Quasi-metric hyper dynamical systems

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Abstract. We start this paper by introducing the concept of quasi-metric hypergroups. We show that the product of two quasi-metric hypergroups is a quasi-metric hypergroup. Quasi-metric hyperdynamical systems are defined, and a method for constructing quasi-metric hyperdynamical systems via two given quasi-metric hyperdynamical systems, is deduced. Attracting sets for quasi-metric hyperdynamical systems are considered. A method for constructing quasi-metric hyperdynamical systems with attracting sets via two given quasi-metric hyperdynamical systems with attracting sets, is presented.

Keywords: quasi-metric hypergroup, quasi-metric hyperdynamical system, time hypergroup, attracting set.

1. Introduction

The notion of quasi-metric spaces has been studied first by Stolenberg as an extension of the metric spaces [7, 8]. Quasi-metric spaces are more compatible with some of realistic structures. In fact symmetry is an ideal property for soft computing, but in the realistic case we do not have such property, for example according to traffic rules the time for going from point a to point b in a city is not equal to the time for going from b to a. In section three we introduce quasi-metric hypergroups by adding a kind of continuity to the join operation via a

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quasi-metric on it, and we show that there is a quasi-metric on the product of two quasi-metric hypergroups to make it a quasi-metric hypergroup. In section four we define quasi-metric hyperdynamical systems via two conditions. The first condition makes it an evolution operator and the second condition put continuity on its time evolution. We show that the product of two quasi-metric hyperdynamical systems is a quasi-metric hyperdynamical system. In section five we consider attracting sets for a quasi-metric hyperdynamical system as an invariant set of it which attracts any bounded set which is near it. We show that if two given quasi-metric hyperdynamical systems have attracting sets, then their product has an attracting set.

2. Basic notions

According to Stolenberg definition [7] if M is a non-empty set, then a function $q : M \times M \longrightarrow [0, \infty)$ is called a quasi-metric if it satisfies the following conditions.

(1) $q(x, y) = 0 \Leftrightarrow x = y$.

(2) $q(x, z) \leq q(x, y) + q(y, z)$, for all $x, y, z \in M$

In this case (M, q) is called a quasi-metric space or a q-metric space.

The topology $\tau(q)$ induced by a quasi-metric q on M is the topology determined by the basis consisting of all r-balls $B^q_r(p)= \{m \in M: \quad q(p, m) < r\ \}$ where $p \in M$ and $r \in [0, \infty)$ (see, [7, 6]).

A join operation on the nonempty set H ([1, 2]) is a mapping from $H \times H$ to the set $P_*(H)$ which is the set of all nonempty subsets of H. If $x, y \in H$, then we denote their joins by xy. A join operation on H creates an operation from $P_*(H) \times P_*(H)$ to $P_*(H)$ by $(X, Y) \mapsto XY$, where $XY = \bigcup_{(x,y)\in X\times Y} xy$. For simplicity $\{x\}Y$ and $Y\{x\}$ are denoted by xY and Yx respectively.

H with a join operation is called a hypergroup ([4]) if for all $x, y, z \in H$ we have $x(yz) = (xy)z$ and $xH = Hx = H$.

3. Quasi-metric hypergroups

We assume H is a hypergroup and q is a quasi-metric on it.

Definition 3.1. (H, q) *is said to be a quasi-metric hypergroup or a q-hypergroup if for given* $x, y \in H$ *and for all* $r > 0$ *and* $z \in xy$*, there is* $d > 0$ *such that* $hs \cap B^q_r(z) \neq \emptyset$*, for all* $h \in B^q_d(x)$ *and* $s \in B^q_d(y)$*.*

Now, we show that the product of two quasi-metric hypergroups is a quasi-metric hypergroup.

Theorem 3.1. *If* (H_1, q_1) *and* (H_2, q_2) *are two quasi-metric hypergroups, then* $(H_1 \times H_2, q)$ *is a quasi-metric hypergroup, where* q *is the following map:*

$$q : (H_1 \times H_2, q) \longrightarrow [0, \infty),$$
$$((h_1, h_2), (s_1, s_2)) \longmapsto q_1(h_1, s_1) + q_2(h_2, s_2),$$

and the join operation of $H_1 \times H_2$ is the following operation:

$$. : (H_1 \times H_2) \times (H_1 \times H_2) \longrightarrow P_*(H_1 \times H_2),$$
$$((h_1, h_2), (s_1, s_2)) \longmapsto (h_1 s_1) \times (h_2 s_2).$$

Proof. We first show that $(H_1 \times H_2, q)$ is a quasi-metric space. We see that:

$$\begin{aligned}
&q((h_1, h_2), (s_1, s_2)) = 0 \Leftrightarrow \\
&q_1(h_1, s_1) + q_2(h_2, s_2) = 0 \Leftrightarrow \\
&q_1(h_1, s_1) = 0 \quad \text{and} \quad q_2(h_2, s_2) = 0 \Leftrightarrow \\
&h_1 = s_1 \quad \text{and} \quad h_2 = s_2.
\end{aligned}$$

For $(h_1, h_2), (t_1, t_2), (s_1, s_2) \in H_1 \times H_2$ we have:

$$\begin{aligned}
&q((h_1, h_2), (s_1, s_2)) = q_1(h_1, s_1) + q_2(h_2, s_2) \\
&\leq q_1(h_1, t_1) + q_1(t_1, s_1) + q_2(h_2, t_2) + q_2(t_2, s_2) \\
&\leq q((h_1, h_2), (t_1, t_2)) + q((t_1, t_2), (s_1, s_2)).
\end{aligned}$$

Thus, $(H_1 \times H_2, q)$ is a quasi-metric space. For all $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in H_1 \times H_2$ we have:

$$\begin{aligned}
&(x_1, x_2)((y_1, y_2)(z_1, z_2)) = (x_1(y_1 z_1), x_2(y_2 z_2)) \\
&= ((x_1 y_1) z_1, (x_2 y_2) z_2) = ((x_1, x_2)(y_1, y_2))(z_1, z_2),
\end{aligned}$$

and

$$(x_1, x_2) H_1 \times H_2 = x_1 H_1 \times x_2 H_2 = H_1 \times H_2 = H_1 x_1 \times H_2 x_2 = H_1 \times H_2 (x_1, x_2).$$

Hence $H_1 \times H_2$ is a hypergroup.

Let $(x_1, x_2), (y_1, y_2) \in H_1 \times H_2$, $r > 0$, and $(z_1, z_2) \in (x_1, x_2)(y_1, y_2)$ be given. Since $z_1 \in x_1 y_1$ and $z_2 \in x_2 y_2$, then there exist $d_1 > 0$ and $d_2 > 0$ such that $h_1 s_1 \cap B^{q_1}_{\frac{r}{2}}(z_1) \neq \emptyset$ and $h_2 s_2 \cap B^{q_2}_{\frac{r}{2}}(z_2) \neq \emptyset$, for all $h_1 \in B^{q_1}_{d_1}(x_1)$, $s_1 \in B^{q_1}_{d_1}(y_1)$, $h_2 \in B^{q_2}_{d_2}(x_2)$, and $s_2 \in B^{q_2}_{d_2}(y_2)$. The definition of q implies that $B^{q_1}_{\frac{r}{2}}(z_1) \times B^{q_2}_{\frac{r}{2}}(z_2) \subseteq B^{q}_{r}(z_1, z_2)$, so, for all $(h_1, h_2) \in B^{q_1}_{d_1}(x_1) \times B^{q_2}_{d_2}(x_2)$ and $(s_1, s_2) \in B^{q_1}_{d_1}(y_1) \times B^{q_2}_{d_2}(y_2)$, we have $(h_1 s_1 \times h_2 s_2) \cap B^{q}_{r}(z_1, z_2) \neq \emptyset$. If we take $d = \min\{d_1, d_2\}$ then $B^{q}_{d}(x_1, x_2) \subseteq B^{q_1}_{d_1}(x_1) \times B^{q_2}_{d_2}(x_2)$ and $B^{q}_{d}(y_1, y_2) \subseteq B^{q_1}_{d_1}(y_1) \times B^{q_2}_{d_2}(y_2)$. Thus, for all $(h_1, h_2) \in B^{q}_{d}(x_1, x_2)$ and $(s_1, s_2) \in B^{q}_{d}(y_1, y_2)$ we have $(h_1 s_1 \times h_2 s_2) \cap B^{q}_{r}(z_1, z_2) \neq \emptyset$. Hence $(H_1 \times H_2, q)$ is a quasi-metric hypergroup. $\square$

4. Quasi-metric hyperdynamical systems

Let (M, q_1) be a q-metric space and let (H, q_2) be a q-hypergroup. Moreover, let $\varphi : H \times M \longrightarrow M$ be a mapping. With these assumptions we have the next definition.

Definition 4.1. *The* 5*-tuple* $(M, q_1, \varphi, H, q_2)$ *is said to be a quasi-metric hyperdynamical system, if it satisfies the following conditions:*

(*i*) *If* $h_1, h_2 \in H$ *and* $m \in M$*, then* $\varphi(h_1, \varphi(h_2, m)) \in \varphi(h_1h_2, m)$*, where* $\varphi(h_1h_2, m) = \{\varphi(h, m): \quad h \in h_1h_2\ \}$;

(*ii*) $\varphi : H \times M \longrightarrow M$ *is a* (q_2, q_1) *continuous map i.e., for all* $V \in \tau_{q_1}$*, there exist* $W \in \tau_{q_2}$ *and* $Z \in \tau_{q_1}$ *such that* $\varphi(W \times Z) \subseteq V$.

Now, we assume that $(M, q_1, \varphi, H, d_1)$ and (N, q_2, ψ, S, d_2) are two quasi-metric hyperdynamical systems. We take quasi-metrics q and d on $M \times N$ and $H \times S$ as in Theorem 3.1 respectively. Moreover, we take the join operation on $H \times S$ as in Theorem 3.1. With these assumptions we have the next theorem.

Theorem 4.1. $(M \times N, q, \varphi \times \psi, H \times S, d)$ *is a quasi-metric hyperdynamical system.*

Proof. If $(h_1, s_1),\ (h_2, s_2) \in H \times S$ and $(m, n) \in M \times N$, then

$$\begin{aligned}
&(\varphi \times \psi)((h_1, s_1), (\varphi \times \psi)(h_2, s_2), (m, n)) = (\varphi \times \psi)((h_1, s_1), (\varphi(h_2, m), \psi(s_2, n)))\\
&= (\varphi(h_1, \varphi(h_2, m)), \psi(s_1, \psi(s_2, n))) \in \varphi(h_1h_2, m) \times \psi(s_1s_2, n)\\
&= (\varphi \times \psi)(h_1h_2 \times s_1s_2, (m, n)).
\end{aligned}$$

Let the nonempty set $V \in \tau_q$ be given. Then, there is $(m, n) \in V$ and $r > 0$ such that $B^q_r(m, n) \subseteq V$. Since $B^{q_1}_{\frac{r}{2}}(m) \in \tau_{q_1}$ and $B^{q_2}_{\frac{r}{2}}(n) \in \tau_{q_2}$, then there exist $W_1 \in \tau_{d_1}$, $Z_1 \in \tau_{q_1}$, $W_2 \in \tau_{d_2}$, and $Z_2 \in \tau_{q_2}$ such that $\varphi(W_1, Z_1) \subseteq B^{q_1}_{\frac{r}{2}}(m)$ and $\psi(W_2, Z_2) \subseteq B^{q_2}_{\frac{r}{2}}(n)$. Hence we have

$$\begin{aligned}
(\varphi \times \psi)(W_1 \times W_2, Z_1 \times Z_2) &= \varphi(W_1, Z_1) \times \psi(W_2, Z\psi_2)\\
&\subseteq B^{q_1}_{\frac{r}{2}}(m) \times B^{q_2}_{\frac{r}{2}}(n) \subseteq B^q_r(m, n) \subseteq V.
\end{aligned}$$

Thus, $\varphi \times \psi$ is a (d, q) continuous map. □

5. Attracting sets

We begin this section by recalling the definition of time hypergroup (see, [5]). A hypergroup H is called a time hypergroup with zero time $e \in H$ ([5]) if $(H, <)$ is a partially ordered hypergroup with the following two properties:

(1) $h \in he$ and $h \in eh$, for all $h \in H$;

(2) If $h \in H$, then $h < e$ or $h > e$ or $h = e$.

In this section we assume that H is a time hypergroup, and $(M, q_1, \varphi, H, q_2)$ is a quasi-metric hyperdynamical system.

A subset A of M is called an invariant set for $(M, q_1, \varphi, H, q_2)$ if $\varphi(H, A) \subseteq A$.

If $A, B \subseteq M$ the distance of A to B is defined by

$$\operatorname{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} q_1(a, b).$$

Definition 5.1. *A non-empty invariant subset A is said to be an attracting set for $(M, q_1, \varphi, H, q_2)$ if there is an open subset B of M containing A such that for each bounded subset C of B we have:* $\lim_{n\to\infty} \operatorname{dist}(\varphi(h^n, C), A) = 0$ *for each $h > e$.*

We see that any attracting set of $(M, q_1, \varphi, H, q_2)$ attracts any bounded subset which is near it, under the iterations of positive times.

Suppose H and S are two time hypergroups with the zero times e_1 and e_2 and partial orders $<_1$ and $<_2$ respectively. On $H \times S$ we define a partial order $<$ by: $(h, s) < (e_1, e_2)$ if $h <_1 e_1$, and $s <_2 e_2$; $(e_1, e_2) < (r, t)$ if $e_1 <_1 r$, or $e_2 <_2 t$; $(h, s) < (r, t)$ if $(h, s) < (e_1, e_2) < (r, t)$.

We see that $H \times S$ with this partial order is a time hypergroup. Now, we also assume that $(M, q_1, \varphi, H, d_1)$ and (N, q_2, ψ, S, d_2) are two quasi-metric hyperdynamical systems with attracting sets A_1 and A_2 respectively. Moreover, we assume that $(M \times N, q, \varphi \times \psi, H \times S, d)$ is the quasi-metric hyperdynamical system which is constructed via the assumptions of Theorem 4.1. With these assumptions we have the next theorem.

Theorem 5.1. *$A_1 \times A_2$ is an attracting set for $(M \times N, q, \varphi \times \psi, H \times S, d)$.*

Proof. There exist open sets U and V in M and N corresponding to $(M, q_1, \varphi, H, d_1)$ and (N, q_2, ψ, S, d_2) for A_1 and A_2 respectively. We show that the open set $U \times V$ satisfies the condition of Definition 5.1 for $A_1 \times A_2$. If C is a bounded set in $U \times V$, then it means there is $r > 0$, and $(m, n) \in M \times N$ such that $C \subseteq B_r^q(m, n)$. Thus, $C \subseteq B_r^{q_1}(m) \times B_r^{q_2}(n)$. Hence $C \subseteq C_1 \times C_2$, where $C_1 = B_r^{q_1}(m) \cap U$ and $C_2 = B_r^{q_2}(n) \cap V$. If $h = (h_1, h_2) > (e_1, e_2)$, then $h_1 > e_1$ and $h_2 > e_2$, and we have:

$$\begin{aligned}
&\lim_{n\to\infty} \operatorname{dist}((\varphi \times \psi)(h^n, C), A_1 \times A_2) \leq \lim_{n\to\infty} \operatorname{dist}((\varphi \times \psi)(h^n, C_1 \times C_2), A_1 \times A_2) \\
&= \lim_{n\to\infty} \operatorname{dist}(\varphi(h_1^n, C_1) \times \psi(h_2^n, C_2), A_1 \times A_2) \\
&= \lim_{n\to\infty} \operatorname{dist}(\varphi(h_1^n, C_1), A_1) + \lim_{n\to\infty} \operatorname{dist}(\psi(h_2^n, C_2), A_2) = 0.
\end{aligned}$$

Thus, the invariant set $A_1 \times A_2$ is an attracting set for $(M \times N, q, \varphi \times \psi, H \times S, d)$. □

Conclusion

We have considered quasi-metric hyperdynamical systems as an extension of metric dynamical systems. In fact we extend the phase spaces to quasi-metric spaces and we also extend the time sets to hypergroups. By using of time hypergroup we have considered attracting sets for the quasi-metric hyperdynamical systems. If H and S are two times hypergroups, then we have introduced a partial order on $H \times S$ which make it a time hypergroup and able us to construct new attracting sets via the product of their quasi-metric hyperdynamical systems. One must pay attention to this point that we can define another partial order on $H \times S$ which make it a time hypergroup, but Theorem 5.1 is not valid for it. For example by refer to [5] one can see the partial order $<$ on $H \times S$ which has been defined by: $(h_1, s_1) < (h_2, s_2)$ if $h_1 <_1 h_2$, and when $h_1 = h_2$ then $s_1 <_2 s_2$, where $<_1$ and $<_2$ are partial orders on H and S respectively. With this partial order Theorem 5.1 is not valid.

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