A note on b-generalized derivations in rings with involution

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Abstract. Let R be a ring with involution $*$. The purpose of this paper is to investigate the special type of mappings defined on $(R, *)$. In fact it is shown that these mappings are actually the b-generalized derivation defined on R.

Keywords: prime ring, derivation, b-generalized derivation, involution.

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1. Introduction

Throughout this article, R be a prime ring with involution and Q_r will be the right Martindale quotient ring of R. For any $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$, while the symbol $x \circ y$ will stand for the anti-commutator $xy+yx$. R is said to be 2-torsion free if whenever $2x = 0$; with $x \in R$ implies $x = 0$. R is prime if $xRy = (0)$, where $x, y \in R$, implies $x = 0$ or $y = 0$ and is called a semiprime ring in case $xRx = (0)$ implies $x = 0$. A derivation on R is an additive mapping $d : R \to R$ such that $d(xy) = d(x)y + xd(y)$, for all $x, y \in R$. A derivation d is said to be inner if there exists $a \in R$ such that $d(x) = ax - xa$, for all $x \in R$. Following Brešar [9], an additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that $F(xy) = F(x)y + xd(y)$, for all $x, y \in R$. Basic examples are derivations and generalized inner derivations are maps of type $x \mapsto ax + xb$ for some $a, b \in R$.

Many results in the literature indicate how the global structure of a ring R is often tightly connected to the behavior of additive mappings defined on R. Many results in this direction can be found in [1, 5, 6, 7, 8, 9, 10, 15, 16, 17]. Very recently Koşan and Lee [14] introduced the new concept of left b-generalized derivation as follows: Let $d : R \to Q_r$ be an additive mapping and $b \in Q_r$. An additive mapping $F : R \to Q_r$ is called a left b-generalized derivation, with an associated mapping d, if $F(xy) = F(x)y + bxd(y)$, for all $x, y \in R$. Moreover, it is prove that if R is a prime ring, then d is a derivation of R. In the present paper, this mapping F will be called a b-generalized derivation with an associated derivation d. It is easy to see that every generalized derivation is a 1-generalized derivation. For instance for any $x \in R$, the mapping $x :\to ax + bxc$ for $a, b, c \in Q$ is a b-generalized derivation of R, which is known as inner b-generalized derivation of R.

An additive mapping $* : R \to R$ is called an involution if $*$ is an anti-automorphism of order 2, that is $(x^*)^* = x$, for all $x \in R$. A ring equipped with an involution $*$ is called an involution ring. Very recently, many authors have studied certain additive mappings like derivations, generalized derivations in the setting of rings with involution (see [2, 3, 4, 11, 12] for references). They not only characterized these mappings but also found that there is a close connection between the commutativity of R and these mappings. Here our emphasis will be more in the direction of the a special type of mapping defined on R, which were first studied in [18].

In fact, our motivations comes from [[13], Theorem 4.1.2], which stated as: Let R be a simple ring with involution of characteristic not 2, such that $dim_Z R > 4$. Let $d : R \to R$ be such that $d(xx^*) = d(x)x^* + xd(x^*)$, for all $x \in R$. Then d is a derivation of R. We prove the following results.

Theorem 1.1. *Let R be a 2-torsion free semiprime $*-$ring with involution such that R has a commutator which is not a zero divisor. If there exists an additive mapping $F : R \to R$ associated with a nonzero derivation $d : R \to R$ such that*

$F(xx^*) = F(x)x^* + bxd(x^*)$, for all $x \in R$, where b is a fixed element of R. Then F is a b-generalized derivation.

Theorem 1.2. *Let R be a 2-torsion free semiprime $*-$ring and let R has a commutator which is not a zero divisor. If there exists an additive mapping $F : R \to R$ associated with a nonzero derivation $d : R \to R$ such that $F(xy^*x) = F(x)y^*x + bxd(y^*)x + bxy^*d(x)$, for all $x, y \in R$ and b is a fixed element of R. Then F is a b-generalized derivation.*

2. Main results

To prove the above results we need the following lemma.

Lemma 2.1. *Let R be a 2-torsion free ring and let $F : R \to R$ be an additive mapping associated with a nonzero derivation $d : R \to R$ such that $F(x^2) = F(x)x + bxd(x)$, where b is a fixed element of R. Then, for all $x, y, z \in R$, the following statements hold:*

(i) $F(xy + yx) = F(x)y + F(y)x + bxd(y) + byd(x)$;

(ii) $F(xyx) = F(x)yx + bxd(y)x + bxyd(x)$;

(iii) $F(xyz + zyx) = F(x)yz + F(z)yx + bxd(y)z + bxyd(z) + bzd(y)x + bzyd(x)$;

(iv) $\delta(x, y)[x, y] = 0$, *where* $\delta(x, y) = F(xy) - F(x)y - bxd(y)$.

Proof. (i) We have

(1) $$F(x^2) = F(x)x + bxd(x), \quad \text{for all} \quad x \in R.$$

Replacing x by $x + y$ and using (1), we get

(2) $$F(xy + yx) = F(x)y + F(y)x + bxd(y) + byd(x), \quad \text{for all} \quad x, y \in R.$$

(ii) Taking $y = xy + yx$ in (2) and using it, we arrive at

(3) $$\begin{aligned} F(x^2y + yx^2) + 2F(xyx) = {} & F(x)xy + F(x)yx + F(x)yx \\ & + F(y)x^2 + bxd(y)x + byd(x)x \\ & + bxd(x)y + bx^2d(y) + bxd(y)x + bxy(x) \\ & + bxyd(x) + byxd(x), \quad \text{for all} \quad x, y \in R. \end{aligned}$$

Replacing x by x^2 in (2) and using (3) and the fact that R is 2-torsion free, we obtain

(4) $$F(xyx) = F(x)yx + bxd(y)x + bxyd(x), \quad \text{for all} \quad x, y \in R.$$

There by proving (ii).

(iii) Replacing x by $x+z$ in (4) and using (4), we get

$$F(xyz+zyx) = F(x)yz+F(z)yx+bxd(y)z+bzd(y)x+bxyd(z)+bzyd(x), \tag{5}$$

for all $x, y \in R$. Thus proves (iii).

(iv) On substituting $xy-yx$ in place of z in (5), we get $\delta(x,y)[x,y]=0$, for all $x, y \in R$, where $\delta(x,y) = F(xy)-F(x)y-bxd(y)$. This completes the proof of Lemma. □

Proof of Theorem 1.1. We have

$$F(xx^*) = F(x)x^* + bxd(x^*), \quad \text{for all} \quad x \in R. \tag{6}$$

On linearizing (6), we get

$$F(xy^*+yx^*) = F(x)y^*+F(y)x^*+byd(x^*)+bxd(y^*), \quad \text{for all} \quad x, y \in R. \tag{7}$$

Taking $y = x^*$ in (7), we have

$$F(x^2+(x^*)^2)+F(x)x+F(x^*)x^*+bx^*d(x^*)+bxd(x), \quad \text{for all} \quad x \in R.$$

This can be further written as

$$B(x)+B(x^*) = 0, \quad \text{for all} \quad x \in R, \tag{8}$$

where $B(x) = F(x^2)-F(x)x-bxd(x)$, for all $x \in R$. Replacing y by xy^*+yx^* in (7), we obtain

$$F(x(y+y^*)x^*) = -B(x)y^*+F(x)(y+y^*)x^*+bxd(y+y^*)x^*, \quad \text{for all} \quad x, y \in R.$$

Using $y-y^*$ for y, we get

$$B(x)y = B(x)y^*, \quad \text{for all} \quad x, y \in R. \tag{9}$$

In view of [[19], Lemma 1], we get $B(x) \in Z(R)$, for all $x \in R$. Taking $y = y^*$ in (7), we arrive at

$$F(xy+y^*x^*) = F(x)y+F(y^*)x^*+by^*d(x^*)+bxd(y), \quad \text{for all} \quad x, y \in R. \tag{10}$$

Replacing y by xy in (10), we get

$$F(x^2y+y^*(x^*)^2) = F(x)xy+F(y^*x^*)x^*+bxd(x)y+bx^2d(y)+by^*x^*d(x^*), \tag{11}$$

for all $x, y \in R$. Taking $x = x^2$ in (10), we obtain

$$F(x^2y+y^*(x^*)^2) = F(x^2)y+F(y^*)(x^*)^2+bx^2d(y)+by^*d((x^*)^2), \tag{12}$$

for all $x, y \in R$. Using (11) and (12), we get

$$(F(x^2)-F(x)x-bxd(x))y+(F(y)x^*-F(y^*x^*)+by^*d(x^*))x^* = 0, \quad \text{for all} \quad x, y \in R.$$

Replacing y by x, we have

$$(F(x^2)-F(x)x-bxd(x))x-(F((x^*)^2-F(x^*)x^*-bx^*d(x^*))x^* = 0, \text{ for all } x \in R.$$

This implies that

$$B(x)x - B(x^*)x^* = 0, \quad \text{for all } \; x \in R.$$

By (8), we arrive at

$$B(x)(x + x^*) = 0, \quad \text{for all } \; x \in R. \tag{13}$$

Taking $y = x$ in (9), we get

$$B(x)(x - x^*) = 0, \quad \text{for all } \; x \in R. \tag{14}$$

Thus in view of (13) and (14), we get $2B(x)x = 0$, for all $x \in R$. Since R is 2-torsion free, we obtain

$$B(x)x = 0, \quad \text{for all } \; x \in R. \tag{15}$$

Since $B(x)$ is in $Z(R)$, this implies that $xB(x) = 0$, for all $x \in R$. Linearizing (15), we get

$$B(x)y + B(y)x + \sigma(x, y)x + \sigma(x, y)y = 0, \quad \text{for all } \; x, y \in R, \tag{16}$$

where $\sigma(x, y) = F(xy + yx) - F(x)y - F(y)x - bxd(y) - byd(x)$, for all $x, y \in R$. Taking $x = -x$ in (16), we have

$$B(x)y - B(y)x + \sigma(x, y)x - \sigma(x, y)y = 0, \quad \text{for all } \; x, y \in R. \tag{17}$$

Using (16) and (17), we arrive at $B(x)y + \sigma(x, y)x = 0$, for all $x, y \in R$. Right multiplying by $B(x)$, we get $B(x)yB(x) + \sigma(x, y)xB(x) = 0$, for all $x, y \in R$. This implies that $B(x)yB(x) = 0$, for all $x, y \in R$. Since R is a semiprime ring, we obtain $B(x) = 0$, for all $x \in R$. This implies that

$$F(x^2) = F(x)x + bxd(x), \quad \text{for all } \; x \in R. \tag{18}$$

Let u, v be fixed element of R such that $w[u, v] = 0$ or $[u, v]w = 0$. Then in view of Lemma 2.1 (iv) and hypothesis

$$\delta(u, v) = 0, \tag{19}$$

we have to show that $\delta(x, y) = 0$, for all $x, y \in R$. Again in view of Lemma 2.1 (iv), we have

$$\delta(x, y)[x, y] = 0. \tag{20}$$

Replacing x by $x+u$ and using (20), we get

$$\delta(x,y)[u,y]+\delta(u,y)[x,y]=0, \quad \text{for all} \quad x,y\in R. \tag{21}$$

On substituting y by $y+v$ and using (19) and (20), we have

$$\delta(x,y)[u,v]+\delta(x,v)[u,y]+\delta(x,v)[u,v]+\delta(u,y)[x,v]=0, \quad \text{for all} \quad x,y\in R. \tag{22}$$

Taking $x=u$ in (22) and making use of (19), we obtain $2\delta(u,y)[u,v]=0$, for all $y\in R$. Since R is 2-torsion free and using the given assumption, we have

$$\delta(u,y)=0, \quad \text{for all} \quad y\in R. \tag{23}$$

Again replacing y by v in (21) and using (19), we get $\delta(x,v)[u,v]=0$, for all $x,y\in R$. Since $[u,v]$ is not a zero divisor, we get

$$\delta(x,v)=0, \quad \text{for all} \quad x\in R. \tag{24}$$

Thus by (22), (23) and (24) we get $\delta(x,y)[u,v]=0$, for all $x,y\in R$. This implies that $\delta(x,y)=0$, for all $x,y\in R$ i.e., $F(xy)=F(x)y+bxd(y)$, for all $x,y\in R$, which completes the proof.

Proof of Theorem 1.2. By the given hypothesis, we have

$$F(xy^*x)=F(x)y^*x+bxd(y^*)x+bxy^*d(x), \quad \text{for all} \quad x,y\in R. \tag{25}$$

On substituting x by $x+z$ and on solving, we have

$$\begin{aligned} F(xy^*z+zy^*x)&=F(x)y^*z+F(z)y^*x+bxd(y^*)z \\ &\quad +bzd(y^*)x+bxy^*d(z)+bzy^*d(x), \end{aligned} \tag{26}$$

for all $x,y,z\in R$. Replacing z by x^2 in (26), we get

$$\begin{aligned} F(xy^*x^2+x^2y^*x)&=F(x)y^*x^2+F(x^2)y^*x+bxd(y^*)x^2+bx^2d(y^*)x \\ &\quad +bxy^*d(x)x+bxy^*xd(x)+bx^2y^*d(x), \quad \text{for all} \quad x,y\in R. \end{aligned} \tag{27}$$

Taking y as x^*y+yx^* in (25), we obtain

$$\begin{aligned} F(xy^*x^2+x^2y^*x)&=F(x)y^*x^2+F(x)xy^*x+bxd(y^*)x^2 \\ &\quad +bxy^*d(x)x+bxd(x)y^*x \\ &\quad +bx^2d(y^*)x+bxy^*xd(x)+bx^2y^*d(x), \quad \text{for all} \quad x,y\in R. \end{aligned} \tag{28}$$

On comparing (27) and (28), we arrive at

$$(F(x^2)-F(x)x-bxd(x))y^*x=0, \quad \text{for all} \quad x,y\in R. \tag{29}$$

This can be further written as

$$\phi(x)y^*x=0, \quad \text{for all} \quad x,y\in R, \tag{30}$$

where $\phi(x) = F(x^2) - F(x)x - bxd(x)$. Replacing y by y^*x^* in (30), we get $\phi(x)xyx = 0$. Now replacing y by $z\phi(x)$, we get $\phi(x)xz\phi(x)x = 0$, for all $x, z \in R$. Using the semiprimeness of R, we obtain

$$\phi(x)x = 0, \quad \text{for all } \ x \in R. \tag{31}$$

Taking $x = x + y$, we get

$$\phi(x)y + \beta(x,y)x + \phi(y)x + \beta(x,y)y = 0, \quad \text{for all } \ x, y \in R, \tag{32}$$

where $\beta(x,y) = F(xy + yx) - F(x)y - F(y)x - bxd(y) - byd(x)$. Replacing x by $-x$ in (32) and making use of (32), we get

$$2(\phi(x)y + \beta(x,y)x) = 0, \quad \text{for all } \ x, y \in R.$$

Since R is 2-torsion free, we arrive at

$$\phi(x)y + \beta(x,y) = 0, \quad \text{for all } \ x, y \in R. \tag{33}$$

Multiplying (33) by $\phi(x)$ on the right side, we get

$$\phi(x)y\phi(x) + \beta(x,y)x\phi(x) = 0, \quad \text{for all } \ x, y \in R. \tag{34}$$

Taking $y = y^*$ in (30), we get $\phi(x)yx = 0$, for all $x, y \in R$. This further implies that $x\phi(x)yx\phi(x) = 0$, for all $x, y \in R$. Thus by the semiprimeness of R, we get $x\phi(x) = 0$, for all $x \in R$. Using this in (34), we obtain $\phi(x) = 0$, for all $x \in R$. Hence $F(x^2) = F(x)x + bxd(x)$, for all $x \in R$. Now, following on similar lines as after (18), we get the required result.

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