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On a class of Lorentzian paracontact metric manifolds

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Abstract. In this present paper, we consider a class of Lorentzian almost paracontact metric manifolds namely Lorentzian para-Kenmotsu (briefly LP-Kenmotsu) manifolds admitting a pseudo-projective curvature tensor $\overline{W}(X,Y)$. We study and have shown that the scalar curvature of Lorentzian para-Kenmotsu manifold is constant if and only if the time like vector field ξ is harmonic, whenever the LP-Kenmotsu manifold satisfying $R(X,Y)\cdot\overline{W}=0$ is not an Einstein manifold. Further we have shown that Lorentzian para-Kenmotsu manifolds admitting an irrotational pseudo-projective curvature tensor and a conservative pseudo-projective curvature tensor are an Einstein manifolds of constant scalar curvature. At the end, we construct an example of a 3-dimensional LP-Kenmotsu manifold admitting a pseudo-projective curvature tensor which verifies the results discussed in the present work.

Keywords: Lorentzian para-Kenmotsu manifolds, pseudo-projective curvature tensor, harmonic vector field, irrotational and conservative vector fields.

1. Introduction

In 1989, Matsumoto [8] introduced the notion of Lorentzian paracontact metric manifolds and defined Lorentzian para-Sasakian (LP-Sasakian) manifolds, which are regarded as a special kind of these Lorentzian paracontact manifolds. Further, these manifolds have been widely studied by many geometers such as

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De, Matsumoto and Shaikh [7], Matsumoto and Mihai [9], Mihai and Rosca [10], Mihai, Shaikh and De [11], Venkatesha and Bagewadi [16], Venkatesha, Pradeep Kumar and Bagewadi [17] and obtained several results on these manifolds.

In 1995, Sinha and Sai Prasad [15] defined a class of almost paracontact metric manifolds namely para-Kenmotsu (briefly P-Kenmotsu) and special para-Kenmotsu (briefly SP-Kenmotsu) manifolds in similar to P-Sasakian and SP-Sasakian manifolds. In 2018, Abdul Haseeb and Rajendra Prasad defined a class of Lorentzian almost paracontact metric manifolds namely Lorentzian para-Kenmotsu (briefly LP-Kenmotsu) manifolds [1] and they studied ϕ-semi-symmetric LP-Kenmotsu manifolds with a quarter-symmetric non-metric connection admitting Ricci solitons [13].

On the other hand, in 1970 [12], Pokhariyal and Mishra introduced new tensor fields, called the Weyl-projective curvature tensor W_2 of type $(1,3)$ and the tensor field E on a Riemannian manifold. In our earlier work, we consider LP-Kenmotsu manifolds admitting the Weyl-projective curvature tensor W_2 and shown that these manifolds admitting a Weyl-flat projective curvature tensor, an irrotational Weyl-projective curvature tensor and a conservative Weyl-projective curvature tensor are an Einstein manifolds of constant scalar curvature [14].

The idea of Weyl-projective curvature tensor has been extended by Bhagawat Prasad [6], and in 2002 he defined the pseudo-projective curvature tensor $\overline{W}$ on a Riemannian manifold M_n of dimension n as:

$$\begin{aligned}\overline{W}(X,Y)Z &= aR(X,Y)Z + b[S(Y,Z)X - S(X,Z)Y]\\ &\quad - \frac{r}{n}\Big[\frac{a}{n-1} + b\Big][g(Y,Z)X - g(X,Z)Y],\end{aligned} \tag{1}$$

where a and b are constants such that $a, b \neq 0$. In the above expression $R(X,Y)$ is known to be the Riemannian curvature tensor, S is the Ricci tensor and r is the scalar curvature with respect to the Levi-Civita connection.

The pseudo-projective curvature tensor on a Riemannian manifold was widely studied by Bagewadi *et al.,* [2], Bagewadi and Venkatesha [3, 4] and by many geometers. In 2008, Bagewadi *et al.,* [5] have extended these concepts to Lorentzian paracontact structures and studied LP-Sasakian manifolds admitting this tensor field of particular type. They have shown that the LP-Sasakian manifold is an Einstein manifold if the pseudo projective curvature tensor admitted by the manifold is irrotational.

Motivated by these studies, in the present paper, we explore the geometrical significance of LP-Kenmotsu manifolds admitting the pseudo-projective curvature tensor. The present paper is organized as follows: Section 2 is equipped with some prerequisites about Lorentzian para-Kenmotsu manifolds. In section 3, we consider Lorentzian para-Kenmotsu manifolds admitting $R(X,Y)\cdot\overline{W} = 0$ and shown that it is an η-Einstein manifold of constant scalar curvature $n(n-1)$. As a special case, we have shown that the scalar curvature of Lorentzian para-Kenmotsu manifold is constant if and only if the time like vector field ξ is harmonic, whenever the LP-Kenmotsu manifold satisfying $R(X,Y)\cdot\overline{W}=0$ is not an Einstein manifold.

In the sections 4 and 5, we study geometrical properties of these manifolds, and in particular, we have shown that Lorentzian para-Kenmotsu manifolds admitting an irrotational pseudo-projective curvature tensor and a conservative pseudo-projective curvature tensor are an Einstein manifolds of constant scalar curvature. Finally, in section 6, we construct an example of a 3-dimensional LP-Kenmotsu manifold admitting pseudo-projective curvature tensor which verifies the results discussed in the present work.

2. Preliminaries

An n-dimensional differentiable manifold M_n admitting a (1, 1) tensor field ϕ, contravariant vector field ξ, a 1-form η and the Lorentzian metric $g(X,Y)$ satisfying

$$\phi^2X = X+\eta(X)\xi,\quad g(\phi X,\phi Y)=g(X,Y)+\eta(X)\eta(Y), \tag{2}$$

and

$$\eta(\xi)=-1,\quad \phi\xi=0,\quad \eta(\phi X)=0,\quad g(X,\xi)=\eta(X),\ rank\ \phi=n-1, \tag{3}$$

for arbitrary vector fields X, Y on M_n, is called Lorentzian almost paracontact manifold [8].

In a Lorentzian almost paracontact manifold, for any vector fields X, Y on M_n, we have

$$\Phi(X,Y)=\Phi(Y,X), \tag{4}$$

where $\Phi(X,Y)=g(X,\phi Y)$ is a symmetric (0, 2) tensor field.

A Lorentzian almost paracontact manifold M_n is called Lorentzian para-Kenmotsu manifold if [1]

$$(\nabla_X\phi)Y=-g(\phi X,Y)\xi-\eta(Y)\phi X, \tag{5}$$

for all X, $Y\in\chi(M_n)$, where $\chi(M_n)$ is the set of all differentiable vector fields on M_n and ∇ is known to be the operator of covariant differentiation with respect to the Lorentzian metric g.

In a Lorentzian para-Kenmotsu manifold, the following relations hold good [1]:

$$\nabla_X\xi=-\phi^2X=-X-\eta(X)\xi, \tag{6}$$

$$(\nabla_X\eta)Y=-g(X,Y)-\eta(X)\eta(Y), \tag{7}$$

$$g(R(X,Y)Z,\xi)=\eta(R(X,Y)Z)=g(Y,Z)\eta(X)-g(X,Z)\eta(Y), \tag{8}$$

$$R(\xi,X)Y=g(X,Y)\xi-\eta(Y)X, \tag{9}$$

$$R(X,Y)\xi=\eta(Y)X-\eta(X)Y, \tag{10}$$

$$S(X,\xi)=(n-1)\eta(X) \tag{11}$$

and

$$S(\phi X, \phi Y) = S(X,Y) + (n-1)\eta(X)\eta(Y), \tag{12}$$

for any vector fields X, Y and Z on M_n.

By putting $Z = \xi$ in (1) and on simplification by using (3), (10) and (11), we get

$$\overline{W}(X,Y)\xi = [a + (n-1)b]\Big[1 - \frac{r}{n(n-1)}\Big][\eta(Y)X - \eta(X)Y]. \tag{13}$$

The above expression can be written as:

$$\overline{W}(X,Y)\xi = k[\eta(Y)X - \eta(X)Y], \tag{14}$$

where

$$k = [a + (n-1)b]\Big[1 - \frac{r}{n(n-1)}\Big].$$

3. Pseudo-projective semisymmetric LP-Kenmotsu manifolds

Let us consider an LP-Kenmotsu manifold (M_n, g) satisfying the condition [3, 4]

$$R(X,Y)\cdot \overline{W} = 0, \tag{15}$$

for any arbitrary vector fields X, Y on M_n. Then the manifold M_n is called as the pseudo-projective semisymmetric LP-Kenmotsu manifold (or) simply called as $\overline{W}$-semisymmetric LP-Kenmotsu manifold.

On the other hand, we have

$$\begin{aligned}(R(X,Y)\cdot \overline{W})(U,V)Z =& R(X,Y)\overline{W}(U,V)Z - \overline{W}(R(X,Y)U,V)Z \\ & - \overline{W}(U,R(X,Y)V)Z - \overline{W}(U,V)R(X,Y)Z,\end{aligned} \tag{16}$$

for any vector fields X, Y, Z, U, $V \in \chi(M_n)$. Then, from (15) and (16), we have

$$\begin{aligned}& g(R(\xi,Y)\overline{W}(U,V)Z,\xi) - g(\overline{W}(R(\xi,Y)U,V)Z,\xi) \\ & - g(\overline{W}(U,R(\xi,Y)V)Z,\xi) - g(\overline{W}(U,V)(R(\xi,Y)Z,\xi)) = 0.\end{aligned} \tag{17}$$

By virtue of (8) and (9), we get each term of the above expression as:

$$\begin{aligned} &(a)\ g(R(\xi,Y)\overline{W}(U,V)Z,\xi) = -\overline{W}'(U,V,Z,Y) - \eta(Y)\eta(\overline{W}(U,V)Z), \\ &(b)\ g(\overline{W}(R(\xi,Y)U,V)Z,\xi) = g(Y,U)\eta(\overline{W}(\xi,V)Z) - \eta(U)\eta(\overline{W}(Y,V)Z), \\ &(c)\ g(\overline{W}(U,R(\xi,Y)V)Z,\xi) = \eta(V)\eta(\overline{W}(U,Y)Z) - g(Y,V)\eta(\overline{W}(U,\xi)Z), \\ &(d)\ g(\overline{W}(U,V)(R(\xi,Y)Z,\xi)) = g(Y,Z)\eta(\overline{W}(U,V)\xi) \\ &\qquad\qquad - \eta(Z)\eta(\overline{W}(U,V)Y) = 0,\end{aligned} \tag{18}$$

for arbitrary vector fields $U,\ V,\ Z,\ Y \in \chi(M_n)$, where

$$\overline{W}'(U,V,Z,Y) = g(\overline{W}(U,V)Z,Y).$$

By substituting the above values from (18) in (17), we obtain that

$$\begin{aligned}(19)\qquad & -\overline{W}'(U,V,Z,Y) - \eta(Y)\eta(\overline{W}(U,V)Z) - g(Y,U)\eta(\overline{W}(\xi,V)Z)\\ & + \eta(U)\eta(\overline{W}(Y,V)Z) - g(Y,V)\eta(\overline{W}(U,\xi)Z) + \eta(V)\eta(\overline{W}(U,Y)Z)\\ & - g(Y,Z)\eta(\overline{W}(U,V)\xi) + \eta(Z)\eta(\overline{W}(U,V)Y) = 0.\end{aligned}$$

Clearly it follows from (13) that

$$(20)\qquad \eta(\overline{W}(U,V)\xi) = 0,$$

where $U,\ V \in \chi(M_n)$.

Now, by using (20) in (19), we get

$$\begin{aligned}(21)\qquad & -\overline{W}'(U,V,Z,Y) - \eta(Y)\eta(\overline{W}(U,V)Z) - g(Y,U)\eta(\overline{W}(\xi,V)Z)\\ & + \eta(U)\eta(\overline{W}(Y,V)Z) - g(Y,V)\eta(\overline{W}(U,\xi)Z) + \eta(V)\eta(\overline{W}(U,Y)Z)\\ & + \eta(Z)\eta(\overline{W}(U,V)Y) = 0,\end{aligned}$$

for any vector fields $U,\ V,\ Z,\ Y \in \chi(M_n)$.

Let $\{e_i = 1 : i = 1,2,3,\cdots,n\}$ be an orthonormal basis of the tangent space at any point of the manifold.

By putting $U = Y = e_i$ in (21) we get that

$$\begin{aligned}(22)\qquad & \overline{W}'(e_i,V,Z,e_i) + g(e_i,e_i)\eta(\overline{W}(\xi,V)Z) + \eta(V)\eta(\overline{W}(e_i,\xi)Z)\\ & - \eta(V)\eta(\overline{W}(e_i,e_i)Z) - \eta(Z)\eta(\overline{W}(e_i,V)e_i) = 0.\end{aligned}$$

On further simplification of the above equation, we have

$$(23)\qquad \overline{W}'(e_i,V,Z,e_i) + g(e_i,e_i)\eta(\overline{W}(\xi,V)Z) - \eta(Z)\eta(\overline{W}(e_i,V)e_i) = 0,$$

as $\eta(\overline{W}(e_i,e_i)Z) = 0$.

Now, by taking summation over $1 \leq i \leq n$ in (23), we get

$$(24)\quad \sum_{i=1}^{n} \epsilon_i \overline{W}'(e_i,V,Z,e_i) + (n-1)\eta(\overline{W}(\xi,V)Z) - \eta(Z)\sum_{i=1}^{n} \epsilon_i \eta(\overline{W}(e_i,V)e_i) = 0,$$

where $\epsilon_i = g(e_i,e_i)$.

Now, by using (9) and (1), the terms of the above expression are obtained as:

$$\begin{aligned}&(a)\ \sum_{i=1}^{n}\epsilon_i\overline{W}'(e_i,V,Z,e_i)=[a+(n-1)b]S(V,Z)\\&\qquad\qquad -\frac{r}{n}\Big[a+(n-1)b\Big]g(V,Z),\\&(b)\ \eta(\overline{W}(\xi,V)Z)=\Big[-a+\frac{r}{n}\Big(\frac{a}{n-1}+b\Big)\Big][g(V,Z)+\eta(V)\eta(Z)\\&\qquad\qquad -bS(V,Z)-b(n-1)\eta(V)\eta(Z)],\\&(c)\ \sum_{i=1}^{n}\epsilon_i\eta(\overline{W}(e_i,V)e_i)=[a-b]\Big[\frac{r}{n}-(n-1)\Big]\eta(V).\end{aligned}\tag{25}$$

By substituting the above values in (24), we get

$$aS(V,Z)-a(n-1)g(V,Z)+b[r-n(n-1)]\eta(V)\eta(Z)=0,\tag{26}$$

which can be written as

$$S(V,Z)=(n-1)g(V,Z)-\frac{b}{a}[r-n(n-1)]\eta(V)\eta(Z),\tag{27}$$

for any vector fields V and Z on M_n. Thus, we have the following assertion.

Theorem 3.1. *An LP-Kenmotsu manifold* (M_n,g) *satisfying the condition* $R(X,Y)\cdot\overline{W}=0$ *is an* η*-Einstein manifold.*

Further, by taking $Z=\xi$ in (27) and on simplification by using (3) and (11), we obtain that $r=n(n-1)$ and this leads to the following assertion.

Corollary 3.1. *An LP-Kenmotsu manifold* (M_n,g) *satisfying the condition* $R(X,Y)\cdot\overline{W}=0$ *is of constant scalar curvature* $n(n-1)$.

Now, let us consider a special case in which the LP-Kenmotsu manifold admitting $R(X,Y)\cdot\overline{W}=0$ is not an Einstein manifold. Then, from (27) it follows that $r\neq n(n-1)$; otherwise it is an Einstein manifold.

On differentiating (27) covariantly along X and then on using (7), we get

$$\begin{aligned}(\nabla_XS)(V,Z)=&-\frac{b}{a}dr(X)\eta(V)\eta(Z)\\&-\frac{b}{a}[r-n(n-1)][g(X,V)\eta(Z)+g(X,Z)\eta(V)+2\eta(X)\eta(V)\eta(Z)].\end{aligned}\tag{28}$$

By putting $X=Z=e_i$ in the above expression and on taking summation for $1\leq i\leq n$, we obtain that

$$dr(V)=\frac{b}{a}\Big[dr(\xi)-[r-n(n-1)\Psi]\Big]\eta(V),\tag{29}$$

where $\Psi=1+\sum_{i=1}^{n}\epsilon_ig(e_i,e_i)$.

On replacing V with ξ in the above expression (29), we get that

$$dr(\xi) = \frac{b}{a+b}[r - n(n-1)]\Psi. \tag{30}$$

From (29) and (30) we obtain

$$dr(V) = \frac{b}{a+b}[n(n-1) - r]\Psi\eta(V). \tag{31}$$

If r is constant then (31) yields either $r = n(n-1)$ or $\Psi = 0$. But as $r \neq n(n-1)$, we must have $\Psi = 0$, which means that the vector field ξ is harmonic.

Again, if $\Psi = 0$, then from (31) it follows that r is constant. Thus we can state the following:

Theorem 3.2. *If the LP-Kenmotsu manifold admitting the condition $R(X,Y) \cdot \overline{W} = 0$ is not an Einstein manifold, then the scalar curvature of the manifold is constant if and only if the time like vector field ξ is harmonic.*

4. Irrotational pseudo-projective curvature tensor in LP-Kenmotsu manifolds

Definition 4.1. *The rotation (curl) of pseudo-projective curvature tensor $\overline{W}$ on a Riemannian manifold is given by [2]*

$$\begin{aligned} Rot\ \overline{W} &= (\nabla_U \overline{W})(X,Y)Z + (\nabla_X \overline{W})(U,Y)Z \\ &\quad + (\nabla_Y \overline{W})(X,U)Z - (\nabla_Z \overline{W})(X,Y)U, \end{aligned} \tag{32}$$

for all X, Y, U, $Z \in \chi(M_n)$.

In virtue of Bianchi's second identity, we have

$$(\nabla_U \overline{W})(X,Y)Z + (\nabla_X \overline{W})(U,Y)Z + (\nabla_Y \overline{W})(X,U)Z = 0. \tag{33}$$

Therefore, (32) reduces to

$$Rot\ \overline{W} = -(\nabla_Z \overline{W})(X,Y)U, \tag{34}$$

for all X, Y, U, $Z \in \chi(M_n)$.

Now, let us suppose that the pseudo-projective curvature tensor is irrotational. Then *curl* $\overline{W} = 0$ and so by (34) we get

$$-(\nabla_Z \overline{W})(X,Y)U = 0,$$

which implies the following:

$$\nabla_Z(\overline{W}(X,Y)U) = \overline{W}(\nabla_Z X, Y) + \overline{W}(X, \nabla_Z Y)U + \overline{W}(X,Y)\nabla_Z U \tag{35}$$

for any arbitrary vector fields X, Y, U, $Z \in \chi(M_n)$.

By replacing $U = \xi$ in (35), we have

$$\nabla_Z(\overline{W}(X,Y)\xi) = \overline{W}(\nabla_Z X,Y)\xi + \overline{W}(X,\nabla_Z Y)\xi + \overline{W}(X,Y)\nabla_Z \xi. \tag{36}$$

Using (14) in (36) and on simplifying by making use of (6), we get

$$\overline{W}(X,Y)\phi^2 Z = -k[g(Z,\phi Y)X - g(Z,\phi X)Y], \tag{37}$$

which on further simplification by using (2) and (14), we get

$$\overline{W}(X,Y)Z = k[g(Y,Z)X - g(X,Z)Y], \tag{38}$$

for any vector fields X, Y, $Z \in \chi(M_n)$. Thus, we can state:

Lemma 4.1. *If the pseudo-projective curvature tensor $\overline{W}$ in an LP-Kenmotsu manifold is irrotational, then $\overline{W}$ is given by the expression* (38).

Further, in view of (1) and (38) we get

$$\begin{aligned} aR(X,Y)W = {} & [a+(n-1)b][g(Y,W)X - g(X,W)Y] \\ & - b[S(Y,W)X - S(X,W)Y], \end{aligned} \tag{39}$$

where X, Y, $Z \in \chi(M_n)$.

Let $\{e_i = 1 : i = 1,2,3,\cdots,n\}$ be an orthonormal basis of the tangent space at any point of the manifold. Then, by putting $Y = Z = e_i$ in (39), we get that

$$\begin{aligned} aR(X,e_i)W = {} & [a+(n-1)b][\eta(W)X - g(X,W)e_i] \\ & - b[S(e_i,W)X - S(X,W)e_i]. \end{aligned} \tag{40}$$

By taking the inner product of (40) with W and on taking summation over $1 \leq i \leq n$ we get

$$S(X,W) = (n-1)g(X,W). \tag{41}$$

This proves that the manifold is Einstein.

Finally, by taking $X = W = e_i$ in (41) and on taking summation from 1 to n we obtain

$$r = n(n-1). \tag{42}$$

Hence we can state that:

Theorem 4.1. *If the pseudo-projective curvature tensor in an LP-Kenmotsu manifold is irrotational, then the manifold is Einstein and the scalar curvature under such conditions is given by $n(n-1)$.*

5. Conservative pseudo-projective curvature tensor in LP-Kenmotsu manifolds

On differentiating (1) with respect to U, we get

$$(\nabla_U \overline{W})(X,Y)Z = a(\nabla_U R)(X,Y)Z + b[(\nabla_U S)(Y,Z)X - (\nabla_U S)(X,Z)Y] - \frac{dr(U)}{n}\Big[\frac{a}{n-1} + b\Big][g(Y,Z)X - g(X,Z)Y], \tag{43}$$

which on contraction with respect to U becomes

$$(\text{div}\,\overline{W})(X,Y)Z = a[(\text{div}\,R)(X,Y)Z] + b[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] - \frac{1}{n(n-1)}[a+(n-1)b][g(Y,Z)dr(X) - g(X,Z)dr(Y)], \tag{44}$$

for arbitrary vector fields X, Y, Z, $U \in \chi(M_n)$.

Let us suppose that the pseudo-projective curvature tensor is conservative, i. e., div $\overline{W} = 0$. Then, (44) can be written as:

$$(a+b)[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] = \frac{1}{n(n-1)}[a+(n-1)b][g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \tag{45}$$

By putting $X = \xi$ in (45), we have

$$(a+b)[(\nabla_\xi S)(Y,Z) - (\nabla_Y S)(\xi,Z)] = \frac{1}{n(n-1)}[a+(n-1)b][g(Y,Z)dr(\xi) - g(\xi,Z)dr(Y)]. \tag{46}$$

On the other hand, since ξ is a Killing vector and the scalar curvature r remains invariant, we have $dr(\xi) = 0$.

Also, we have

$$(\nabla_\xi S)(Y,Z) = \xi S(Y,Z) - S(\nabla_\xi Y, Z) - S(Y, \nabla_\xi Z),$$

and

$$(\nabla_Y S)(\xi,Z) = \nabla_Y S(\xi,Z) - S(\nabla_Y \xi, Z) - S(\xi, \nabla_Y Z),$$

for any vector fields Y, $Z \in \chi(M_n)$.

By virtue of the above, the relation (46) becomes

$$(a+b)[-\nabla_Y(S(\xi,Z)) + S(\nabla_Y \xi, Z) + S(\xi, \nabla_Y Z)] = \frac{1}{n(n-1)}[a+(n-1)b][-\eta(Z)dr(Y)], \tag{47}$$

which on using (6) reduces to

$$(a+b)[-\nabla_Y\{(n-1)\eta(Z)\} + S(-\phi^2 Y, Z) + (n-1)\eta(\nabla_Y Z)] = \frac{1}{n(n-1)}[a+(n-1)b][-\eta(Z)dr(Y)], \tag{48}$$

and further it is simplified to

$$\begin{aligned}&(a+b)[-(n-1)\nabla_Y\{\eta(Z)\}-S(\phi Y,\phi Z)+(n-1)\eta(\nabla_Y Z)]\\ &=\frac{1}{n(n-1)}[a+(n-1)b][-\eta(Z)dr(Y)],\end{aligned} \tag{49}$$

for arbitrary vector fields Y, $Z \in \chi(M_n)$.

By putting $Z=\phi Z$ in (49), we get

$$(a+b)[-S(\phi Y,\phi^2 Z)+(n-1)\eta(\nabla_Y(\phi Z))]=0. \tag{50}$$

If $a+b\neq 0$, then (50) becomes

$$S(\phi Y,Z)=(n-1)g(\phi Y,Z). \tag{51}$$

By putting $Z=\phi Z$ in (51), we get

$$S(\phi Y,\phi Z)=(n-1)g(\phi Y,\phi Z), \tag{52}$$

and this implies that

$$S(Y,Z)=(n-1)g(Y,Z), \tag{53}$$

which on contracting gives

$$r=n(n-1),\ \text{where} \tag{54}$$

$$r=\sum_{i=1}^{3}\epsilon_i\ S(e_i,e_i)\ \text{and}\ \epsilon_i=g(e_i,e_i),\ \text{which is constant.} \tag{55}$$

So, one can state that:

Theorem 5.1. *An LP-Kenmotsu manifold admitting a conservative pseudo-projective curvature tensor is an Einstein manifold and it is of constant scalar curvature.*

6. Example

Example 6.1. We consider a 3-dimensional manifold $M_3=\{(x,y,z)\in R^3\}$, where (x,y,z) are the standard coordinates in R^3. Let e_1, e_2 and e_3 be the vector fields on M_3 given by

$$e_1=x\frac{\partial}{\partial x}=\xi,\quad e_2=x\frac{\partial}{\partial y},\quad e_3=x\frac{\partial}{\partial z}. \tag{56}$$

Clearly, $\{e_1,e_2,e_3\}$ is a set of linearly independent vectors for each point of M_3 and hence form a basis of $\chi(M_3)$.

The Lorentzian metric $g(X, Y)$ is defined by:

$$g(e_i, e_j) = \begin{cases} -1, & \text{if } i = j = 1 \\ 1, & \text{if } i = j = 2 \text{ or } 3 \\ 0, & \text{if } i \neq j;\ i,\ j = 1, 2, 3. \end{cases}$$

Let η be the 1-form defined by:

$$\eta(Z) = g(Z, e_1), \text{for any } Z \in \chi(M_3).$$

Let ϕ be a (1, 1)-tensor field on M_3 defined by:

$$\phi(e_1) = 0, \phi(e_2) = -e_2, \phi(e_3) = -e_3 \text{ and } \phi^2(e_1) = 0, \phi^2(e_2) = e_2, \phi^2(e_3) = e_3.$$

The linearity of ϕ and $g(X, Y)$ yields that

$$\eta(e_1) = -1,\ \phi^2(Z) = Z + \eta(Z)e_1 \text{ and } g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y),$$

for any vector fields $X,\ Y,\ Z \in \chi(M_3)$. Thus, for $e_1 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian almost paracontact structure on M_3.

Let ∇ be the Levi-Civita connection with respect to the Lorentzian metric g. Then, we have [14]

$$[e_1, e_2] = e_2, [e_1, e_3] = e_3, [e_2, e_3] = 0.$$

The Koszul's formula is defined by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(Z, X) - Zg(X, Y) \\ &\quad - g(X, [Y, Z]) - g(Y, [X, Z]) + g(Z, [X, Y]). \end{aligned} \tag{57}$$

By using the above Koszul's formula and on taking $e_1 = \xi$, we get the following [14]:

$$\begin{aligned} &\nabla_{e_1} e_1 = 0, \nabla_{e_1} e_2 = 0, \nabla_{e_1} e_3 = 0, \\ &\nabla_{e_2} e_1 = -e_2, \nabla_{e_2} e_2 = -e_1, \nabla_{e_2} e_3 = 0, \\ &\nabla_{e_3} e_1 = -e_3, \nabla_{e_3} e_2 = 0, \nabla_{e_3} e_3 = -e_1. \end{aligned} \tag{58}$$

From the above calculations, we see that the manifold under consideration satisfies all the properties of Lorentzian para-Kenmotsu manifold i.e., $\nabla_X \xi = -\phi^2 X = -X - \eta(X)\xi$ and $(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X$, for all $e_1 = \xi$. Thus, the manifold M_3 under consideration with the structure (ϕ, ξ, η, g) is a 3-dimensional Lorentzian para-Kenmotsu manifold [14].

It is known that

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{59}$$

Then, by using (58) and (59), the non-vanishing components of the curvature tensor are obtained as [14]:

$$\begin{aligned} &R(e_1, e_2)e_1 = e_2, R(e_1, e_2)e_2 = e_1, R(e_1, e_3)e_1 = e_3,\\ &R(e_1, e_3)e_3 = e_1, R(e_2, e_3)e_2 = -e_3, R(e_2, e_3)e_3 = e_2. \end{aligned} \tag{60}$$

With the help of above expressions of the curvature tensors, it follows that

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y. \tag{61}$$

This proves that the 3-dimensional manifold M_3 under consideration is an LP-Kenmotsu manifold and it admits a pseudo-projective curvature tensor.

Let X, Y and Z be any three vector fields given by:

$$X = a_1e_1 + a_2e_2 + a_3e_3, Y = b_1e_1 + b_2e_2 + b_3e_3, Z = c_1e_1 + c_2e_2 + c_3e_3; \tag{62}$$

where a_i, b_i, c_i are all non-zero real numbers, for all $i = 1, 2, 3$.

By putting $Z = \xi = e_1$ in (61) and on using (62), we get that

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y = a_1b_2e_2 + a_1b_3e_3 - a_2b_1e_2 - a_3b_1e_3.$$

Further, in view of (61) and (62), we get

$$\begin{aligned} R(X, Y)Z = g(Y, Z)X - g(X, Z)Y &= (c_1e_2 + c_2e_1)(a_1b_2 - a_2b_1)\\ &+ (a_1b_3 - a_3b_1)(c_1e_3 + c_3e_1) + (a_2b_3 - a_3b_2)(c_3e_2 - c_2e_3) \end{aligned}$$

and hence from (1) we have

$$\begin{aligned} \overline{W}(X, Y)Z = [a + (n-1)b]\Big[1 - \frac{r}{n(n-1)}\Big](c_1e_2 + c_2e_1)(a_1b_2 - a_2b_1)\\ + (a_1b_3 - a_3b_1)(c_1e_3 + c_3e_1) + (a_2b_3 - a_3b_2)(c_3e_2 - c_2e_3), \end{aligned} \tag{63}$$

and

$$\begin{aligned} &\overline{W}(X, Y)\xi\\ &= [a + (n-1)b]\Big[1 - \frac{r}{n(n-1)}\Big](a_1b_2e_2 + a_1b_3e_3 - a_2b_1e_2 - a_3b_1e_3). \end{aligned} \tag{64}$$

Hence, we can say that $\overline{W}(X, Y)Z = 0$ (or) $\overline{W}(X, Y)\xi = 0$, only if $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$.

This proves that the manifold M_3 under consideration is an LP-Kenmotsu manifold and it admits a flat pseudo-projective curvature tensor, provided the above condition is satisfied.

Further, by using (60), we obtain the Ricci tensors and scalar curvatures as follows: $S(e_1, e_1) = -2$, $S(e_2, e_2) = 2$, $S(e_3, e_3) = 2$ and $r = 6$, where

$$S(X, Y) = \sum_{i=1}^{3} \epsilon_i \, g(R(e_i, X)Y, e_i),$$

$$r = \sum_{i=1}^{3} \epsilon_i \, S(e_i, e_i) \text{ and } \epsilon_i = g(e_i, e_i).$$

The above arguments verifies the results discussed in sections 4 and 5.

7. Conclusions

The present work explores the geometrical significance of a new class of Lorentzian paracontact metric manifolds namely Lorentzian para-Kenmotsu manifolds whenever a pseudo-projective curvature tensor admitted by these manifolds exhibits the physical phenomena, *i.e.*, the curvature tensor is either irrotational or conservative.

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