Efficient block approach for the numerical integration of higher-order ordinary differential equations with initial values

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Abstract. This paper considered an innovative procedure to numerically approximate higher–order Initial Value Problems (IVPs) of Ordinary Differential Equations (ODEs). The proposed method is a one-step, self-starting Block integrator method employed to approximate higherorder (Third, Fourth, and Fifth-order) IVPs without reduction to lower order. The method was developed through collocation and interpolation approach. The basic properties of the method such as convergence, consistency, zero stability, order and error constant are well investigated. The accuracy of the method over existing methods are validated by numeral experiments. The method produces more interesting and superior results when compared to some existing numerical methods in terms of accuracy and absolute errors.

Keywords: accuracy, block method, collocation and interpolation, higher-order ODEs, Initial Value Problems (IVPs).

1. Introduction

The pursuit of translating various scientific, engineering, modeling and real life problems to differential equations which can be ordinary differential equations or partial differential equations in nature has given rise to the development of several numerical methods to provide an approximate solution to resulting higher order differential equations coupled with its initial or boundary conditions that may be rigorous to solve or has no analytical solutions. Researchers have suggested methods of solving higher order ODEs directly (Jena and Mohanty 2019, Yap and Ismail 2015, Waeleh et al., 2011, Suleiman et al., 2011). Oftentimes, developing numerical methods has been to convalesce the efficiency

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and convergence of the method. Adeyefa & Kuboye (2020)

$$
\begin{aligned}
y''(x) &= f(x,y,y',y''), y(x_0)=y_0, y'(x_0)=y_1, y''(x_0)=y_2,\\
y'^{v}(x) &= f(x,y,y',y'',y'''), y(x_0)=y_0, y'(x_0)=y_1, y''(x_0)\\
&= y_2, y'''(x_0)=y_3,\\
y^{v}(x) &= f(x,y,y',y'',y''',y'^{v}), y(x_0)=y_0, y'(x_0)=y_1, y'(x_0)\\
&= y_2, y'''(x_0)=y_3, y'^{v}(x_0)=y_4.
\end{aligned} \tag{1}
$$

In literature, researchers have proposed block methods for the direct integration of (1) without necessarily reducing it to system of first order ODEs (Khataybeh et al., 2019, Adoghe et al., 2016, Agboola et al., 2015, Kuboye and Omar 2015a, Kuboye and Omar 2015b, Hussain et al. 2015). These authors developed different method to handle various higher-order of ODEs with focus only on p^{th} order of equation. Adeyefa & Kuboye (2020), developed a method that is capable of handling two different orders of ODEs using the block approach. The developed method was capable of solving second and third-order ODEs. Our interest in this work is to develop a method that will be capable of handling three different systems of p^{th}, $(p+1)^{th}$ and $(p+2)^{th}$ orders IVPs for direct solution of ODEs where $p=3$.

2. The method

We consider the formulation of our proposed method for p^{th}, $(p+1)^{th}$ and

$$
y(x) = \sum_{j=0}^{k+9} a_j x^j \tag{2}
$$

$(p+2)^{th}$ by adopting a power series of the form: for $k=1$, as the approximate solution of (1), interpolating (2) at $x = x_{n+\tau}, \tau = \frac{2}{5}, \frac{3}{5}, \frac{4}{5}$, and collocating the third, fourth, and fifth derivative of (2) at $x = x_{n+\varsigma}, \varsigma = 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1, x = x_{n+\varepsilon}, \varepsilon = 0$ and $x = x_{n+\varpi}, \varpi = 1$, respectively. This subsequently resulted into systems of linear equations of the form:

$$
\begin{aligned}
&\sum_{j=0}^{k+9} a_j x^{n+\tau} = y_{n+\tau}, \quad \sum_{j=2}^{k+9} j(j-1)(j-2) a_j x_{n+\varsigma}^{j-2} = f_{n+\varsigma},\\
&\sum_{j=3}^{k+9} j(j-1)(j-2)(j-3) a_j x_{n+\varepsilon}^{j-5} = g_{n+\varepsilon},\\
&\sum_{j=4}^{k+9} j(j-1)(j-2)(j-3)(j-4) a_j x_{n+\omega}^{j-5} = m_{n+\omega},
\end{aligned} \tag{3}
$$

where f, g and m are the third, fourth, and fifth derivatives of (2) respectively. Gaussian's elimination method is applied to find the values of as' in (3) and then substituted into (2) to produce a continuous implicit method:

$$\text{(4)}\quad \alpha_{\frac{2}{5}}(t)y_{n\frac{2}{5}} + \alpha\frac{3}{5}(t)y_{n+\frac{3}{5}} + \alpha_{\frac{4}{5}}(t)y_{n+\frac{4}{5}} = h^3(\beta_0(t)f_n + \beta_{\frac{1}{5}}(t)f_{n+\frac{1}{5}} + \beta_{\frac{2}{5}}(t)f_{n+\frac{2}{5}}$$
$$+\beta_{\frac{3}{5}}(t)f_{n+\frac{3}{5}} + \beta_{\frac{4}{5}}(t)f_{n+\frac{4}{5}} + \beta_1(t)f_{n+1}) + h^4(\gamma_0(t)g_n) + h^5(\omega_{\frac{1}{5}}(t)m_{n+\frac{1}{5}}),$$

for $t = \frac{x-x_n}{h}$, $\begin{pmatrix}\alpha_{\frac{2}{5}}\\ \alpha_{\frac{3}{5}}\\ \alpha_{\frac{4}{5}}\end{pmatrix}(t) = [\Phi]\begin{pmatrix}t^0\\ t^1\\ t^2\end{pmatrix}$,

$$\begin{pmatrix}\beta_0\\ \beta_{\frac{1}{5}}\\ \beta_{\frac{2}{5}}\\ \beta_{\frac{3}{5}}\\ \beta_{\frac{4}{5}}\\ \beta_1\end{pmatrix}(t){=}h^3[\gamma]\begin{pmatrix}t^0\\ t^1\\ t^2\\ t^3\\ t^4\\ t^5\\ t^6\\ t^7\\ t^8\\ t^9\\ t^{10}\end{pmatrix}(\gamma_0)(t){=}h^4[\Psi]\begin{pmatrix}t^0\\ t^1\\ t^2\\ t^3\\ t^4\\ t^5\\ t^6\\ t^7\\ t^8\\ t^9\\ t^{10}\end{pmatrix}, \left(\omega_{\frac{1}{5}}\right)(t){=}h^5[\Pi]\begin{pmatrix}t^0\\ t^1\\ t^2\\ t^3\\ t^4\\ t^5\\ t^6\\ t^7\\ t^8\\ t^9\\ t^{10}\end{pmatrix}$$

where

$$\gamma = \begin{pmatrix}
\frac{499699}{37800000} & -\frac{8987}{315000} & -\frac{2099}{630000} & -\frac{457}{33750} & \frac{559}{2520000} & \frac{61}{1575000}\\
-\frac{20688119}{453600000} & \frac{99251}{756000} & \frac{49547}{1512000} & \frac{31447}{567000} & \frac{4327}{6048000} & \frac{383}{2700000}\\
\frac{90163}{90163} & -\frac{15367}{43200} & \frac{4397}{120960} & -\frac{15917}{226800} & \frac{1067}{345600} & -\frac{107}{302400}\\
\frac{1}{6} & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0\\
-\frac{208879}{8640} & \frac{1075}{24} & -\frac{575}{24} & \frac{425}{108} & -\frac{125}{192} & \frac{7}{120}\\
\frac{924719}{6912} & -\frac{924719}{576} & \frac{78025}{576} & -\frac{19075}{864} & \frac{8375}{2304} & -\frac{187}{576}\\
-\frac{1872425}{6048} & \frac{1170875}{2016} & -\frac{22625}{72} & \frac{152875}{3024} & -\frac{2375}{288} & \frac{1475}{2016}\\
\frac{2194375}{6048} & \frac{3659375}{5376} & \frac{246875}{672} & -\frac{1403125}{24192} & \frac{3125}{336} & -\frac{625}{768}\\
-\frac{3855625}{18144} & \frac{9640625}{24192} & -\frac{1296875}{6048} & \frac{171875}{5184} & -\frac{15625}{3024} & \frac{10625}{24192}\\
\frac{2050625}{41472} & -\frac{640625}{6912} & \frac{171875}{3456} & -\frac{78125}{10368} & \frac{15625}{15625} & -\frac{625}{6912}
\end{pmatrix},$$

$$\Psi = \begin{pmatrix} \frac{281}{450000} \\ -\frac{11941}{5400000} \\ \frac{431}{86400} \\ 0 \\ \frac{1}{24} \\ -\frac{917}{720} \\ \frac{3787}{576} \\ -\frac{1075}{72} \\ \frac{625}{36} \\ -\frac{4375}{432} \\ \frac{8125}{3456} \end{pmatrix}, \Pi = \begin{pmatrix} -\frac{43750}{13} \\ \frac{39}{35000} \\ -\frac{373}{126000} \\ 0 \\ 0 \\ \frac{1}{2} \\ -\frac{137}{48} \\ \frac{375}{56} \\ -\frac{10625}{1344} \\ \frac{3125}{672} \\ -\frac{625}{576} \end{pmatrix}$$

and

$$\Phi = \begin{pmatrix} 3 & -8 & 6 \\ -\frac{25}{2} & 30 & \frac{35}{1} \\ \frac{25}{2} & -25 & \frac{25}{2} \end{pmatrix}.$$

Evaluating (4) at $x = x_{n+\eta}, \eta = 1, \frac{1}{5}, 0$ to produce a discrete scheme of the form:

$$\begin{pmatrix} y_{n+1} \\ y_{n+\frac{1}{5}} \\ y_n \end{pmatrix} - \begin{pmatrix} 3 & 1 & 3 \\ -3 & -3 & -8 \\ 1 & 3 & 6 \end{pmatrix} \begin{pmatrix} y_{n+\frac{4}{5}} \\ y_{n+\frac{3}{5}} \\ y_{n+\frac{2}{5}} \end{pmatrix}$$

$$= h^3[A] \begin{pmatrix} f_{n+1} \\ f_{n+\frac{4}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{2}{5}} \\ f_n \end{pmatrix} + h^4 \begin{pmatrix} \frac{301}{900000} \\ \frac{31}{100000} \\ \frac{281}{450000} \end{pmatrix} (g_n) + h^5 \begin{pmatrix} -\frac{167}{1050000} \\ -\frac{31}{210000} \\ -\frac{13}{43750} \end{pmatrix} \left(m_{n+\frac{1}{5}}\right) \tag{5}$$

for

$$A = \begin{pmatrix} \frac{3546}{75600000} & -\frac{482}{25200000} & -\frac{1464}{37800000} \\ \frac{293745}{75600000} & \frac{3585}{25200000} & \frac{8385}{37800000} \\ \frac{259420}{75600000} & -\frac{123340}{25200000} & -\frac{511840}{37800000} \\ \frac{509220}{75600000} & \frac{62660}{25200000} & -\frac{125940}{37800000} \\ -\frac{999030}{75600000} & -\frac{310290}{25200000} & -\frac{107844}{37800000} \\ \frac{537899}{75600000} & \frac{166267}{25200000} & \frac{499699}{37800000} \end{pmatrix}.$$

The first two derivative of equation (4) yields equations (6) and (7) respectively.

$$\alpha'_{\frac{2}{5}}(t)y_{n+\frac{2}{5}}+\alpha'_{\frac{3}{5}}(t)y_{n+\frac{3}{5}}+\alpha'_{\frac{4}{5}}(t)=h^2(\beta'_0(t)f_n+\beta'_{\frac{1}{5}}(t)f_{n+\frac{1}{5}}+\beta'_{\frac{2}{5}}(t)f_{\frac{2}{5}}$$
$$+\beta'_{\frac{3}{5}}(t)f_{n+\frac{3}{5}}+\beta'_{\frac{4}{5}}(t)f_{n+\frac{4}{5}}+\beta'_1 f_{n+1})+h^3(\gamma_0(t)g_n)+h^4(\omega'_{\frac{1}{5}}(t)m_{n+\frac{1}{5}}) \tag{6}$$

$$\alpha''_{\frac{2}{5}}(t)y_{n+\frac{2}{5}}+\alpha''_{\frac{3}{5}}(t)y_{n+\frac{3}{5}}+\alpha''_{\frac{4}{5}}(t)=h(\beta''_0(t)f_n+\beta''_{\frac{1}{5}}(t)f_{n+\frac{1}{5}}+\beta''_{\frac{2}{5}}(t)f_{\frac{2}{5}}$$
$$+\beta''_{\frac{3}{5}}(t)f_{n+\frac{3}{5}}+\beta''_{\frac{4}{5}}(t)f_{n+\frac{4}{5}}+\beta''_1 f_{n+1})+h^2(\gamma_0(t)g_n)+h^3(\omega''_{\frac{1}{5}}(t)m_{n+\frac{1}{5}}) \tag{7}$$

Evaluating equations (6) and (7) at $x=x_{n+\theta},\theta=1,\frac{4}{5},\frac{3}{5},\frac{2}{5},\frac{1}{5},0$ and $x=x_{n+\psi},\psi=1,\frac{4}{5},\frac{3}{5},\frac{2}{5},\frac{1}{5},0$ to produce the derivatives as:

$$\begin{pmatrix} y'_{n+1}\\ y'_{n+\frac{4}{5}}\\ y'_{n+\frac{3}{5}}\\ y'_{n+\frac{2}{5}}\\ y'_{n+\frac{1}{5}}\\ y'_n\end{pmatrix}-\frac{1}{h}[B]\begin{pmatrix} y_{n+\frac{4}{5}}\\ y_{n+\frac{3}{5}}\\ y_{n+\frac{2}{5}}\end{pmatrix}=h^2[C]\begin{pmatrix} f_{n+1}\\ f_{n+\frac{4}{5}}\\ f_{n+\frac{3}{5}}\\ f_{n+\frac{3}{5}}\\ f_{n+\frac{2}{5}}\\ f_n\end{pmatrix}$$
$$+h^3\begin{pmatrix}-\frac{3040044}{453600000}\\ -\frac{32768}{151200000}\\ -\frac{104412}{453600000}\\ -\frac{332556}{453600000}\\ -\frac{238980}{151200000}\\ -\frac{1003044}{453600000}\end{pmatrix}(g_n)+h^4\begin{pmatrix}\frac{1419840}{453600000}\\ -\frac{154464}{151200000}\\ \frac{48816}{453600000}\\ \frac{159264}{453600000}\\ \frac{112224}{151200000}\\ \frac{505440}{453600000}\end{pmatrix}\left(m_{n+\frac{1}{5}}\right) \tag{8}$$

where

$$B=\begin{pmatrix}\frac{5670000000}{453600000} & \frac{113400000}{151200000} & \frac{1134000000}{453600000} & -\frac{1134000000}{453600000} & -\frac{1134000000}{151200000} & -\frac{5670000000}{453600000}\\ -\frac{9072000000}{453600000} & -\frac{1512000000}{151200000} & 0 & \frac{4536000000}{453600000} & \frac{3024000000}{151200000} & \frac{13608000000}{453600000}\\ \frac{3402000000}{453600000} & \frac{37800000}{151200000} & -\frac{1134000000}{453600000} & -\frac{3402000000}{453600000} & -\frac{189000000}{151200000} & -\frac{7938000000}{453600000}\end{pmatrix}$$

$$C=\begin{pmatrix}
\frac{1280844}{453600000} & -\frac{31384}{151200000} & \frac{11712}{453600000} & \frac{1965}{453600000} & \frac{1478}{151200000} & \frac{64344}{453600000}\\
\frac{17710725}{453600000} & \frac{613125}{151200000} & -\frac{229575}{453600000} & -\frac{135375}{453600000} & \frac{85425}{151200000} & -\frac{324525}{453600000}\\
\frac{2211460}{453600000} & \frac{692000}{151200000} & -\frac{235120}{453600000} & \frac{528400}{453600000} & -\frac{5128200}{151200000} & \frac{25157600}{453600000}\\
-\frac{63855900}{453600000} & \frac{6749500}{151200000} & -\frac{2390700}{453600000} & -\frac{5080500}{453600000} & \frac{118270}{151200000} & \frac{14864100}{453600000}\\
\frac{120281100}{453600000} & \frac{12957000}{151200000} & \frac{4144800}{453600000} & \frac{13047000}{453600000} & \frac{9920700}{151200000} & \frac{59550600}{453600000}\\
-\frac{64267369}{453600000} & \frac{694975}{151200000} & -\frac{220903}{453600000} & -\frac{7086781}{453600000} & -\frac{5072955}{151200000} & -\frac{20688119}{453600000}
\end{pmatrix}$$

$$\begin{pmatrix} y''_{n+1}\\ y''_{n+\frac{4}{5}}\\ y''_{n+\frac{3}{5}}\\ y''_{n+\frac{2}{5}}\\ y''_{n+\frac{1}{5}}\\ y''_{n}\end{pmatrix} - \frac{1}{h^2}[D]\begin{pmatrix} y_{n+\frac{4}{5}}\\ y_{n+\frac{3}{5}}\\ y_{n+\frac{2}{5}}\end{pmatrix} = h[E]\begin{pmatrix} f_{n+1}\\ f_{n+\frac{4}{5}}\\ f_{n+\frac{3}{5}}\\ f_{n+\frac{3}{5}}\\ f_{n+\frac{2}{5}}\\ f_{n}\end{pmatrix}$$

$$(9)\qquad + h^2\begin{pmatrix} -\frac{146874}{9072000}\\ \frac{697116}{18144000}\\ -\frac{89796}{9072000}\\ \frac{253596}{18144000}\\ -\frac{49476}{9072000}\\ \frac{181020}{18144000}\end{pmatrix}(g_n) + h^3\begin{pmatrix} \frac{688788}{9072000}\\ -\frac{32860}{18144000}\\ \frac{42516}{9072000}\\ -\frac{12124}{18144000}\\ \frac{25236}{9072000}\\ -\frac{107424}{18144000}\end{pmatrix}\begin{pmatrix} m_{n+\frac{1}{5}}\end{pmatrix}$$

where

$$D=\begin{pmatrix}
\frac{226800000}{9072000} & \frac{453600000}{18144000} & \frac{226800000}{9072000} & \frac{453600000}{18144000} & \frac{226800000}{9072000} & \frac{453600000}{18144000}\\
-\frac{453600000}{9072000} & -\frac{907200000}{18144000} & -\frac{453600000}{9072000} & -\frac{907200000}{18144000} & -\frac{453600000}{9072000} & -\frac{907200000}{18144000}\\
\frac{226800000}{9072000} & \frac{453600000}{18144000} & \frac{226800000}{9072000} & \frac{453600000}{18144000} & \frac{226800000}{9072000} & \frac{453600000}{18144000}
\end{pmatrix}$$

$$E=\begin{pmatrix}
\frac{604455}{9072000} & -\frac{68136}{18144000} & \frac{7431}{9072000} & -\frac{16296}{18144000} & \frac{3111}{9072000} & -\frac{12840}{18144000}\\
\frac{1954455}{9072000} & \frac{1701795}{18144000} & -\frac{106185}{9072000} & \frac{150915}{18144000} & -\frac{50025}{9072000} & \frac{112035}{18144000}\\
\frac{4632890}{9072000} & \frac{463840}{18144000} & \frac{217850}{9072000} & -\frac{2788640}{18144000} & -\frac{635590}{9072000} & -\frac{2546720}{18144000}\\
-\frac{30561900}{9072000} & \frac{14308260}{18144000} & -\frac{1756140}{9072000} & \frac{3577380}{18144000} & -\frac{3190380}{9072000} & \frac{1313700}{18144000}\\
\frac{58092735}{9072000} & -\frac{27561720}{18144000} & \frac{3544095}{9072000} & -\frac{9953400}{18144000} & \frac{1310655}{9072000} & -\frac{12908280}{18144000}\\
-\frac{31093835}{9072000} & \frac{14784761}{18144000} & -\frac{1907051}{9072000} & \frac{5401241}{18144000} & -\frac{1066571}{9072000} & \frac{3155705}{18144000}
\end{pmatrix}.$$

The fusing of equations (5), (8) and (9) in matrix form by matrix inversion gives a block method written explicitly as

$$I\begin{bmatrix} y_{n+\frac{1}{5}} \\ y_{n+\frac{2}{5}} \\ y_{n+\frac{3}{5}} \\ y_{n+\frac{4}{5}} \\ y_{n+1} \end{bmatrix} = L\begin{bmatrix} y_{n-\frac{1}{5}} \\ y_{n-\frac{2}{5}} \\ y_{n-\frac{3}{5}} \\ y_{n-\frac{4}{5}} \\ y_{n} \end{bmatrix} + h(K)\begin{bmatrix} y'_{n-\frac{1}{5}} \\ y'_{n-\frac{2}{5}} \\ y'_{n-\frac{3}{5}} \\ y'_{n-\frac{4}{5}} \\ y'_{n} \end{bmatrix} + h^2(R)\begin{bmatrix} y''_{n-\frac{1}{5}} \\ y''_{n-\frac{2}{5}} \\ y''_{n-\frac{3}{5}} \\ y''_{n-\frac{4}{5}} \\ y''_{n} \end{bmatrix}$$

$$+ h^3(P)\begin{bmatrix} f_{n-\frac{1}{5}} \\ f_{n-\frac{2}{5}} \\ f_{n-\frac{3}{5}} \\ f_{n-\frac{4}{5}} \\ f_{n} \end{bmatrix} + h^3(V)\begin{bmatrix} f_{n+\frac{1}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1} \end{bmatrix} + h^4(T)\begin{bmatrix} g_{n-\frac{1}{5}} \\ g_{n-\frac{2}{5}} \\ g_{n-\frac{3}{5}} \\ g_{n-\frac{4}{5}} \\ g_{n} \end{bmatrix} + h^5(Z)\begin{bmatrix} m_{n-\frac{1}{5}} \\ m_{n-\frac{2}{5}} \\ m_{n-\frac{3}{5}} \\ m_{n-\frac{4}{5}} \\ m_{n} \end{bmatrix} \tag{10}$$

$$I\begin{bmatrix} y'_{n+\frac{1}{5}} \\ y'_{n+\frac{2}{5}} \\ y'_{n+\frac{3}{5}} \\ y'_{n+\frac{4}{5}} \\ y'_{n+1} \end{bmatrix} = L\begin{bmatrix} y'_{n-\frac{1}{5}} \\ y'_{n-\frac{2}{5}} \\ y'_{n-\frac{3}{5}} \\ y'_{n-\frac{4}{5}} \\ y'_{n} \end{bmatrix} + h(K)\begin{bmatrix} y''_{n-\frac{1}{5}} \\ y''_{n-\frac{2}{5}} \\ y''_{n-\frac{3}{5}} \\ y''_{n-\frac{4}{5}} \\ y''_{n} \end{bmatrix} + h^2(\bar{N})\begin{bmatrix} f_{n-\frac{1}{5}} \\ f_{n-\frac{2}{5}} \\ f_{n-\frac{3}{5}} \\ f_{n-\frac{4}{5}} \\ f_{n} \end{bmatrix}$$

$$+ h^2(\bar{W})\begin{bmatrix} f_{n+\frac{1}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1} \end{bmatrix} + h^3(\bar{D})\begin{bmatrix} g_{n-\frac{1}{5}} \\ g_{n-\frac{2}{5}} \\ g_{n-\frac{3}{5}} \\ g_{n-\frac{4}{5}} \\ g_{n} \end{bmatrix} + h^4(\bar{E})\begin{bmatrix} m_{n-\frac{1}{5}} \\ m_{n-\frac{2}{5}} \\ m_{n-\frac{3}{5}} \\ m_{n-\frac{4}{5}} \\ m_{n} \end{bmatrix} \tag{11}$$

$$I\begin{bmatrix} y''_{n+\frac{1}{5}} \\ y''_{n+\frac{2}{5}} \\ y''_{n+\frac{3}{5}} \\ y''_{n+\frac{4}{5}} \\ y'_{n+1} \end{bmatrix} = L\begin{bmatrix} y''_{n-\frac{1}{5}} \\ y''_{n-\frac{2}{5}} \\ y''_{n-\frac{3}{5}} \\ y''_{n-\frac{4}{5}} \\ y''_{n} \end{bmatrix} + h(\hat{S})\begin{bmatrix} f_{n-\frac{1}{5}} \\ f_{n-\frac{2}{5}} \\ f_{n-\frac{3}{5}} \\ f_{n-\frac{4}{5}} \\ f_{n} \end{bmatrix}$$

$$+ h(\hat{Q})\begin{bmatrix} f_{n+\frac{1}{5}} \\ f_{n+\frac{2}{5}} \\ f_{n+\frac{3}{5}} \\ f_{n+\frac{4}{5}} \\ f_{n+1} \end{bmatrix} + h^2(\hat{Z})\begin{bmatrix} g_{n-\frac{1}{5}} \\ g_{n-\frac{2}{5}} \\ g_{n-\frac{3}{5}} \\ g_{n-\frac{4}{5}} \\ g_{n} \end{bmatrix} + h^3(\hat{P})\begin{bmatrix} m_{n-\frac{1}{5}} \\ m_{n-\frac{2}{5}} \\ m_{n-\frac{3}{5}} \\ m_{n-\frac{4}{5}} \\ m_{n} \end{bmatrix} \tag{12}$$

(see Appendix for the values of the terms).

Equations (10) to (12) are applied as the block integrators to provide solutions to problems.

3. Investigation of the basic properties of the block method

We investigate the order, error constant, and zero stability as well as the consistency of the block method as demonstrated below:

3.1 Order of accuracy and Local Truncation Error (LTE)

We describe a linear difference operators related to the block in (10), described as

$$
\begin{aligned}
L[y(x):h] = \sum_{j=0}^{k} [a_j y(x_n + jh) - h^3 \beta_j f(x_n + jh) \\
- h^4 \gamma_j g(x_n + jh) - h^5 \omega_j m(x_n + jh)],
\end{aligned} \tag{13}
$$

where $y(x)$ is continuously differentiable function on $[a, b]$. Expanding (10) by Taylor series, collecting their terms in powers of h produces

$$
L[y(x):h] = C_0 y(x) + C_1 h y'(x) C_2 h y''(x) + \cdots + C_q h^q(x) + (h^{q+1}),
$$

where C_i are constants. Then, if the first C_{p+2} disappears, we have $C_o = C_1 = C_2 = \cdots = C_p = C_{p+2} = 0$ and $C_{p+3} \neq$. Thus, where p is called the order of the method. Then, $C_{p+3} h^{p+3} y^{p+3}(x)$ is the principal Local Truncation Error if the order p and error constant C_{p+3} are known. Lambert (1973). Therefore, for our block method,$p = [8, 8, 8, 8, 8]^T$ and

$$
\begin{aligned}
C_{p+3} = \Bigg[\frac{5323}{413437500000000}, -\frac{26141}{58179453100}, -\frac{17291}{53634509200}, \\
-\frac{22341}{8172494600}, -\frac{23451}{817965200} \Bigg]^T.
\end{aligned}
$$

3.2 Zero stability

The block method (10) is said to be zero stable if no root of the first characteristic polynomial $\rho(r)$ has modulus greater than one and if every root of modulus one has $\rho(r) = det(rA^0 - A^1) = r^m(r-1)$ multiplicity not greater than the order of the differential equation (Lambert 1973). It follows that the integrators are normalized to give the first characteristic polynomial $\rho(r)$ as with

$$A^0 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$A^i = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The roots of $\rho(r) = 0$ satisfy $|r_j| \leq 1$. Hence, the block method is zero-stable.

3.3 Consistency

A block method is said to be consistent if it has order $p \geq 1$. Hence, our block method is consistent, since$p = 8$ A linear multistep method is said to be convergent, if it is consistent and zero stable (see Lambert, 1973). Hence, our method is convergent satisfying the above condition.

4. Numerical experiments

Experiment 1

$y''' = 3\sin x \ y(0) = 1, y'(0) = 0, y''(0) = -2, h = 0.1$. The experiment above which is a special third-order initial value problem was solved by Kuboye & Omar (2015a) using a block method of step-length $k = 7$ order $p = 8$ and $h = 0.1$

The analytical solution:

$y(x) = 3\cos x + \frac{x^2}{2} - 2$ proposed block method is also used to solve the same experiment. The results are displayed in Table 1 and comparison of their absolute errors shown in Fig. 1.

Experiment 2

$y''' = y'(2xy'' + y'), y(0) = 1, y'(0) = \frac{1}{2}, y''(0) = 0, h = 0.01.$

Analytical solution

$y(x) = 1 + \frac{1}{2}\ln\left(\frac{2+x}{2-x}\right)$

Experiment 2 solved by our new block method is a non-linear third-order problem previously solved by Adoghe et al., (2016) using Taylor's series approach with step-length $k = 5$,order $p = 9$ and $h = 0.1$. The numerical results are displayed in Table 2 and their absolute errors shown in Fig. 2.

Experiment 3

$y''' = y'' - y' + y$ $y(0) = 1, y'(0) = 0, y''(0) = -1, h = 0,01;$ $0 \leq x \leq 1$

Analytical solution

$y(x) = \cos x$ Experiment 3 above is a linear third-order initial value problem solved by our proposed block method and Khataybeh et al., (2019). Table 3 displayed the numerical results and the comparison of their absolute errors can be seen in Fig. 3.

Experiment 4

$y'^{v} = (y')^2 - y(y'') - 4x^2 + e^x\left(1 - 4x + x^2\right)$
$y(0) = 1, y'(0) = 1, y''(0) = 3,$
$y'''(0) = 1, h = 0.01; 0 \leq x \leq 1.$

Analytical solution

$y(x) = x^2 + e^x$ The experiment above is a non-linear third-order problem solved by Kuboye & Omar (2015b) where a 6-step bock method of order $p = 7$ with $h = 0.01$ was developed. Our proposed method is used to solve the same experiment and our results are compared together as displayed in Table 4 and the absolute error for both methods can be seen in Fig. 4.

Experiment 5

$y^{v} = -(\cos x + \sin x),$
$y(0) = 1, y'(0) = 1, y''(0) = -2$
$y'''(0) = 1, y'^{v}(0) = 2, h = 0.1; 0 \leq x \leq 1.$

Exact solution

$y(x) = 2x - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cos x - \sin x$

The special fifth order experiment above was solved by Jena & Mohanty (2019) with a step -length of $k = 7$,order $p = 7$ and $h - 0.1$. We also used our proposed method to solve the same experiment with the results generated compared together (see Table 5) and the graphical performance of their absolute error demonstrated in Fig. 5.

5. Numerical results and comparison

Table 1: Comparison of the new block method results with the results of Kuboye & Omar (2015a)

x	Analytical Results	Numerical Results	Error in New Method	Error in Adoghe et al.,2016
0.1	0.99001249583 407730	0.990012495834 079360	2.0600E - 15	1.7430E-14
0.2	0.96019973352 372489	0.96019973352 3687974	2.9051E - 14	1.0824E-13
0.3	0.91100946737 681806	0.91100946737 6764056	5.4004E-14	2.7111E-13
0.4	0.84318298200 865525	0.84318298200 8574846	8.0404E-14	5.0792E-13
0.5	0.75774768567 111815	0.7577476856 71090550	2.7600E-14	8.1645E-13
0.6	0.65600684472 903489	0.65600684472 9236145	2.0125E-13	1.1997E-12
0.7	0.53952656185 346528	0.53952656185 4197778	7.3249E-13	1.6543E-12
0.8	0.41012012804 149626	0.41012012804 3217003	1.7207E-12	1.6746E-10
0.9	0.26982990481 199337	0.2698299048 15340641	3.3472E-12	3.3363E-10
1.0	0.12090691760 441915	0.12090691761 0237425	5.8182E-12	5.0017E-10

Table 2: Comparison of the new block method results with the results of Adoghe et al., (2016)

x	Analytical Results	Numerical Results	Error in New Method	Error in Adoghe et al.,2016
0.1	0.91482908192 435238	0.91482908192 4352375	5.0000E-18	0.9148E +00
0.2	0.85859724183 983017	0.85859724183 9830167	3.0000E-18	0.8585E +00
0.3	0.83014119242 399690	0.83014119242 3996896	4.0000E-18	0.8301E+00
0.4	0.82817530235 872968	0.82817530235 8729681	1.0000E-18	0.8281E +00
0.5	0.8512787292 9987185	0.85127872929 9871852	2.0000E-18	0.8512E +00
0.6	0.89788119960 949103	0.8978811996 09491024	6.0000E-18	0.8978E +00
0.7	0.96624729252 952348	0.9662472925 29523477	3.0000E-18	0.9662E +00
0.8	1.05445907150 753240	1.0544590715 07532390	1.0000E-17	0.1054E +01
0.9	1.16039688884 305034	1.1603968888 43050330	1.0000E-17	0.1160E +01
1.0	1.28171817154 095476	1.281718171 54095475	1.0000E-17	0.1281E +01

Table 3: Comparison of the new block method results with the results of Khataybeh et al., (2019)

x	Analytical Results	Numerical Results	Error in New Method	Khataybeh et al., (2019)
0.1	0.99995000041 6665278	0.99995000041 6665282	4.0000E-18	5.84E-18
0.2	0.99980000666 6577778	0.99980000666 6577783	5.0000E-18	2.85E-17
0.3	0.99955003374 8987516	0.99955003374 8987521	5.0000E-18	6.86E-17
0.4	0.99920010666 0977940	0.99920010666 0977944	4.0000E-18	1.23E-16
0.5	0.99875026039 4966247	0.99875026039 4966250	3.0000E-18	1.83E-16
0.6	0.99820053993 5204166	0.99820053993 5204169	3.0000E-18	2.24E-16
0.7	0.99755100025 3279575	0.99755100025 3279578	3.0000E-18	1.97E-16
0.8	0.99680170630 2619385	0.99680170630 2619386	1.0000E-18	3.01E-17
0.9	0.99595273301 1994253	0.99595273301 1994252	1.0000E-18	8.53E-16
1.0	0.99500416527 8025766	0.995004165278 025764	2.0000E-18	3.45E-15

Table 4: Comparison of the new block method results with the results of Kuboye & Omar (2015b)

x	Analytical Results	Numerical Results	Error in New Method	Error in Kuboye & Omar (2015b)
0.01	0.00020000000000000	0.00020000000000000	0.00000000E+00	2.220446E-16
0.02	0.00040000000000000	0.00040000000000000	0.00000000E+00	0.000000E+00
0.03	0.00060000000000001	0.00060000000000001	0.00000000E+00	0.000000E+00
0.04	0.00080000000000003	0.00080000000000003	0.00000000E+00	0.000000E+00
0.05	0.00100000000000008	0.00100000000000008	0.00000000E+00	0.000000E+00
0.06	0.00120000000000021	0.00120000000000010	1.12757026E-17	0.000000E+00
0.07	0.00140000000000045	0.00140000000000020	2.51534904E-17	1.043610E-14
0.08	0.00160000000000087	0.00160000000000042	4.53196508E-17	3.463896E-14
0.09	0.00180000000000157	0.00180000000000089	6.83047369E-17	7.882583E-14
0.10	0.00200000000000267	0.00200000000000168	9.84455573E-17	1.505462E-13

Table 5: Comparison of the new block method results with the results of Jena & Mohanty (2019)

x	Analytical Results	Numerical Results	Error in New Method	Error in Jena & Mohanty (2019)
0.1	1.00025993240292980	1.00025993240292980	0.0000E+00	4 .8849E-15
0.2	1.00027992160365910	1.00027992160365930	2.2204E-16	9.1038E-15
0.3	1.00029991000450070	1.00029991000450090	2.2204E-16	1.7763E-15
0.4	1.00031989760546210	1.00031989760546240	2.2204E-16	7.7049E-14
0.5	1.00033988440655190	1.00033988440655210	2.2204E-16	9.9920E-15
0.6	1.00035987040777760	1.00035987040777790	2.2204E-16	1.1990E-14
0.7	1.00037985560914700	1.00037985560914740	4.4408E-16	1.9095E-14
0.8	1.00039984001066880	1.00039984001066950	6.6613E-16	1.11022E-15
0.9	1.00041982361235050	1.00041982361235140	8.8817E-16	5.9952E-15
1.0	1.00043980641420040	1.00043980641420150	1.1102E-15	6.1062E-14

6. Conclusion

A new single step self-starting implicit block method has been presented in this paper for solving higher-order (Third, Fourth, and Fifth-order) initial value problems of ordinary differential equations. The solutions of the numerical experiments obtained are accurate and superior to the existing methods in terms of performance and accuracy as demonstrated in Tables (1 to 5) and Figures (1 to 5). Hence, this method has given rise to more accurate and superior convergent results which is a good approach to provide solution to higher order initial value problems.

Appendix

Appendix 1

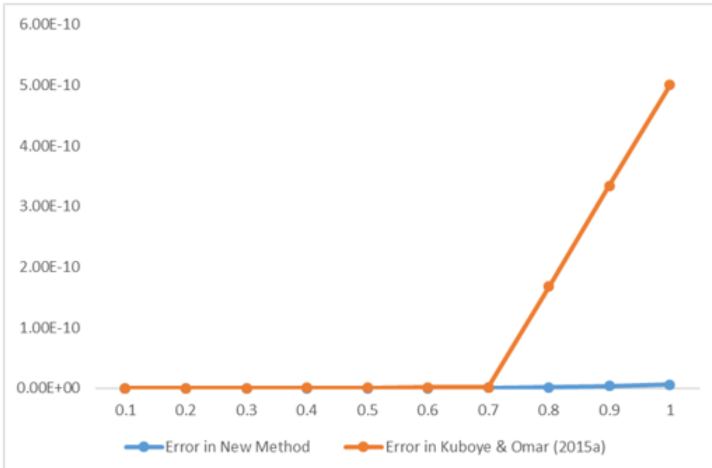


Figure 1: Comparison of Absolute Error with Error in Kuboye & Omar (2015a)

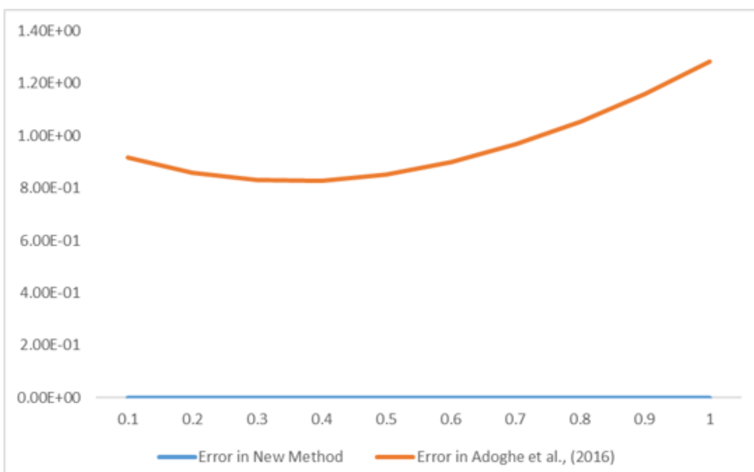


Figure 2: Comparison of Absolute Error with Adoghe et al., (2016)

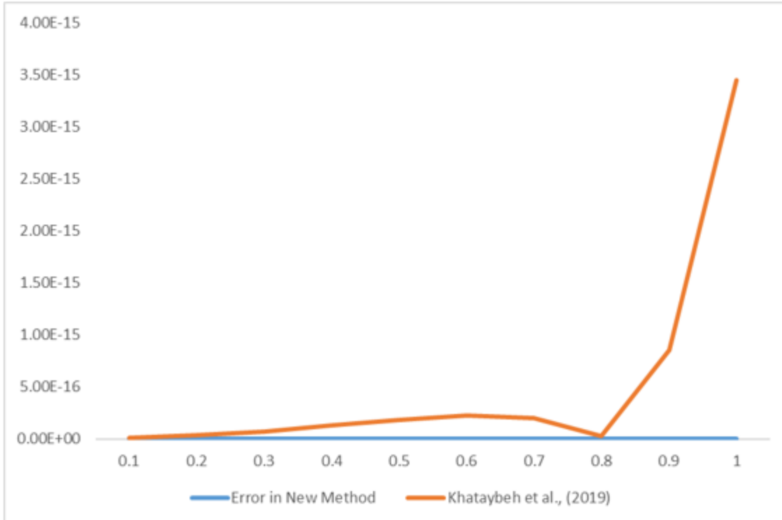


Figure 3: Comparison of Absolute Error with Khataybeh et al., (2019)

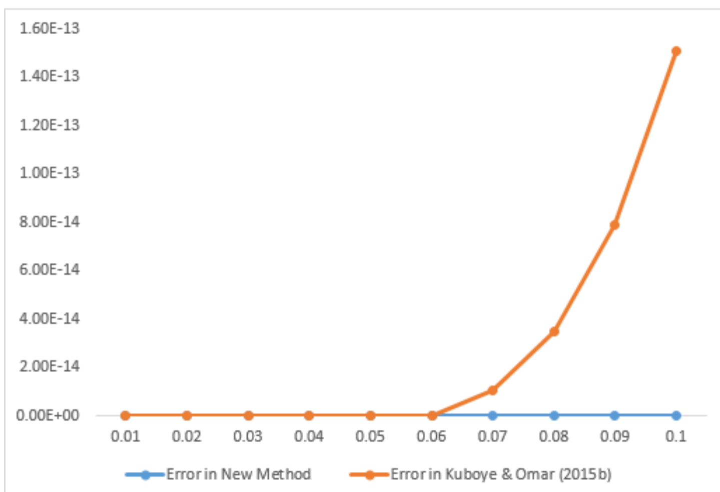
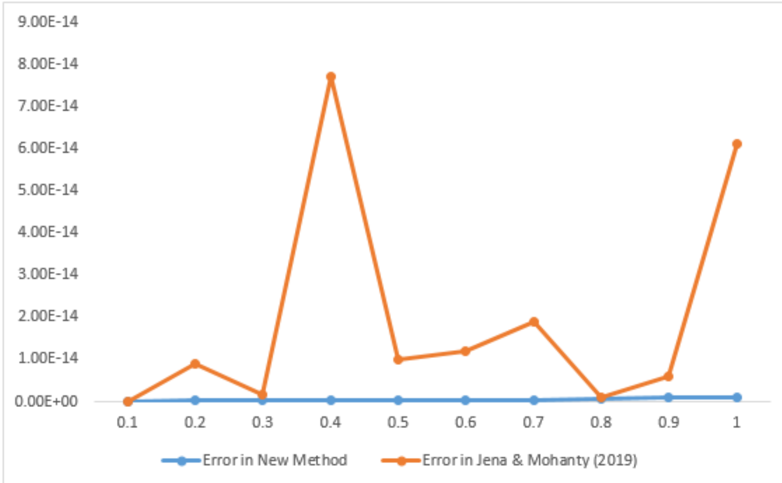


Figure 4: Comparison of Absolute Error with Kuboye & Omar (2015b)

Appendix 2

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, L = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, K = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & \frac{2}{5} \\ 0 & 0 & 0 & 0 & \frac{3}{5} \\ 0 & 0 & 0 & 0 & \frac{4}{5} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, R = \begin{bmatrix} 0 & 0 & 0 & 0 & \frac{1}{50} \\ 0 & 0 & 0 & 0 & \frac{2}{25} \\ 0 & 0 & 0 & 0 & \frac{9}{50} \\ 0 & 0 & 0 & 0 & \frac{8}{25} \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{bmatrix},$$

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{493123}{504000000} \\ 0 & 0 & 0 & 0 & -\frac{315043}{35437500} \\ 0 & 0 & 0 & 0 & -\frac{961011}{56000000} \\ 0 & 0 & 0 & 0 & -\frac{13664}{421875} \\ 0 & 0 & 0 & 0 & -\frac{9841}{207360} \end{bmatrix}, V = \begin{bmatrix} \frac{377}{90000} & \frac{778}{23625} & \frac{21789}{280000} & \frac{1984}{13125} & \frac{725}{3024} \\ -\frac{33017}{15120000} & -\frac{613}{39375} & -\frac{2349}{80000} & -\frac{1088}{23625} & \frac{475}{8064} \\ \frac{13}{36000} & \frac{916}{354375} & \frac{387}{70000} & \frac{1664}{118125} & \frac{575}{18144} \\ -\frac{403}{6720000} & -\frac{29}{67500} & -\frac{81}{89600} & -\frac{64}{39375} & \frac{125}{96768} \\ \frac{509}{94500000} & \frac{38}{984375} & \frac{81}{1000000} & \frac{64}{421875} & \frac{1}{3360} \end{bmatrix}, T = \begin{bmatrix} 0 & 0 & 0 & 0 & -\frac{1291}{18000000} \\ 0 & 0 & 0 & 0 & -\frac{227}{421875} \\ 0 & 0 & 0 & 0 & -\frac{2187}{2000000} \\ 0 & 0 & 0 & 0 & -\frac{32}{15625} \\ 0 & 0 & 0 & 0 & -\frac{53}{17280} \end{bmatrix},$$



$$Z=\begin{bmatrix}0&0&0&0&\frac{71}{1575000}\\0&0&0&0&\frac{64}{196875}\\0&0&0&0&\frac{243}{350000}\\0&0&0&0&\frac{256}{196875}\\0&0&0&0&\frac{1}{504}\end{bmatrix},\ \bar{N}=\begin{bmatrix}0&0&0&0&-\frac{1472753}{64800000}\\0&0&0&0&-\frac{140279}{3543750}\\0&0&0&0&-\frac{356253}{56000000}\\0&0&0&0&-\frac{84284}{1771875}\\0&0&0&0&-\frac{39191}{145152}\end{bmatrix},\ \bar{W}=\begin{bmatrix}\frac{16549}{216000}&\frac{4301}{23625}&\frac{2133}{7000}&\frac{1664}{4725}&-\frac{10222}{12096}\\-\frac{11923}{302400}&-\frac{1723}{23625}&-\frac{4563}{56000}&-\frac{1088}{23625}&-\frac{425}{1728}\\\frac{14803}{2268000}&\frac{874}{70875}&\frac{33}{1400}&\frac{4352}{70875}&\frac{2425}{18144}\\-\frac{937}{864000}&-\frac{97}{47250}&-\frac{783}{224000}&-\frac{4}{23625}&-\frac{1625}{48384}\\\frac{3683}{37800000}&\frac{109}{590625}&\frac{27}{87500}&\frac{128}{590625}&-\frac{41}{12096}\end{bmatrix},$$

$$\bar{D}=\begin{bmatrix}0&0&0&0&-\frac{7369}{5400000}\\0&0&0&0&-\frac{212}{84375}\\0&0&0&0&-\frac{801}{200000}\\0&0&0&0&-\frac{304}{84375}\\0&0&0&0&-\frac{25}{1728}\end{bmatrix},\bar{E}=\begin{bmatrix}0&0&0&0&\frac{1279}{1575000}\\0&0&0&0&\frac{316}{196875}\\0&0&0&0&\frac{891}{350000}\\0&0&0&0&\frac{512}{196875}\\0&0&0&0&-\frac{1}{126}\end{bmatrix},\ \hat{S}=\begin{bmatrix}0&0&0&0&-\frac{1762949}{6048000}\\0&0&0&0&\frac{2599}{21000}\\0&0&0&0&-\frac{86047}{224000}\\0&0&0&0&\frac{15142}{23625}\\0&0&0&0&-\frac{19361}{5376}\end{bmatrix},\hat{Z}=$$

$$\begin{bmatrix}0&0&0&0&-\frac{1111}{72000}\\0&0&0&0&\frac{1}{250}\\0&0&0&0&-\frac{159}{8000}\\0&0&0&0&\frac{32}{1125}\\0&0&0&0&-\frac{11}{64}\end{bmatrix},\ \hat{Q}=\begin{bmatrix}\frac{57517}{67200}&\frac{57}{350}&\frac{24687}{22400}&-\frac{424}{525}&-\frac{6375}{896}\\-\frac{14249}{33600}&\frac{131}{1050}&-\frac{2979}{11200}&\frac{376}{525}&-\frac{4625}{1344}\\\frac{3037}{43200}&-\frac{1}{75}&\frac{263}{1600}&\frac{112}{675}&\frac{125}{192}\\-\frac{1571}{134400}&\frac{3}{1400}&-\frac{801}{44800}&\frac{46}{525}&\frac{375}{1792}\\\frac{353}{336000}&\frac{1}{5250}&\frac{171}{112000}&-\frac{8}{2625}&-\frac{181}{2688}\end{bmatrix},\ \hat{P}=\begin{bmatrix}0&0&0&0&\frac{731}{84000}\\0&0&0&0&-\frac{2}{2625}\\0&0&0&0&\frac{297}{28000}\\0&0&0&0&-\frac{32}{2625}\\0&0&0&0&\frac{55}{672}\end{bmatrix}$$

References

[1] S. R. Jena, M. Mohanty, *Numerical treatment for ODEs (fifth order)*, International Journal on Emerging Technologies, 10 (2019), 191-196.

[2] L. K. Yap, F. Ismail, *Block hybrid collocation method with application to fourth order differential equations*, Mathematical Problems in Engineering, (2015), 1-6.

[3] N. Waeleh, Z. A. Majid, F. Ismail, M. Suleiman, *Numerical solution of higher order ordinary differential equations by direct block code*, Journal of Mathematics and Statistics, 8 (2011), 77-81.

[4] M. Suleiman, Z. B. Ibrahim, A.F.N. BinRasedee, *Solution of higher-order ODEs using backward difference method*, Mathematical Problems in Engineering, (2011), Article ID 810324, 18 pages

[5] E. O. Adeyefa, J. O. Kuboye, *Derivation of new numerical model capable of solving second and third order ordinary differential equations directly*, IAENG International Journal of Applied Mathematics, 50 (2020), 9 pages.

[6] S. O. Fatunla, *Block methods for second order ODEs*, International Journal of Computer Mathematics, 41 (1991), 55-63.

[7] S. N. Khataybeh, I. Hashim, M. Alshbool, *Solving directly third-order ODEs using operational matrices of bernstein polynomials method with applications to fluid flow equations*, Journal of King Saud University-Science, 31 (2019), 822-826.

[8] L. O. Adoghe, B. G. Ogunware, E. O. Omole, *A family of symmetric implicit higher order methods for the solution of third order initial value problems in ordinary differential equations*, Theoretical Mathematics & Applications, 6 (2016), 67-84.

[9] O. O. Agboola, A. A. Opanuga, J. A. Gbadeyan, *Solution of third order ordinary differential equations using differential transform method*, Global Journal of Pure and Applied Mathematics, 11 (2015), 2511-2517.

[10] J. O. Kuboye, Z. Omar, *Numerical solution of third order ordinary differential equations using a seven-step block method*, International Journal of Mathematical Analysis, 9 (2015a), 743-754.

[11] J. O. Kuboye, Z. Omar, *New zero-stable block method for direct solution of fourth order ordinary differential equations*, Indian Journal of Science and Technology, 8 (2015b), 70647.

[12] K. Hussain, F. Ismail, N. Senu, *Runge-Kutta type methods for directly solving special fourth-order ordinary differential equations*, Mathematical Problems in Engineering, (2015), Article ID 893763, 11 pages.

[13] J. D. Lambert, *Computational methods in ordinary differential equations*, (1973), John Wiley & Sons Inc.

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