On the primary-like dimension of modules

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Abstract. Let R be a ring and let M be a left R-module. In this article, we introduce and study the primary-like dimension of M was defined to be the supremum of the lengths of all strong-like chains of primary-like submodules of M and denoted by $P.L.dim(M)$.

Keywords: primary-like dimension, virtually-like Noetherian, virtually-like Artinian.

1. Introduction

In this paper, all rings are associative rings with identity, and all modules are unital and left modules. The symbol $\subseteq$ denotes containment and $\subset$ proper containment for sets. If Q is a submodule of M, then we denote the left annihilator of a factor module M/N of M by $(Q : M)$. We call M faithful if $(0 : M) = 0$. Recall that a left R-module M is said to be prime if $Ann(Q) = Ann(M)$ for every nonzero submodule Q of M. A proper submodule Q of M is called a prime submodule if the quotient module M/Q is a prime module, i.e., if $IN \subseteq Q$, where N is a submodule of M and I is an ideal of R, then either $N \subseteq Q$ or $IM \subseteq Q$. The collection of all prime submodules of M is denoted by $Spec(M)$. This notion of prime submodule was first introduced and systematically studied in [4] and recently it has received a good deal of attention from several authors, see, for example, [1, 2, 10, 11, 15, 18, 20] and many others. There is already a generalization of classical Krull dimension for modules via prime dimension. In fact, the notion of prime dimension of a module $dim(M)$ over a commutative ring $dim(M)$ (denoted by $dim(M)$), was introduced by Marcelo and Masqué [14], as the maximum length of the chains of prime submodules of M (see also [13, 19] for some known results about the prime dimension of modules). A submodule Q of M is said to be primary-like if $Q \neq M$ and whenever $rm \in Q$ (where $r \in R$ and $m \in M$) implies $r \in (Q : M)$ or $m \in radQ$ [5, 6]. An R-module M is said to be primeful if either $M = (0)$ or $M \neq (0)$ and the map $\psi : Spec(M) \longrightarrow Spec(R/Ann(M))$ defined by $Q \longmapsto (Q : M)/Ann(M)$ is surjective[12]. If M/Q is a primeful over R, then $\sqrt{(Q : M)} = (radQ : M)$ [12, Proposition 5.3]. It is easily seen that, if Q is a primary-like submodule of Q such that M/Q is a primeful over R, then $(Q : M)$ is a primary ideal of R and so $P = \sqrt{(Q : M)}$ is a prime ideal of R, and in this case Q is called a P-primary-like submodule of M. The primary-like spectrum of M denoted by $Spec_L(M)$ is defined to be the set of all primary-like submodules Q of M, where M/Q is primeful. In this article, when we say that Q is a primary-like submodue of M, it means that Q is primary-like submodule of M, where M/Q is primeful; i. e. $Q \in Spec_L(M)$. Let M be a left R-module and Q, Q' be two submodules of M. We say that Q is strongly-like properly contained in Q', and write $Q \subset_{sl} Q'$, if $Q \subset Q'$ and also $\sqrt{(Q : M)} \subset \sqrt{(Q' : M)}$. In this case, we also say that Q' strongly-like properly contains Q. Also, $Q \subseteq_{sl} Q'$ means that $Q \subset_{sl} Q'$ or $Q = Q'$. A submodule Q of M will be called virtually maximal primary-like if Q is primary-like and there is no primary-like submodule Q' such that $Q \subset_{sl} Q'$.

Let R be a ring and M be a left R-module such that every primary-like submodule of M is contained in a virtually maximal primary-like submodule. We define, by transfinite induction, sets X_α of primary-like submodules of M. To start with, let X_{-1} be the empty set. Next, consider an ordinal $\alpha \geq 0$; if X_β has been defined, for all ordinals $\beta < \alpha$, let X_α be the set of those primary-like submodules Q in M such that all primary-like submodules strongly-like properly containing Q belong to $\bigcup_{\beta<\alpha} X_\beta$. (In particular, X_0 is the set of virtually maximal primary-like submodules of M.) If some X_γ contains all primary-like submodules of M, we say that $P.L.dim(M)$ exists, and we set $P.L.dim(M)$-the primary-like dimension of M-equal to the smallest such γ. We write $P.L.dim(M) = \gamma$ as an abbreviation for the statement that $P.L.dim(M)$ exists and equals γ.

In Section 2, we introduce the notion of a virtual-like chain condition on submodules of a module. In Section 3, the meaning of the primary-like dimension of modules and related topics are studied.

2. Virtual-like chain conditions

In this section we introduce the notion of virtual-like chain condition on submodules of a module.

Definition 2.1. *Let R be a ring and M be a left R-module. A submodule Q of M will be called:*

(1) *maximal primary-like if Q is a primary-like submodule of M and there is no primary-like submodule Q' of M such that $Q \subset Q'$;*

(2) *virtually maximal primary-like if Q is a primary-like submodule of M and there is no primary-like submodule Q' of M such that $Q \subset_{sl} Q'$ (i.e., Q is a primary-like submodule of M and for any primary-like submodule Q' of M, such that $Q \subseteq Q'$, we have $\sqrt{(Q : M)} = \sqrt{(Q' : M)}$);*

(3) *virtually maximal if the factor module M/Q is a homogeneous semisimple module (see also [16], for definition).*

Example 2.1. Let $M = \mathbb{Q} \bigoplus \mathbb{Z}_p$, where $\mathbb{Z}_p$ is the cyclic group of order p. Then $Spec(M) = \{\mathbb{Q} \bigoplus 0, 0 \bigoplus \mathbb{Z}_p\}$ by [17, Example 2.6]. Clearly, if N is a submodule of M such that $N \nsubseteq \mathbb{Q} \bigoplus 0$ or $N \nsubseteq 0 \bigoplus \mathbb{Z}_p$, then N does not satisfy the primeful property. Also, If $N \subseteq 0 \bigoplus \mathbb{Z}_p$, then $(N : M) = 0$ and so N dose not satisfy the primeful property. Consider the only remaining case $N \subseteq \mathbb{Q} \bigoplus 0$. In this case, if $(N : M) = p\mathbb{Z}$, then $N = \mathbb{Q} \bigoplus 0$ and so $\mathbb{Q} \bigoplus 0 \in Spec_L(M)$. If $(N : M) = 0$, then N does not satisfy the primeful property. The finial case is $0 \subset (N : M) \subset p\mathbb{Z}$. In this case if N is a primary-like submodule satisfying the primeful property, then $(N : M) = p^i\mathbb{Z}$ for some $i \geq 1$, since $(N : M)$ is a primary ideal of R. Assume $i \neq 1$ and $(0, b) \in M \backslash \mathbb{Q} \bigoplus 0$. Now, $p(0, b) = (0, 0)$, follows $p \in p^i\mathbb{Z}$ which is a contradiction. Therefore, $Spec_L(M) = \{\mathbb{Q} \bigoplus 0\}$. Hence, $\mathbb{Q} \bigoplus 0$ is maximal primary-like and virtually maximal primary-like submodule.

Definition 2.2. *Let R be a ring and M be a left R-module. Then, the chain $Q_1 \subseteq_{sl} Q_2 \subseteq_{sl} Q_3 \subseteq_{sl} \cdots$ of submodules of M is called a strong-like ascending chain. Also, the chain $Q_1 \ {}_{sl}{\supseteq}\ Q_2 \ {}_{sl}{\supseteq}\ Q_3 \ {}_{sl}{\supseteq}\ \cdots$ of submodules of M is called a strong-like descending chain.*

Definition 2.3. *Let R be a ring. A left R-module M is said to satisfy the virtual-like ascending chain condition on submodules (or to be virtually-like Noetherian or virtual-like acc) if for every strong-like chain $Q_1 \subseteq_{sl} Q_2 \subseteq_{sl} Q_3 \subseteq_{sl} \cdots$ of submodules of M, there is an integer n such that $Q_i = Q_n$, for all $i \geq n$. Also, a left R-module M is said to satisfy the virtual-like descending chain condition on submodules (or to be virtually-like Artinian or virtual-like dcc) if for every strong-like chain $Q_1 \ {}_{sl}{\supseteq}\ Q_2 \ {}_{sl}{\supseteq}\ Q_3 \ {}_{sl}{\supseteq} \cdots$ of submodules of M, there is an integer n such that $Q_i = Q_n$, for all $i \geq n$.*

It is clear that every Noetherian (respectively, Artinian) module is virtually-like Noetherian (respectively, virtually-like Artinian). In general, the converse is not true. See the following example

Example 2.2. 1) Let R be a commutative Noetherian (respectively, Artinian) ring. Then, every R-module is virtually-like Noetherian (respectively, virtually-like Artinian).

2) For a prime number p, $\mathbb{Z}(p^\infty)$ as a $\mathcal{Z}$-module is virtually-like Noetherian, since every proper submodule of $\mathbb{Z}$-module $\mathbb{Z}(p^\infty)$ is primary-like. However $Spec_L(\mathbb{Z}(p^\infty)) = Spec(\mathbb{Z}(p^\infty)) = \emptyset$. But it is not a Noetherian $\mathbb{Z}$-module.

3) For $\mathbb{Z}$-module $\mathbb{Q}$, $Spec(\mathbb{Q}) = \{0\}$ and $Spec_L(\mathbb{Q}) = \emptyset$, because $\mathbb{Q}$ have no submodules satisfying the primeful property. Therefore, $\mathbb{Q}$ as a $\mathbb{Z}$-module is virtually-like Artinian, but it is not an Artinian $\mathbb{Z}$-module.

4) For a vector space V over a field F, $Spec_L(V) = Spec(V)$= the set of all proper vector subspaces of V. Hence, every vector space over a field is both virtually-like Noetherian and virtually-like Artinian.

Proposition 2.1. *Let M be a left R-module and Q be a proper submodule of M. Then*

$$\begin{array}{ccc} Q \text{ is max and } M/Q \text{ is primeful} & \Longrightarrow & Q \text{ is max primary-like} \\ \Downarrow & & \Downarrow \\ Q \text{ is virtually max} & \Longrightarrow & Q \text{ is max virtually primary-like} \\ & & \Downarrow \\ & & Q \text{ is primary-like} \end{array}$$

Proof. Assume that Q is maximal. Then M/Q is a simple module, and it follows that Q is a maximal primary-like submodule. Also, it is clear that every maximal submodule of M is virtually maximal but, the converse is not true (for example, every proper submodule of a homogeneous semisimple module is virtually maximal but it is not necessarily maximal). Clearly, if Q is a maximal primary-like submodule of M, then Q is virtually maximal primary-like. Now, if Q is virtually maximal, then M/Q is a homogeneous semisimple module. Clearly, for every proper submodule Q' of M, $\sqrt{(Q:M)} = \sqrt{(Q':M)}$ and it follows that Q is a virtually maximal primary-like submodule. Finally, it is clear that every virtually maximal primary-like submodule is primary-like. □

Let M be a left R-module and $N, L \leq M$. We say that N is strongly properly contained in L, and write $N \subset_s L$, if $N \subset L$ and also $(N : M) \subset (L : M)$. A submodule Q of is said to be virtually maximal prime if Q is a prime submodule of M and there is no prime submodule Q' of Q' such that $Q \subset_s\subset Q'$ (i.e., Q is a prime submodule of M and for any prime submodule Q' of M, such that $Q \subseteq Q'$, we have $(Q : M) = (Q' : M)$). A left R-module M is said to satisfy the virtual ascending chain condition on submodules (or to be virtually Noetherian or virtual acc) if for every strong chain $Q_1 \subseteq_s Q_2 \subseteq_s Q_3 \subseteq_s \cdots$ of submodules of M, there is an integer n such that $Q_i = Q_n$, for all $i \geq n$. Also, a left R-module M is said to satisfy the virtual descending chain condition on submodules (or to be virtually Artinian or virtual dcc) if for every strong chain $Q_1 \ {}_s\!\supseteq Q_2 \ {}_s\!\supseteq Q_3 \ {}_s\!\supseteq \cdots$ of submodules of M, there is an integer n such that $Q_i = Q_n$, for all $i \geq n$ (see [3]).

Proposition 2.2. *Let R be a ring. Then, the following statements are equivalent:*

1) *R has acc (respectively, dcc) on two-sided ideals;*

2) *each R-module is virtually-like Noetherian (respectively, virtually-lkie Artinian);*

3) *the left R-module R is virtually-like Noetherian (respectively, virtually-like Artinian);*

4) *the left R-module R is virtually Noetherian* (*respectively, virtually Artinian*)*;*

5) *each R-module is virtually-like Noetherian* (*respectively, virtually-like Artinian*)*;*

6) *each R-module is virtually Noetherian* (*respectively, virtually Artinian*)*.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) is clrear.
(1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) follows from [3, Proposition 2.1]. □

Corollary 2.1. *Let R be a commutative ring. Then, the following statements are equivalent:*

1) *R is Noetherian* (*respectively, Artinian*)*;*

2) *each R-module is virtually-like Noetherian* (*respectively, virtually-like Artinian*)*;*

3) *the R-module R is virtually-like Noetherian* (*respectively, virtually-like Artinian*)*;*

4) *the R-module R is virtually Noetherian* (*respectively, virtually Artinian*)*;*

5) *each R-module is virtually-like Noetherian* (*respectively, virtually-like Artinian*)*;*

6) *each R-module is virtually Noetherian* (*respectively, virtually Artinian*)*.*

Proof. Follows from Proposition 2.2. □

Definition 2.4. *An R-module M is said to satisfy the virtual-like maximum condition* (*respectively, virtual-like minimum condition*) *on submodules if every nonempty set of submodules of M contains a maximal* (*respectively, minimal*) *element with respect to strong inclusion $\subseteq_{sl}$* (*respectively, ${}_{sl}\supseteq$*)*.*

Proposition 2.3. *An R-module M is virtually-like Noetherian* (*respectively, virtually-like Artinian*) *if and only if M satisfies virtual-like maximum condition* (*respectively, virtual-like minimum condition*) *on submodules.*

Proof. Is clear. □

Proposition 2.4. *Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of modules. Then, M_2 is virtually-like Noetherian (respectively, virtually-like Artinian) if and only if M_1 and M_2 are virtually-like Noetherian (respectively, virtually-like Artinian).*

Proof. Is clear. □

Corollary 2.2. *Let N be a submodule of an R-module M. Then, M satisfies the strong-like ascending (respectively, descending) chain condition if and only if so do N and M/N.*

Proof. Apply Proposition 2.4 to the sequence

$$0 \to N \overset{\subseteq}{\to} M \to M/N \to 0. \qquad \square$$

Corollary 2.3. *Let $M_1, M_2, \cdots, M_n$ be modules. Then, the direct sum $M_1 \oplus M_2 \oplus \cdots \oplus M_n$ satisfies the strong-like ascending (respectively, descending) chain condition on submodules if and only if so does each M_i.*

Proof. Use induction on n. If $n = 2$, apply Proposition 2.4 to the following sequence

$$0 \to M_1 \overset{\iota_1}{\to} M_1 \oplus M_2 \overset{\pi_2}{\to} M_2 \to 0. \qquad \square$$

3. Primary-like dimension for modules

In this section, we introduce and study a new generalization of the Krull dimension for modules.

Definition 3.1. *Let R be a ring and M be a left R-module such that every primary-like submodule of M is contained in a virtually maximal primary-like submodule. We define, by transfinite induction, sets X_α of primary-like submodules of M. To start with, let X_{-1} be the empty set. Next, consider an ordinal $\alpha \geq 0$; if X_β has been defined, for all ordinals $\beta < \alpha$, let X_α be the set of those primary-like submodules Q in M such that all primary-like submodules strongly-like properly containing Q belong to $\bigcup_{\beta<\alpha} X_\beta$. (In particular, X_0 is the set of virtually maximal primary-like submodules of M.) If some X_γ contains all primary-like submodules of M, we say that $P.L.dim(M)$ exists, and we set $P.L.dim(M)$-the primary-like dimension of M-equal to the smallest such γ. We write $P.L.dim(M) = \gamma$ as an abbreviation for the statement that $P.L.dim(M)$ exists and equals γ.*

Proposition 3.1. *Let R be a ring and M be a left R-module with the virtual-like acc on primary-like submodules. Then $P.L.dim(M)$ exists.*

Proof. Define the sets X_γ of primary-like submodules as in the definition above of primary-like dimension. Since there is a bound the cardinalities of these sets (e.g., 2^{cardM}), the transfinite chain $X_{-1} \subseteq X_0 \subseteq X_1 \subseteq \cdots$ cannot be properly increasing forever. Hence, there exists an ordinal γ such that $X_\gamma = X_{\gamma+1}$. If $P.L.dim(M)$ dose not exist, then X_γ dose not contain all the primary-like submodules of M. Using the virtual-like acc on primary-like submodules, there is a primary-like submodule Q of M virtually maximal with respect to the property $Q \notin X_\gamma$. Hence, all primary-like submodules strongly-like properly containing Q lie in X_γ. But, then $Q \in X_{\gamma+1} = X_\gamma$, a contradiction. $\square$

Corollary 3.1. *Let R be aring and M be a left R-module such that the set $\{P \in Spec(R) | P = \sqrt{(Q : M)}, Q \in Spec_L(M)\}$ has acc. Then $P.L.dim(M)$ exists.*

Proof. Follows from Proposition 3.1. □

Lemma 3.1. *Let M be an R-module for which $P.L.dim(M)$ exists. Then, for any submodule N of M, $P.L.dim(M/N)$ exists and is no larger than $P.L.dim(M)$.*

Proof. Note submodule Q/N of M/N is primary-like if and only if submodule Q of M is primary-like and $N \subseteq Q$. □

Corollary 3.2. *Let M be an R-module for which $P.L.dim(M)$ exists. If Q and Q' are primary-like submodules of M such that $Q \subset_{sl} Q'$, then $P.L.dim(M/Q') \leq P.L.dim(M/Q)$.*

Proof. Follows from Lemm 3.1. □

Theorem 3.1. *Let M be a left R-module. Then, $P.L.dim(M)$ exists if and only if M has virtual-like acc on primary-like submodules.*

Proof. Suppose that $P.L.dim(M) = \gamma$, where γ is an ordinal number. If $Q_1 \subset_{sl} Q_2 \subset_{sl} Q_3 \subset_{sl} \cdots$ is a strong-like assenting chain of primary-like submodules of M, then by Lemma 3.1 and Corollary 3.2, we have

$$\cdots < P.L.dim(M/Q_3) < P.L.dim(M/Q_2) < P.L.dim(M/Q_1) < \gamma,$$

which is impossible. Therefore, M has virtual-like acc on primary-like submodules. The converse is immediate from Proposition 3.1. □

Suppose that the module M contains a primary-like submodule Q. Then, the virtual-like height of Q, denoted by $vl.ht(Q)$, is the greatest nonnegative integer n such that there exists a strong-like chain of primary-like submodules of M

$$Q_0 \subset_{sl} Q_1 \subset_{sl} \cdots \subset_{sl} Q_n = Q,$$

and $vl.ht(Q) = \infty$ if no such n exists.

A prime ring R is called left bounded if for each regular element r in R there exists an ideal I of R and a regular element s such that $Rs \subseteq I \subseteq Rr$. A general ring R is called left fully bounded if every prime homomorphic image of R is left bounded. A ring R is called a left FBN-ring if R is left fully bounded and left Noetherian. It is well known that if R is a PI-ring (ring with polynomial identity) and P is a prime ideal of R, then the ring R/P is (left and right) bounded and (left and right) Goldie [18, 13.6.6].

Proposition 3.2. *Let R be a PI-ring (or an FBN-ring) and let M be an R-module such that every primary-like submodule of M is contained in a maximal submodule of M. If $P.L.dim(M) = n < \infty$, then for each primary-like submodule Q of M such that $vl.ht(Q) = n$, the factor module M/Q is homogeneous semisimple.*

Proof. Suppose that Q is a primary-like submodule of M with $vl.ht(Q) = n$ and Q' is a maximal submodule of M such that $Q \subseteq Q'$. Since $P.L.dim(M) = n$, so that $P = \sqrt{(Q : M)} = \sqrt{(Q' : M)}$ is a maximal ideal of R and M/Q' is a faithful simple R/P-module. The ring R/P is left bounded, left Goldie, thus, [7, Proposition 8.7] gives that R/P embeds as a left R-module in a finite direct sum of copies of M/Q'. It follows that the ring R/P is left Artinian, and, hence, R/P is simple Artinian. Thus, the left R/P-module M/Q is a direct sum of isomorphic simple modules. It follows that M/Q is a homogeneous semisimple R-module. □

Corollary 3.3. *Let R be a PI-ring and M be a finitely generated R-module such that $P.L.dim(M) = n < \infty$. Then, for each primary-like submodule Q of M such that $vl.ht(Q) = n$, the factor module M/Q is homogeneous semisimple.*

Proof. It follows from Proposition 3.2. □

Lemma 3.2. *Let M be an R-module. Then, $P.L.dim(M) = 0$ if and only if $Spec_L(M) \neq \emptyset$; and every primary-like submodule of M is a virtually maximal primary-like submodule.*

Proof. Is clear. □

A ring R is called a left FBN-ring if R is left fully bounded and left Noetherian.

A submodule Q of M is said to be virtually maximal prime if Q is a prime submodule of M and there is no prime submodule Q' of M such that $Q \subset_s Q'$ (i.e., Q is a prime submodule of M and for any prime submodule Q' of M, such that $Q \subseteq Q'$, we have $(Q : M) = (Q' : M)$).

Lemma 3.3. *Let R be a PI-ring (or an FBN-ring) and let M be an R-module in which every proper submodule is contained in a maximal submodule. Then, for each proper submodule Q of M such that M/Q is primeful, the following statements are equivalent.*

1) *Q is a virtually maximal submodule.*

2) *Q is a virtually maximal prime submodule.*

3) *Q is a virtually maximal primary-like submodule.*

Proof. (1) ⇒ (2) ⇒ (3) is clear.

(3) ⇒ (1) Assume that Q is a virtually maximal primary-like submodule of M. Then, there exists a maximal submodule Q' of M such that $Q \subset Q'$. It follows that $\sqrt{(Q : M)} = \sqrt{(Q' : M)} = P$ and M/Q' is a simple R/P-module. Since R is a PI-ring (or an FBN-ring), then the ring R/P is a left bounded, left Goldie ring. Now, by [7, Proposition 8.7] we have that R/P embeds as a left R-module in a finite direct sum of copies of M/Q'. It follows that the ring R/P

is left Artinian, and, hence, R/P is simple Artinian. Thus, the left R/P-module M/Q is a direct sum of isomorphic simple modules. It follows that M/Q is a homogeneous semisimple R-module; i.e., Q is a virtually maximal submodule of M. □

Corollary 3.4. *Let R be a PI-ring (or an FBN-ring) and let M be an R-module in which every proper submodule is contained in a maximal submodule and $Spec_L(M) \neq \emptyset$. Then, for each proper submodule Q of M such that M/Q is primeful, the following statements are equivalent.*

1) *Q is a virtually maximal submodule.*

2) *Q is a virtually maximal prime submodule.*

3) *Q is a virtually maximal primary-like submodule.*

4) *$P.L.dim(M) = 0$.*

Proof. Follows from Lemmas 3.2 and 3.3. □

4. Conclusion

In this paper, we introduced the notion of virtual-like ascending and descending chains condition on submodules of a module where every Noetherian (respectively, Artinian) module is virtually-like Noetherian (respectively, virtually-like Artinian) and it is shown that the converse is not generally true Example 2.2.

The connections between maximal, virtually maximal, maximal primary-like, maximal virtually primary-like and primary-like submodules are investigated Proposition 2.1. Also, exact sequences of modules, the quotient structure and the direct sum of modules are considered and studied under this concept Proposition 2.4 and Corollaries 2.2 and 2.3.

Moreover, the primary-like dimension of a module is defined and shown that it there exists for every left R-module with the virtual-like acc on primary-like submodules Proposition 3.1. Furthermore, links of the primary-like dimension of a module and the related quotient structure and also primary-like submodules are investigated and it is shown that existence of the primary-like dimension of a module is depended to existence of virtual-like acc on primary-like submodules Theorem 3.1. And the connection between the finiteness of the primary-like dimension of modules and homogeneity and semi-simplicity of the related factor modules Proposition 3.2. Finally the connection between virtually maximal, virtually maximal prime and virtually maximal primary-like submodules in R-modules with a PI-ring (or an FBN-ring) R is indicated Proposition 3.3.

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