trianglelefteq G$ and $\theta \in Irr(N)$, then the inertia group of θ in G is $I_G(\theta)$=$\{g \in G \mid \theta^g$=$\theta\ \}$.

Lemma 2.1 (Thompson, [14], Lemma 2.3)**.** *Suppos that p is a prime and $p \mid \chi(1)$ for every nonlinear $\chi \in Irr(G)$. Then, G has a normal p-complement.*

Lemma 2.2 (Ghallgher's Theorem, [8], Corollary 6.17)**.** *Let $N \trianglelefteq G$ and let $\chi \in Irr(G)$ be such that $\chi_N = \theta \in Irr(N)$. Then, the characters $\beta\chi$ for $\beta \in Irr(\frac{G}{N})$ are irreducible and distinct for distinct β and are all of the irreducible constituents of θ^G.*

Lemma 2.3 (Ito's Theorem, [3], Corollary 6.15). *Let $A \trianglelefteq G$ be abelian. Then, $\chi(1)$ divides $|G:A|$ for all $\chi \in Irr(G)$.*

Lemma 2.4 ([3], Theorems 6.2, 6.8, 11.29). *Let $N \trianglelefteq G$ and let $\chi \in Irr(G)$. Let θ be an irreducible constituent of χ_N, and suppose $\theta_1 = \theta,..,\theta_t$ are the distinct conjugates of θ in G. Then, $\chi_N = e\sum_{i=1}^{t} e_i\chi_i$, where $e=[\chi_N,\ \theta]$ and $t=[G:I_G(\theta)]$. Also, $\theta(1)|\chi(1)$ and $\chi(1)/\theta(1)||G:N|$.*

Lemma 2.5 ([17], Lemma). *Let G be nonsolvable group. Then, G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of isomorphic nonabelian simple group and $|G/K|||Out(K/H)|$.*

Lemma 2.6 ([3], Lemma 12.3 and Theorem 12.4). *Let $N \trianglelefteq G$ be maximal such that G/N is solvable and nonabelian. Then, one of the following holds.*

(i) G/N is a r-group for some prime r. If $\chi \in Irr(G)$ and $r \mid \chi(1)$, then $\chi\tau \in Irr(G)$ for all $\tau \in Irr(G/N)$.

(ii) G/N is a Frobenius group with an elementary abelian Frobenius kennel F/N.

Thus, $|G:F| \in cd\ (G)$, $|F:N| = r^{\alpha}$, where a is the smallest integer such that $|G:F| \mid r^{\alpha}-1$. For every $\psi \in Irr(F)$, either $|G:F|\psi(1) \in cd(G)$ or $|F:N||\psi(1)^2$. If no proper multiple of $|G:F|$ is in $cd(G)$, then $\chi(1)||G:F|$ for all $\chi \in Irr(G)$ such that $r \mid \chi(1)$.

Lemma 2.7 ([16], Lemma 2.3). *In the context of (ii) of Lemma 2.5, we have*

(i) If $\chi \in Irr(G)$ such that lcm$(\chi(1),|G:F|)$ does not divide any character degree of G, then $r^{\alpha} \mid \chi(1)^2$

(ii) If $\chi \in Irr(G)$ such that no proper multiple of $\chi(1)$ is a degree of G, then either $|G:F| \mid \chi(1)$ or $r^{\alpha}|\chi(1)^2$. Moreover if $\chi(1)$ is divisible by no nontrivial proper character degree in G then $|G:F| = \chi(1)$ or $r^{a}|\chi(1)^2$.

3. Proof of the main theorem

In this section we present the proof of Main theorem. In fact, we prove this theorem by two steps:

Step 1. First we prove that G is a nonsolvable group.We show that $G' = G''$. Assume by contradiction that $G' \neq G''$ and let $N \trianglelefteq G$ be maximal such that G/N is solvable and nonabelian.By Lemma 2.6, G/N is an r-group for some prime r or G/N is a Frobenius group with an elementary abelian Frobenius kernel F/N.

Case 1. G/N is an r-group for some prime r. Since G/N is nonabelian, there is $\psi \in Irr(G/N)$ such that $\psi(1) = r^a > 1$. From the classification of prime power degree representations of quasi-simple group in [12], we deduce that $\psi(1) = r^a$ must be equal to the degree of the Steinberg character of H of degree p^{12} and thus $r^a = p^{12}$, which implies that $r = p$. By Lemma 2.1, G possesses a nontrivial irreducible character χ with $p \mid \chi(1)$. Lemma 2.4 implies that $\chi_N \in Irr(N)$.

Using Ghallagher's lemma, we deduce that $\chi(1)\psi(1) = p^{12}\chi(1)$ is a character degree of G, which is impossible with the condition (ii) of main theorem.
Case 2. G/N is a Frobenius group whit an elementary abelian Frobenius kernel F/N. Thus according to Lemma 2.6, $|G : F| \in cd(G)$, $|F : N| = r^a$, where a is the smallest integer such that $|G : F||r^a - 1$. Let χ be a character of G of degree p^{12}. As no proper multiple of p^{12} is in $cd(G)$, Lemma 2.6 implies that either $|G : F||p^{12}$ or $r = p$. We consider two following subcases.

(a) $|G : F||p^6$. Then, $|G : F| \in cd(G)$, by the assumption of the theorem, this implies that no multiple of $|G : F|$ is in $cd(G)$. Therefore, by Lemma 2.6, for every $\psi \in Irr(G)$ either $\psi(1)|p^{12}$ or $r|\psi(1)$. Taking ψ to be characters of degree $p^2\Phi_3$ and $p^2\Phi_2^2\Phi_4$, we obtain that $r|\psi(1)$.This implies that r divides both $p^2\Phi_3$ and $p^2\Phi_2^2\Phi_4$. This leads us to a contradiction since (Φ_3 , $\Phi_2^2\Phi_4$)=1.

(b) $r = p$. Thus $|F : N| = p^a$ and $|G : F||p^a - 1$. Let χ be a character of G of degree $p^2\Phi_2^2\Phi_4$ and ψ be a character of degree $\Phi_2\Phi_4$). It follows that $\psi(1)|\chi(1)$ so that by Lemma 2.7, $|G : F| = p^2\Phi_2^2\Phi_4$ or $p^a|p^4\Phi_2^2\Phi_4^2$ which implies that $a \leq 4$, $|G : F| \leq p^4 - 1$. This leads us to a contradiction since $min\{\chi(1)|\chi(1) > 1, \chi \in Irr(G)\} = p^2\Phi_3$.

Therefore, G is not a solvable group.
Step 2. Now, we prove that G is isomorphic to $PSL(4, p^2)$.

By the above discussion and using Lemma 2.5, we get that G has a normal series $1 \trianglelefteq H \trianglelefteq K \trianglelefteq G$ such that K/H is a direct product of m copies of a nonabelian simple group S and $|G/K|||Out(\mathrm{K/H})|$. Also, p is a prime divisor of $|G|$ such that $p^{12}\||G|$

First, we prove that $p \nmid |G/K|$. On the contrary, let $p||G/K|$. We know that $Out(K/H) \cong Out(S) \wr S_m$, which implies that $p||S_m|$ or $p||Out(S)|$. If $P||S_m|$, then $m \geq p$ and so $p^{12}(p^4-1)(p^6-1)(p^8-1) \geq |K/H| \geq 60^p$, which is impossible. Hence $p||Out(S)|$. According to the orders of automorphism group of alternating group and sporadic simple group, we implies that S is a simple group of Lie type over $GF(q)$, where $q = p_0^f$. By assumption, $p||Out(S)| = dfg$, where d, f, and $g \leq 3$ are the orders of diagonal, field, and graph automorphisms of S respectively. Using [2], we know that if S is a simple group of Lie type over $GF(q)$, then $q(q^2-1) \leq S$ and so if $p|f$, then $2^p(2^{2p}-1) \leq q(q^2-1) \leq |S| \leq p^{12}(p^4-1)(p^6-1)(p^8-1)$, which is a contradiction. Hence $p|d$. Since $p > 7$, we get that $S = A_n(q)$ and $d = (n+1, q-1)$ or $S = ^2A_n(q)$ and $d = (n+1, q+1)$. In each case we get that $p|q-1$ and $n \geq 6$ or $p|q+1$ and $n \geq 6$. Then, $p^{13}||S|$, which is a contradiction. Therefore, $p \nmid |G/K|$.

Now, we prove that $p \nmid |H|$. On the contrary, let $p||H|$. So there exist twelve possibilities, $p^i\||H|$ where $1 \leq i \leq 12$.
Case 1. First, suppose that $p\||H|$. Using the classification of finite simple group we determine all simple groups S such that $p^5||S|^5$. Now, we consider two subcases:

(i) Let $m = 1$. Then, $p^{11}||S|$ and $|S||p^{11}(p^4-1)(p^6-1)(p^8-1)$.

If $S \cong A_n$, then $p \leq n$ and $n!|p^{12}(p^4-1)(p^6-1)(p^8-1)$. Which is impossible since $p > 7$. Also, there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple groups, we get that, there is no Lie group satisfying these conditions.

Since the proofs for the other simple groups are similar, we state the proof only for a few of them for convenience.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions.

If $S \cong B_n(q)$, where $n \geq 2$, then $p|q^{2j} - 1$, for some $1 \leq j \leq n$. Therefore, $p \leq q^n + 1$. Then, since $q^{2i-1} \leq q^{2i} - 1$, we get that

$$q^{n^2}.q^{2(1+2+...+n)-n} \leq |S| < p^{23} \leq (q^n + 1)^{23} \leq q^{23n+23},$$

which implies that $2n^2 < 23(n + 1)$. Therefore, $n \in \{2, 3, 4, ..., 12\}$. First let $n = 2$. Then, $p^{11}|q^4(q^2 - 1)(q^4 - 1)$.It implies that $p^{11}|(q - 1)^2$ or $p^{11}|(q + 1)^2$ or $p^{11}|q^2 + 1$, and so $p^{11} < 2q^2$. On the other hand $q^4|(p - 1)^3$ or $q^4|(p + 1)^3$ or $q^4|(p^2+1)^2$ or $q^4|(p^2+p+1)$ or $q^4|(p^2-p+1)$, and so $q^4 < p^5$. Therefore, easily we get a contradiction. If $n \in \{3, 4, 5, ..., 12\}$, similarly we get a contradiction. If $S \cong C_n(q)$, where $n \geq 4$, then withe the same manner we get a contradiction.

If $S \cong A_n(q)$, then similarly to the above, we get $n \in \{1, 2, ..., 15\}$. For example, let $n = 5$. Then,

$$p^{11}|(q - 1)^5(q + 1)^3(q^2 + q + 1)^2(q^2 - q + 1)(q^4 + q^3 + q^2 + q + 1)$$

so, $p^{11} < 5q^4$. On the other hand $q^{15}|(p-1)^3(p+1)^3(p^2+1)^2(p^2-p+1)(p^2+p+1)$ so $q^{15} < p^7$. Therefore, we get a contradiction. For other case, similarly we get a contradiction. If $S \cong^2 A_n(q)$, with the same manner we get a contradiction.

If $S \cong D_n(q)$, where $n \geq 4$, then $p^{11}||S|$, Therefore, $p|q^{2i} - 1$, for some $1 \leq i \leq n-1$ or $p|(q^n - 1)$. Therefore, $p < q^n$, and since $q^{2i-1} < q^{2i} - 1$, we get that

$$q^{n(n-1)}q^{n-1}(q^{2(1+2+...+(n-1)-(n-1))} < |S| < p^{23}$$

and so $q^{(2n(n-1)} < |S| < p^{23}$. On the other hand, $p < q^n$ and hence $2(n-1) < 23$. Therefore, $n \in \{4, 5, 6, ..., 12\}$. Let $n = 6$. Then, $p^{11}|(q - 1)^6(q + 1)^6(q^2 + q + 1)^2(q^2 - q + 1)^2(q^2 + 1)^2(q^4 + 1)(q^4 + q^3 + q^2 + q + 1)(q^4 - q^3 + q^2 - q + 1)$ and so $p^{11} < q^7$. On the other hand

$$q^{30}|(p - 1)^3(p + 1)^3(p^2 + 1)^2(p^2 + p + 1)(p^2 - p + 1)$$

and so, $q^{30} < p^7$. Therefore, we get a contradiction. Fore some other cases, similarly we get a contradiction. If $S \cong^2 D_n(q)$, with the same manner we get a contradiction.

If $S \cong G_2(q)$, then $p^{11}||S|$, and hence $p^{11} < q^3$. On the other hand,

$$q^6|(p - 1)^3(p + 1)^3(p^2 + 1)^2(p^2 + p + 1)(p^2 - p + 1)$$

so, $q^6 < p^7$. Therefore, we get a contradiction. If $S \cong F_4(q),^2 F_4(q), E_6(q), E_7(q)$ or $E_8(q)$, we get a contradiction similarly.

If $S \cong^2 B_2(q)$, where $q = 2^{2n+1}$, then $p^{11}|q-1$ or $p^{11}|q^2+1$. If $p^{11}|q-1$, then $|S| < p^{23} < (q-1)^5$, wiche is impossible. If $p^{11}|(q^2+1)$, then $p^{11}|(q^2+1)/5$, so $p^{11} < q^2$. On the other hand

$$q^2|4(p-1)^3(p+1)^3(\frac{p^2+1}{2})^2(p^2+p+1)(p^2-p+1)$$

therefore, $q^2|16(p-1)^3$ or $q^2|16(p+1)^3$, so $q < p^3$, which is impossible.

If $S \cong^2 G_2(q)$, where $q = 3^{2n+1}$, then $p^{11}||S|$, therefore $p^{11}|q-1$ or $p^{11}|q+1$ or $p^{11}|q^2-q+1$ or$p^{11}|q^2+q+1$, it follows that $p^{11} < q^2$. On the other hand, $q^3|6(p-1)^3(p+1)^3$ or $q^3|(p^2+1)/2$ or $q^3|(p^2+p+1)$ or$q^3|(p^2-p+1)$, it follows that $q^3 < p^7$, which is impossible.

Therefore, $m \neq 1$.

(ii) $m = 11$. Then, $p||S|$ and $|S|^{11}|p^{11}(p^4-1)(p^6-1)(p^8-1)$.

Similarly, to the previous case we get a contradiction.

Case 2. Suppose that $p^2|||H|$. Therefore, $p^{10}||K/H|$, since K/H is m is a direct product of m copies of a nonabelian simple group S, it follows that, $m \in \{1, 2, 5, 10\}$. Now we consider four subcases:

(i) Let $m = 1$. Then, $p^{10}||S|$ and $|S||p^{10}(p^4-1)(p^6-1)(p^8-1)$. We claim that there is no simple group satisfying these conditions.

If $S \cong A_n$, then $p < n$ and $n!|p^{10}(p^4-1)(p^6-1)(p^8-1)$, which is impossible since $p > 7$. Also, there is no sporadic simple group satisfying these conditions.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above conditions.

Similarl to case 1, we deduce that, there is no nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$, satisfying the above conditions.

Hence, $m \neq 1$.

(ii) Let $m = 2$

Similarly to last case, we deduce $S \ncong A_n$. Also, there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the order of the simple group, we get that, there is no simple group satisfying the above conditions.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case1$), we obtain a contradiction.

Hence $m \neq 2$

(iii) Let $m = 5$. Then, $p^2||S|$ and $|S|^5|p^{10}(p^4-1)(p^6-1)(p^8-1)$. Using the classification of finite simple group, we show that, there is no simple group satisfying these conditions. If S is a nonabelian simple group of Lie type over a field of characteristic p, using the order of the simple group, we get that, there is no simple group satisfying the above conditions.

If $S \cong A_n$, then $p \leq n$ and $(n!)^5|p^{10}(p^4-1)(p^6-1)(p^8-1)$, which is impossible since $p > 7$. Also, there is no sporadic simple group satisfying these conditions.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in $(case1)$, we obtain a contradiction.

Hence, $m \neq 5$.

(iv) Let $m = 10$. Then, $p||S|$ and $|S|^{10}|p^{12}(p^4-1)(p^6-1)(p^8-1)$.

If $S \cong A_n$, then $p \leq n$ and $(n!)^{10}|p^{12}(p^4-1)(p^6-1)(p^8-1)$, which is impossible since $p > 7$. Also, there is no sporadic simple group satisfying these conditions.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple groups, we get that, the only possibility cases are $A_1(p)$ and $A_2(p)$.

(A) If $S \cong A_1(p)$, then $p^{10}(p^2-1)^{10}|p^{12}(p-1)^3(p+1)^3(p^2+1)^2(p^2+p+1)(p^2-p+1)$, therefore $(p-1)^7(p+1)^7|(p^2+1)^2(p^2+p+1)(p^2-p+1)$, which is impossible.

(B) If $S \cong A_2(p)$, then $|S|^{10} \leq p^{12}(p^4-1)(p^6-1)(p^8-1)$, which is impossible.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in $(case1)$, we obtain a contradiction.

Hence $m \neq 10$.

Case 3. If $p^3\|\,|H|$. Therefore, $p^9||K/H|$, since K/H is m is a direct product of m copies of a nonabelian simple group S, it follows that, $m \in \{1, 3, 9\}$. Now we consider three subcases:

(i) Let $m = 1$. Then, $p^9\|\,|S|$ and $|S||p^3(p^4-1)(p^6-1)(p^8-1)$.

If $S \cong A_n$ Similarly to the case1, we get a contradiction. Also, there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above conditions.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in $(case1)$, we obtain a contradiction.

Hence, $m \neq 1$.

(ii) Let $m = 3$. Then, $p^3||S|$ and $|S|^3|p^3(p^4-1)(p^6)(p^8-1)$

If $S \cong A_n$ Similarly to the case1, we get a contradiction. Also, there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above condition.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in $(case1)$, we obtain a contradiction.

Hence $m \neq 3$.

(iii) Let $m = 9$. Then, $p||S|$ and $|S|^9|p^{11}(p^4-1)(p^6-1)(p^8-1)$.

If $S \cong A_n$ Similarly to the case1, we get a contradiction. Also, there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above condition.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case1$), we obtain a contradiction.

Hence, $m \neq 9$.

Case 4. If $p^4 \| |H|$. Therefore, $p^8 | |K/H|$, since K/H is m is a direct product of m copies of a nonabelian simple group S, it follows that, $m \in \{1, 2, 4, 8\}$. Now we consider two subcases:

(i) Let $m = 1$. Then, $p^8 \| |S|$ and $|S| | p^4(p^4 - 1)(p^6 - 1)(p^8 - 1)$.

If $S \cong A_n$, then similar to Case 1, we get a contradiction. Also, there is no sporadic simple group satisfying these conditions.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above conditions.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case1$), we obtain a contradiction.

Hence, $m \neq 1$.

(ii) Let $m = 2$. Then, $p^6 \| |S|$ and $|S|^2 | p^6(p^4 - 1)(p^6 - 1)(p^8 - 1)$.

If $S \cong A_n$ Similarly to the case1, we get a contradiction. Also, there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above conditions.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case1$), we obtain a contradiction.

Hence, $m \neq 2$.

(iii) Let $m = 4$. Then, $p^3 | |S|$ and $|S|^3 | p^9(p^4 - 1)(p^6 - 1)(p^8 - 1)$.

If $S \cong A_n$ Similarly to the case1, we get a contradiction. Also, there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above conditions.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case1$), we obtain a contradiction.

Hence, $m \neq 4$.

(iv) Let $m = 8$. Then, $p \| |S|$ and $|S|^8 | p^{11}(p^4 - 1)(p^6 - 1)(p^8 - 1)$.

If $S \cong A_n$ Similarly to the case1, we get a contradiction. Also, there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above conditions.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case$1), we obtain a contradiction.

Hence, $m \neq 8$.

Case 5. If $p^5 \| |H|$. Therefore, $p^7 | |K/H|$, since K/H is m is a direct product of m copies of a nonabelian simple group S, it follows that, $m \in \{1, 7\}$.

(i) Let $m = 1$. Then, $p^7 \| |S|$ and $|S| | p^5(p^4 - 1)(p^6 - 1)(p^8 - 1)$.

If $S \cong A_n$ Similarly to the case1, we get a contradiction. Also, there is no sporadic simple group satisfying these condition.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above conditions

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case$1), we obtain a contradiction. (ii)Let $m = 7$. Then, $p \| |S|$ and $|S|^7 | p^{11}(p^4 - 1)(p^6 - 1)(p^8 - 1)$.

If $S \cong A_n$, then similar to Case 1, we get a contradiction. Also, there is no sporadic simple group satisfying these conditions.

If S is a nonabelian simple group of Lie type over a field of characteristic p, using the orders of the simple group, we get that, there is no simple group satisfying the above conditions.

If S be a nonababelian simple group of Lie type over a field $GF(q)$, where $p \nmid q$. We claim that there is no simple group satisfying the above conditions. Now argue as in ($case$1), we obtain a contradiction. Where $6 \leq i \leq 11$, then withe the same manner we get contradiction.

If $i = 12$ then $p^{12} | |H|$, choos $\chi \in Irr(G)$, such that $\chi(1) = p^{12}$. Let θ be an irreducible constituent of χ_H, then $\chi(1)/\theta(1) | |G : H|$, which implies that $\theta(1) = p^{12}$. Therefore, $\chi_H = \theta$ and by Gallagher's theorem $\beta\chi \in Irr(G)$, for each $\beta \in Irr(G/H)$. Hence $p^{12}\beta(1) \in$ cd(G), which is contradiction.

By the above discussion, we get that $p^{12} | |K/H|$. Since $p^{12} \| |G|$, it follows that K/H is a nonabelian simple group say S, such that $p^{12} \| |S|$ and $|S| | p^{12}(p^4 - 1)(p^6 - 1)(p^8 - 1)$ or $K/H \cong S \times S$ and $p^6 \| |S|$ and $|S|^2 | p^{12}(p^4 - 1)(p^6 - 1)(p^8 - 1)$ or $K/H \cong \prod_{i=1}^{3} S$ and $|S|^4 | p^{12}(p^4 - 1)(p^6 - 1)(p^8 - 1)$ or $K/H \cong \prod_{i=1}^{4} S$ and $p^3 \| |S|$ and $|S|^4 | p^{12}(p^4 - 1)(p^6 - 1)(p^8 - 1)$ or $K/H \cong \prod_{i=1}^{6} S$ and $p^2 \| |S|$ and $|S|^6 | p^{12}(p^4 - 1)(p^6 - 1)(p^8 - 1)$ or $K/H \cong \prod_{i=1}^{12} S$ and $p \| |S|$ and $|S|^{12} | p^{12}(p^4 - 1)(p^6 - 1)(p^8 - 1)$.

Now, using the classification of finite simple groups and similar to the above argument, we get $K/H \cong PSL(4, p^2)$. Therefore, $|H||G/K| = 1$, and hence, $H = 1$ and $G/K = 1$. Hence $G \cong PSL(4, p^2)$, and the main theorem is proved.

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References

[1] R. Brauer, *Representations of finite group*, Lectures on Modern Mathematics, 1963.

[2] J.H. Conway, S.P. Norton, R.A. Parker, R.A. Wilson, *Atlas of finite groups*, Clarendon Press, Oxford, 1985.

[3] D.Gorenstein, R. Lyons and R. Solomon. *The classifications of the finite simple groups*, Math. Surveys Monogr., No. 3, Amer. Math. Soc., Providence, 1998.

[4] W. Feit, *Extending Steinberg characters*, Contemp. Math., 153 (1993), 1-9.

[5] B. Huppert, *Character theory of finite groups*, Berlin/New York: de Gruyter, 1998.

[6] B. Huppert, *Some simple groups which are determined by the set of their character degrees*, II. Rend. Sem. Mat. Univ. Padova, 115 (2006), 1-13.

[7] I.M. Isaacs, *Character theory of finite groups*, Dover Publications, Inc., New York, 1994.

[8] I.M. Isaacs, *Character degree graphs and normal subgroups*, Trans. Am. Math. Soc., 356 (2004), 1155-1183.

[9] B. Khosravi, B. Khosravi, B. Khosravi, Z. Momen, *A new characterization for the simple group $PSL(2,p^2)$ by order and some character degrees*, Czechoslovak Math. J., 65 (2015), 271-280.

[10] . B. Khosravi, B. Khosravi, B. Khosravi, Z. Momen, *A new characterization for the simple group $PSL(2,p)$ by order and some information on its character degrees were p is a prime*, Monatsh. Math, 175 (2014), 277-282.

[11] M.L. Lewis, D.L. White, *Nonsolvable groups with no prime dividing three character degrees*, J. Algebra, 336 (2011), 58-183.

[12] G. Malle, A.E. Zalesskii, *Prime power degree representations of quasi-simple groups*, Arch. Math, 77 (2001), 461-468.

[13] H.N. Nguyen, H.P. Tong-viet, T.P. Wakefield, *Projective special linear groups $PSL_4(q)$ are determined by the set of their character degrees*, J. Algebra and Its Applications, 11 (2012), 1250108.

[14] Y. Rezayi, A. Iranmanesh, *A characterization of PSL(4,p) by some character degree*, Facta Universitatis' Series Mathematics and Informatics, 34 (2019), 679-688.

[15] H.P. Tong -Viet, *Simple classical groups of Lie type are detemined by their character degrees*, J. Algebra, 357 (2012), 61-68.

[16] H.P. Tong-Viet, P. Hung, *The simple Ree groups*${}^2F_4(q^2)$ *are determined by the set of their character degrees*, J. Algebra, 339 (2011), 357-369.

[17] H. Xu, Y. Yan, G.Y. Chen , *A new characterization of Mathieu-groups by the order and one irreducible character degree*, J. Ineq. App., 209 (2013), 1-6.

[18] H. Xu, Y.Yan, G.Y. Chen, *A new characterization of simple* K_3*-groups by their orders and large degrees of their irreducible characters*, Comm. Algebra, 42 (2013), 1-6.

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