Roughness of soft sets over a semigroup

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Abstract. In this study we discuss the concept of rough soft sets over a semigroup. Basic results of the lower and upper approximations of soft semigroups , soft ideals, soft bi-ideals and soft interior ideals over a semigroup with a congruence relation are introduced. Finally, topological structures of rough soft sets are presented.
Keywords: soft sets, soft semigroups, soft ideals, rough soft sets.

1. Introduction

A lot of concepts like fuzzy sets [9], rough sets [14], soft sets [8] are presented to deal mathematically with uncertain knowledge. In 1982, Pawlak [14] introduced the rough set theory as an extension of ordinary set theory, in which a pair of ordinary sets namely the lower approximation and upper approximation are associated to a subset of a universe. A connection between algebraic systems and rough sets are studied by some authors. One of these algebraic systems is semigroup theory, which is the main interest of this study. Kuroki in [7], presented the concept of a rough ideal in a semigroup. Since Molodtsov [8] intoduced the theory of soft sets, the literature of soft algebraic systems has grown rapidly. For example, Aktas and Cagman [1] initiated the concept of soft groups, Ali et al. [2] applied the notion of soft sets to the semigroup theory, and

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introduced the concept of soft semigroups and soft ideals. Following, the present authors defined several types of soft ideals over a semigroup in [10, 11, 12]. In 2010, Feng et al. [3] studied the relation between soft sets and rough sets by introducing the concept of rough soft set. Ghosh and Samanta [4] discussed the main properties of rough soft sets and defined rough soft groups. In this paper, we deal with rough soft set theory by giving the universal set the structure of a semigroup and examine some basic properties of rough soft sets with illustrative examples. Then, we study the roughness of soft semigroups, soft left (right) ideals, soft bi-ideals and soft interior ideals over a semigroup. Finally, some topological spaces induced by rough soft sets are studied.

2. Preliminaries

A nonempty subset $\phi \neq B \subseteq S$ of a semigroup S is called a bi-ideal (an interior ideal) of S if it is subsemigroup of S and $BSB \subseteq B(SBS \subseteq B)$ (see, [5]).

Definition 2.1 ([8]). *Let E be a set of parameters, $P(S)$ the power set of S and $A \subseteq E$. The pair (F, A) is called a soft set over S, where F is a mapping $F : A \longrightarrow P(S)$.*

The soft set (F, A) is called a nonempty soft set,$(F, A) \neq \phi$, if and only if $F(a) \neq \phi$, for all $a \in A$. Here, we fix A as the set of parameters and denote the set of all soft sets over a semigroup S by $\mathcal{T}(S)$.

Definition 2.2 ([2]). *Let $(F, A), (G, A) \in \mathcal{T}(S)$, then (G, A) is called a soft subset of (F, A), denoted by $(G, A) \sqsubseteq (F, A)$ if $G(a) \subseteq F(a)$, for all $a \in A$. The two sets (F, A) and (G, A) are equal iff $(G, A) \sqsubseteq (F, A)$ and $(F, A) \sqsubseteq (G, A)$.*

Definition 2.3 ([2]). *Let $(F, A), (G, A) \in \mathcal{T}(S)$, the intersection of (F, A) and (G, A) is the soft set $(F \sqcap G, A)$ such that $F \sqcap G(a) = F(a) \cap G(a)$, for all $a \in A$.*

Definition 2.4 ([2]). *Let $(F, A), (G, A) \in \mathcal{T}(S)$, the union of (F, A) and (G, A) is the soft set $(F \sqcup G, A)$ such that $F \sqcup G(a) = F(a) \cup G(a)$ for all $a \in A$.*

Definition 2.5 ([2]). *Let $(F, A), (G, A) \in \mathcal{T}(S)$. The soft product $(F, A) \bullet (G, A)$ is defined as the soft set (FG, A) where $FG(a) = F(a)G(a)$, for all $a \in A$.*

Definition 2.6 ([2]). *A soft set (F, A) over a semigroup S is called a soft semigroup if $(F, A) \bullet (F, A) \sqsubseteq (F, A)$.*

Definition 2.7 ([2]). *A soft set (F, A) over S is called a soft left [right] ideal over S, if $(S, A) \bullet (F, A) \sqsubseteq (F, A)$ $[(F, A) \bullet (S, A) \sqsubseteq (F, A)]$, where (S, A) is a soft set over S defined by $S(a) = S$ $\forall a \in A$. A soft set (F, A) is called a soft ideal if it is both a soft left and a soft right ideal over S.*

Proposition 2.1 ([2]). *A soft set (F, A) is a soft semigroup (ideal) over S if and only if $\forall a \in A$, $F(a) \neq \phi$ is a subsemigroup (an ideal) of S.*

Definition 2.8 ([2, 11]). *A soft set (F, A) over a semigroup S is called a soft bi-ideal (interior ideal) if and only if $\forall a \in A$, $F(a) \neq \phi$ is a bi-ideal (interior ideal) of S.*

Definition 2.9 ([13]). *A collection $\mathcal{T}$ of soft sets over S is called a soft topology on S if:*

(*i*) $(\phi, A), (S, A) \in \mathcal{T}$ *where* $\phi(a) = \phi$ *and* $S(a) = S$, *for all* $a \in A$,

(*ii*) *the intersection of any two soft sets in* $\mathcal{T}$ *belongs to* $\mathcal{T}$,

(*iii*) *the union of any number of soft sets in* $\mathcal{T}$ *belongs to* $\mathcal{T}$.

The triplet $(S, \mathcal{T}, A)$ is called a soft topological space over S.

3. Approximation of soft sets over a semigroup

Let (S, θ) be a Pawlak approximation space (PAS), that is, θ is an equivalence relation on a semigroup S. The lower approximation $\theta_\star(X)$ and upper approximation $\theta^\star(X)$ of $X \subseteq S$ are defined by[14]

$$\theta_\star(X) = \{x \in S : [x] \subseteq X\},$$

$$\theta^\star(X) = \{x \in S : [x] \cap X \neq \phi\}.$$

Definition 3.1 ([3]). *Let (S, θ) be PAS and (F, A) be a soft set over S. The lower approximation $(\underline{F}, A)$ and upper approximation $(\overline{F}, A)$ of (F, A) are soft sets over S defined by*

$$\underline{F}(a) = \theta_\star(F(a)) = \{x \in S : [x] \subseteq F(a)\},$$
$$\overline{F}(a) = \theta^\star(F(a)) = \{x \in S : [x] \cap F(a) \neq \phi\},$$

for all $a \in A$. If $(\underline{F}, A) = (\overline{F}, A)$, then (F, A) is called definable; otherwise (F, A) is called a rough soft set.

The following properties of rough soft sets are due to [3]. We shall give a proof for completeness.

Theorem 3.1. *Suppose that (S, θ) is PAS. If $(F, A), (G, A) \in \mathcal{T}(S)$, then the following hold:*

(1) $(\underline{F}, A) \sqsubseteq (F, A) \sqsubseteq (\overline{F}, A)$;

(2) $(\overline{F \sqcup G}, A) = (\overline{F}, A) \sqcup (\overline{G}, A)$;

(3) $(\underline{F \sqcap G}, A) = (\underline{F}, A) \sqcap (\underline{G}, A)$;

(4) *If* $(F, A) \sqsubseteq (G, A) \Longrightarrow (\underline{F}, A) \sqsubseteq (\underline{G}, A)$ *and* $(\overline{F}, A) \sqsubseteq (\overline{G}, A)$;

(5) $(\overline{F \sqcap G}, A) \sqsubseteq (\overline{F}, A) \sqcap (\overline{G}, A)$;

(6) $(\underline{F \sqcup G}, A) \sqsupseteq (\underline{F}, A) \sqcup (\underline{G}, A)$;

(7) $(\underline{\underline{F}}, A) = (\underline{F}, A)$;

(8) $(\overline{\underline{F}}, A) = (\underline{F}, A)$.

Proof. (1) Let $u \in \underline{F}(a)$ then $u \in [u] \subseteq F(a)$, for all $a \in A$. Thus, $(\underline{F}, A) \sqsubseteq (F, A)$. If $u \in F(a) \Rightarrow [u] \cap F(a) \neq \phi$, that is, $u \in \theta^{\star}(F(a)) = \overline{F}(a)$. Hence, $(F, A) \sqsubseteq (\overline{F}, A)$.

(2) Let $u \in \overline{F \sqcup G}(a) = \theta^{\star}((F \sqcup G)(a)) = \theta^{\star}(F(a) \cup G(a)) = \theta^{\star}(F(a)) \cup \theta^{\star}(G(a)) = \overline{F}(a) \cup \overline{G}(a)$. Thus, $(\overline{F \sqcup G}, A) = (\overline{F}, A) \sqcup (\overline{G}, A)$.

(3) Let $u \in \underline{F \sqcap G}(a) = \theta_{\star}((F \sqcap G)(a)) = \theta_{\star}(F(a) \cap G(a)) = \theta_{\star}(F(a)) \cap \theta_{\star}(G(a)) = \underline{F}(a) \cap \underline{G}(a)$. Thus, $(\underline{F \sqcap G}, A) = (\underline{F}, A) \sqcap (\underline{G}, A)$.

(4) Let $u \in \underline{F}(a) = \theta_{\star}(F(a)) \subseteq \theta_{\star}(G(a)) = \underline{G}(a)$, since $F(a) \subseteq G(a)$, for all $\in A$. Hence, $(\underline{F}, A) \sqsubseteq (\underline{G}, A)$. Similarly, we show that $(\overline{F}, A) \sqsubseteq (\overline{G}, A)$.

(5) Let $u \in \overline{F \sqcap G}(a) = \theta^{\star}(F \sqcap G(a)) = \theta^{\star}(F(a) \cap G(a)) \subseteq \theta^{\star}(F(a)) \cap \theta^{\star}(G(a)) = \overline{F}(a) \cap \overline{G}(a)$. Therefore, $(\overline{F \sqcap G}, A) \sqsubseteq (\overline{F}, A) \sqcap (\overline{G}, A)$.

(6) Let $u \in \underline{F \sqcup G}(a) = \theta_{\star}(F \sqcup G(a)) = \theta_{\star}(F(a) \cup G(a)) \supseteq \theta_{\star}(F(a)) \cup \theta_{\star}(G(a)) = \overline{F}(a) \cup \overline{G}(a)$. Therefore, $(\underline{F \sqcup G}, A) \sqsupseteq (\underline{F}, A) \sqcup (\underline{G}, A)$.

(7), (8) see items (4,6) in Theorem 3 [3]. □

Definition 3.2 ([7]). *An equivalence relation θ on a semigroup S is called a congruence on S if $(a, b) \in \theta$ implies $(ax, bx) \in \theta$ and $(xa, xb) \in \theta$, for all $x \in S$. A congruence θ on S is called complete if $[x][y] = [xy]$, for all $x, y \in S$.*

Theorem 3.2. *Let θ be a congruence relation on S. If (F, A) and (G, A) are nonempty soft sets over S, then*

$$(\overline{F}, A) \bullet (\overline{G}, A) \sqsubseteq (\overline{FG}, A).$$

Proof. By applying Theorem 2.2 in [7], we have

$$\begin{aligned}\overline{F}(a)\overline{G}(a) &= \theta^{\star}(F(a))\theta^{\star}(G(a)) \\ &\subseteq \theta^{\star}(F(a)G(a)) = \theta^{\star}(FG(a)) = \overline{FG}(a).\end{aligned}$$

Then, $\overline{F}(a)\overline{G}(a) \subseteq \overline{FG}(a)$, for all $a \in A$. Thus, we have $(\overline{F}, A) \bullet (\overline{G}, A) \sqsubseteq (\overline{FG}, A)$. □

The following theorem comes as a direct application of Theorem 2.3 in [7].

Theorem 3.3. *Let θ be a complete congruence relation on S. If (F, A) and (G, A) are nonempty soft sets over S, then*

$$(\underline{F}, A) \bullet (\underline{G}, A) \sqsubseteq (\underline{FG}, A).$$

Proposition 3.1 ([2]). *A soft set (F, A) is a soft semigroup over S if and only if $\forall a \in A$, $F(a) \neq \phi$ is a subsemigroup of S.*

Theorem 3.4. *Let θ be a congruence relation on a semigroup S. Then:*

(1) *(F, A) is a soft semigroup $\Longrightarrow$ $(\overline{F}, A)$ is a soft semigroup;*

(2) *(F, A) is a soft ideal $\Longrightarrow$ $(\overline{F}, A)$ is a soft ideal.*

Proof. (1) Assume that (F, A) is a soft semigroup over S, then $F(a)$ is a subsemigroup of S, for all $a \in A$. Since $(\overline{F}, A)$ is nonempty soft set then Theorem 3.2 implies that

$$(\overline{F}, A) \bullet (\overline{F}, A) \sqsubseteq (\overline{FF}, A).$$

That is, $\overline{F}(a)\overline{F}(a) \subseteq \overline{FF}(a)$, for all $a \in A$. By Theorem 3.1, we obtain

$$\overline{F}(a)\overline{F}(a) \subseteq \overline{FF}(a) = \theta^{\star}(FF(a)) = \theta^{\star}(F(a)F(a)) \subseteq \theta^{\star}(F(a)) = \overline{F}(a).$$

Thus, $\overline{F}(a)$ is a subsemigroup of S, for all $a \in A$. Therefore, $(\overline{F}, A)$ is a soft semigroup over S.

(2) Assume that (F, A) is a soft left ideal over S, then $F(a)$ is a left ideal of S, for all $a \in A$. Note that $\theta^{\star}(S) = S$. Then, Theorem 3.2 implies

$$(\overline{S}, A) \bullet (\overline{F}, A) \sqsubseteq (\overline{SF}, A).$$

That is, $\overline{S}(a)\overline{F}(a) \subseteq \overline{SF}(a)$, for all $a \in A$. By Theorem 3.1, we obtain

$$\overline{S}(a)\overline{F}(a) \subseteq \overline{SF}(a) = \theta^{\star}(SF(a)) \subseteq \theta^{\star}(F(a)) = \overline{F}(a).$$

Thus, $\overline{F}(a)$ is a left ideal of S, for all $a \in A$. Therefore, $(\overline{F}, A)$ is a soft left ideal over S. In a similar way, it can be shown that $(\overline{F}, A)$ is a soft right ideal over S whenever (F, A) is. This completes the proof. □

Theorem 3.4 shows that every soft semigroup (soft ideal) (F, A) over a semigroup S can be extended to the largest soft semigroup (soft ideal) $(\overline{F}, A)$. Generally, the converse of the above theorem does not hold, as shown in the following example.

Example 3.1. *Let $S = \{a, b, c, d\}$ be a semigroup with the following table* [7]:

*	a	b	c	d
a	a	b	c	d
b	b	b	b	b
c	c	c	c	c
d	d	c	b	a

Suppose S is partitioned by a congruence relation θ into the classes: $\{a\}, \{d\}, \{b, c\}$
Let $A = \{e_1, e_2\}$ and (F, A) be a soft set over S defined by

$$F(e_1) = \{b\}, \quad F(e_2) = S.$$

Then, the upper approximation $(\overline{F}, A)$ of (F, A) is defined as follows:

$$\overline{F}(e_1) = \theta^{\star}(\{b\}) = \{b, c\}, \quad \overline{F}(e_2) = \theta^{\star}(S) = S.$$

It is clear that $(\overline{F}, A)$ is a soft ideal over S while (F, A) is not.

The below result is an application to Theorem 3.2 in [7].

Theorem 3.5. *Let θ be a complete congruence relation on S. Then*

(1) (F, A) *is a soft semigroup* $\Longrightarrow$ $(\underline{F}, A) \neq \phi$ *is a soft semigroup;*

(2) (F, A) *is a soft ideal* $\Longrightarrow$ $(\underline{F}, A) \neq \phi$ *is a soft ideal.*

Proof. (1) Suppose that (F, A) is a soft semigroup over S, then $F(a)$ is a subsemigroup of S, for all $a \in A$. Since $(\underline{F}, A)$ is nonempty soft set then Theorem 3.3 implies that

$$(\underline{F}, A) \bullet (\underline{F}, A) \sqsubseteq (\underline{FF}, A).$$

That is, $\underline{F}(a)\underline{F}(a) \subseteq \underline{FF}(a)$, for all $a \in A$. By Theorem 3.1, we obtain

$$\underline{F}(a)\underline{F}(a) \subseteq \underline{FF}(a) = \theta_\star(FF(a)) = \theta_\star(F(a)F(a)) \subseteq \theta_\star(F(a)) = \underline{F}(a).$$

Thus, $\underline{F}(a)$ is a subsemigroup of S, for all $a \in A$. Therefore, $(\underline{F}, A)$ is a soft semigroup over S.

(2) Assume that (F, A) is a soft left ideal over S, then $F(a)$ is a left ideal of S, for all $a \in A$. Note that $\theta_\star(S) = S$. Then, Theorem 3.1 implies

$$(\underline{S}, A) \bullet (\underline{F}, A) \sqsubseteq (\underline{SF}, A).$$

That is, $\underline{S}(a)\underline{F}(a) \subseteq \underline{SF}(a)$, for all $a \in A$. By Theorem 3.3, we obtain

$$\underline{S}(a)\underline{F}(a) \subseteq \underline{SF}(a) = \theta_\star(SF(a)) \subseteq \theta_\star(F(a)) = \underline{F}(a).$$

Thus, $\underline{F}(a)$ is a a left ideal of S, for all $a \in A$. Therefore, $(\underline{F}, A)$ is a soft left ideal over S. In a similar way, $(\underline{F}, A)$ is a soft right ideal over S whenever (F, A) is. This completes the proof. □

Theorem 3.6. *Let θ be a congruence relation on S. Then, $(\overline{F}, A)$ is a soft bi-ideal over S if (F, A) is a soft bi-ideal.*

Proof. Let (F, A) be a soft bi-ideal over S then (F, A) is a soft semigroup over S and by theorem 3.4, $(\overline{F}, A)$ is a soft semigroup. By applying Theorem 3.2 more times, we have

$$\begin{aligned}\overline{F}(a)\overline{S}(a)\overline{F}(a)) &= \theta^\star(F(a))\theta^\star(S(a))\theta^\star(F(a)) \\ &\subseteq \theta^\star(F(a))\theta^\star(S(a)F(a)) \\ &\subseteq \theta^\star(F(a)S(a)F(a)) \\ &\subseteq \theta^\star(F(a)) = \overline{F}(a).\end{aligned}$$

Thus $\overline{F}(a)$ is a bi-ideal of S, for all $a \in A$. Therefore,$(\overline{F}, A)$ is a soft bi-ideal over S. □

Theorem 3.7. *Let θ be a complete congruence relation on S. Then, $(\underline{F}, A) \neq \phi$ is a soft bi-ideal over S if (F, A) is a soft bi-ideal.*

Proof. It is immediate by applying Theorem 3.3 and Theorem 3.5. □

Theorem 3.8. *Let θ be a congruence relation on S. Then, $(\overline{F}, A)$ is a soft interior ideal over S if (F, A) is a soft interior ideal.*

Proof. Let (F, A) be a soft interior ideal over S then (F, A) is a soft semigroup over S and by theorem 3.4, $(\overline{F}, A)$ is a soft semigroup. By applying Theorem 3.2 , we have

$$\begin{aligned} S\overline{F}(a)S = \overline{S}(a)\overline{F}(a)\overline{S}(a) &= \theta^\star(S(a))\theta^\star(F(a))\theta^\star(S(a)) \\ &\subseteq \theta^\star(S(a))\theta^\star(F(a)S(a)) \\ &\subseteq \theta^\star(S(a)F(a)S(a)) \\ &\subseteq \theta^\star(F(a)) = \overline{F}(a). \end{aligned}$$

Thus, $\overline{F}(a)$ is an interior ideal of S, for all $a \in A$. Therefore, $(\overline{F}, A)$ is a soft interior ideal over S. □

Theorem 3.9. *Let θ be a complete congruence relation on S. Then, $(\underline{F}, A) \neq \phi$ is a soft interior ideal over S if (F, A) is a soft interior ideal.*

Proof. It is immediate by applying Theorem 3.3 and Theorem 3.5. □

Theorem 3.10. *Let θ be a congruence relation on S. (F, A) and (G, A) are a soft right ideal and a soft left ideal over S, respectively, then:*

(i) $(\overline{FG}, A) \sqsubseteq (\overline{F}, A) \sqcap (\overline{G}, A)$*;*

(ii) $(\underline{FG}, A) \sqsubseteq (\underline{F}, A) \sqcap (\underline{G}, A)$*.*

Proof. (i) By hypotheses, $F(a)$ is a right ideal of S and $G(a)$ is a left ideal of S, for all $a \in A$. Then, we have

$$F(a)G(a) \subseteq F(a)S \subseteq F(a),\ F(a)G(a) \subseteq SG(a) \subseteq G(a).$$

Thus, $F(a)G(a) \subseteq F(a) \cap G(a)$, for all $a \in A$. Theorem 3.1 verifies that

$$(\overline{FG}, A) \sqsubseteq (\overline{F \sqcap G}, A) \sqsubseteq (\overline{F}, A) \sqcap (\overline{G}, A).$$

(ii) Similar to item (i). This completes the proof. □

Assume θ and ρ are congruence relations on S. Then, the composition

$$\theta \circ \rho = \{(x, y) : (x, z) \in \theta,\ (z, y) \in \rho\}$$

is a congruence relation on S iff $\theta \circ \rho = \rho \circ \theta$ (see, [7]).

Theorem 3.11. *Let θ and ρ be congruence relations on S such that $\theta \circ \rho = \rho \circ \theta$. If (F, A) is a soft semigroup over S, then*

$$(\overline{F}_\theta, A) \bullet (\overline{F}_\rho, A) \sqsubseteq (\overline{F}_{\theta\circ\rho}, A).$$

Proof. Let $c \in \overline{F}_\theta(a)\overline{F}_\rho(a)$ then there exist $x \in \overline{F}_\theta(a) = \theta^\star(F(a))$ and $y \in \overline{F}_\rho(a) = \rho^\star(F(a))$ such that $c = xy \in \theta^\star(F(a))\rho^\star(F(a)) \subseteq S$. From definition of the upper approximation, there exist $z, w \in S$ such that

$$z \in [x]_\theta \cap F(a), \quad w \in [y]_\rho \cap F(a).$$

That is, $(z, x) \in \theta$ and $(w, y) \in \rho$. Since θ and ρ are congruence relations on S, then $(zw, xw) \in \theta$ and $(xw, xy) \in \rho$. This implies that $(zw, xy) \in \theta \circ \rho$. Since $F(a)$ is a subsemigroup of S, for all $a \in A, zw \in F(a)$. Therefore we have $zw \in [xy]_{\theta\circ\rho} \cap F(a)$ which implies $xy \in (\theta \circ \rho)^\star(F(a)) = \overline{F}_{\theta\circ\rho}(a)$. Thus, we have

$$(\overline{F}_\theta, A) \bullet (\overline{F}_\rho, A) \sqsubseteq (\overline{F}_{\theta\circ\rho}, A).$$

4. Topological structures on rough soft sets

In this section, some topological spaces induced by rough soft sets are discussed. Throughout this section, S is a semigroup, θ is an equivalence relation on S.

4.1 Topological semigroups Vs soft sets

Topologies versus rough sets are studied by M. Kondo [6].

Proposition 4.1 ([6]). *$T_\theta = \{X \subseteq S : \theta_\star(X) = X\}$ is a topology on S.*

Furthermore, we show that the pair (S, T_θ) is a topological semigroup.

Theorem 4.1. *If θ is a complete congruence relation on S, (S, T_θ) is a topological semigroup.*

Proof. Let $x, y \in S$ and $U \in T_\theta$ be an open set containing the element $p = xy$. Then, $xy \in U = \theta_\star(U)$ which implies $[xy] \subseteq U$. By completeness of θ, we have $[x][y] = [xy] \subseteq U$. Since, $[x], [y]$ are open sets containing x, y respectively such that $[x][y] \subseteq U$ then we conclude that the multiplication $. : S \times S \to S$ of S is a continuous mapping. Therefore, (S, T_θ) is a topological semigroup. □

Theorem 4.2. *Let θ be a complete congruence relation on S, and (F, A) be a soft semigroup over S. Then, for all $a \in A$, the pair $(F(a), T_{F(a)})$ is a topological semigroup, where $T_{F(a)}$ is the relative topology on $F(a)$ induced from T_θ.*

Proof. By Theorem 4.1,(S, T_θ) is a topological semigroup. Since (F, A) is a soft semigroup over S, then $F(a)$ is a subsemigroup of S, for all $a \in A$. Let $x, y \in F(a)$ and $W \in T_{F(a)}$ containing the element xy. From definition of $T_{F(a)}$, there exists an open set $U \in T_\theta$ such that $xy \in W = F(a) \cap U$. Hence, $xy \in U = \theta_\star(U)$ that is, $[xy] \subseteq U$. Since the classes $[x], [y] \in T_\theta$, the two sets $F(a) \cap [x]$, $F(a) \cap [y]$ are open sets in $F(a)$ and containing x, y respectively. Then, we obtain

$$(F(a) \cap [x])(F(a) \cap [y]) \subseteq F(a) \cap [x][y] = F(a) \cap [xy] \subseteq F(a) \cap U = W.$$

Thus, for all $a \in A$, the pair $(F(a), T_{F(a)})$ is a topological semigroup. □

4.2 Soft topologies Vs rough soft sets

Let (F, A) be a soft set over S. Denote

$$\mathcal{T}_B = \{(F, A) \in \mathcal{T}(S) : (\underline{F}, A) = (\overline{F}, A)\},$$

$$\mathcal{T}_E = \{(F, A) \in \mathcal{T}(S) : (\underline{F}, A) = (F, A)\},\ \mathcal{T}_L = \{(\underline{F}, A) : (F, A) \in \mathcal{T}(S)\}.$$

By using items (7),(8) in Theorem 3.1, the proof of the following result is a clear matter.

Proposition 4.2. $\mathcal{T}_B = \mathcal{T}_E = \mathcal{T}_L$.

Notatios:. The complement of $(F, A) \in \mathcal{T}(S)$ is a soft set $(H, A) \in \mathcal{T}(S)$ such that $H(a) = F(a)^c = S - F(a)$, for all $a \in A$.

Proposition 4.3. *Let (S, θ) be PAS. Then, for all $(F, A) \in \mathcal{T}(S)$,*

$$(\underline{F}, A) = (F, A) \Longleftrightarrow (H, A) = (\underline{H}, A).$$

Proof. $(\Rightarrow)$ Suppose that $(\underline{F}, A) = (F, A)$. That is, $F(a) = \theta_\star(F(a))$, for all $a \in A$. From Propostion 5 in [6], we have

$$H(a) = F(a)^c = \theta_\star(F(a)^c) = \theta_\star(H(a)) = \underline{H}(a).$$

This means that $(H, A) = (\underline{H}, A)$.

$(\Leftarrow)$ Proof of the converse statement is similar. □

Theorem 4.3. *Let (S, θ) be PAS, then:*

(i) $(S, \mathcal{T}_E, A)$ is a soft topological space.

(ii) $(F, A) \in \mathcal{T}_E \Longleftrightarrow (H, A) \in \mathcal{T}_E$.

Proof. (i) Since $\underline{S}(a) = \theta_\star(S(a)) = \theta_\star(S) = S$, for all $a \in A$, then the whole soft set $(S, A) \in \mathcal{T}_E$. Similarly the empty soft set $(\phi, A) \in \mathcal{T}_E$. Assume that (F, A) and (G, A) are arbitrary elements in $\mathcal{T}_E$, then, by Theorem 3.1, we have

$$\underline{F \sqcap G}(a) = \underline{F} \sqcap \underline{G}(a) = \underline{F}(a) \cap \underline{G}(a) = F(a) \cap G(a) = F \sqcap G(a)$$

Thus, $(F, A) \sqcap (G, A)$ is an element in $\mathcal{T}_E$.

Let $\{(F_i, A) : i \in J\} \subset \mathcal{T}_E$ be a family of soft open sets over S, then, by item(1) in Theorem 3.1, we have

$$(\underline{\sqcup_{i\in J} F_i}, A) \sqsubseteq (\sqcup_{i\in J} F_i, A).$$

Since $(\underline{F_i}, A) = (F_i, A)\, \forall i \in J$ and by Propostion 2 in [6], we have

$$\sqcup_{i\in J} F_i(a) = \bigcup_{i\in J} F_i(a) \subseteq \theta_\star(\bigcup_{i\in J} F_i(a)) = \theta_\star(\sqcup_{i\in J} F_i(a)) = \underline{\sqcup_{i\in J} F_i}(a),$$

for all $a \in A$. Then, $(\underline{\sqcup_{i\in J} F_i}, A) = (\sqcup_{i\in J} F_i, A)$. So $(\sqcup_{i\in J} F_i, A)$ is an element in $\mathcal{T}_E$. Therefore, $(S, \mathcal{T}_E, A)$ is a soft topological space.

(ii) Let $(F, A) \in \mathcal{T}_E \Leftrightarrow (\underline{F}, A) = (F, A)$. Then, Proposition 4.3 implies that $(\underline{H}, A) = (H, A)$ and so $(H, A) \in \mathcal{T}_E$. Thus, (F, A) is soft closed.

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