The structure of (θ_1, θ_2)-isoclinism classes of groups

Ali Mohammad Z. Mehrjerdi
Department of Mathematics
Mashhad Branch
Islamic Azad University
Mashhad
Iran
and
Department of Pure Mathematics
Farhangian University
Iran
alizare.math@gmail.com

Mohammad Reza R. Moghaddam*
Department of Mathematics
Khayyam University
Mashhad
Iran
and
Department of Pure Mathematics
Centre of Excellence in Analysis on Algebraic Structures (CEAAS)
Ferdowsi University of Mashhad
P.O.Box 1159, Mashhad, 91775
Iran
m.r.moghaddam@khayyam.ac.ir
rezam@ferdowsi.um.ac.ir

Mohammad Amin Rostamyari
Department of Mathematics
Khayyam University
Mashhad
Iran
m.a.rostamyari@khayyam.ac.ir

Abstract. In 1940, Philip Hall introduced the concept of isoclinism among all groups, and it is generalized to a more general notion called isologism. This concept is isoclinism with respect to a given variety of groups. The equivalence relation of isologism partitions the class of all groups into families.

In this article, we introduce a kind of isoclinism with respect to θ-centre, $Z^\theta(G)$, and right θ-commutator subgroup $K^\theta(G)$, for some automorphism θ of the group G, and we investigate some of its properties.

Keywords: right and left θ-commutator, central automorphism, absolute centre, θ-centre, θ-commutator subgroup.

∗. Corresponding author

1. Introduction

One of the most classical notions playing a fundamental role in classifying groups is the notion of isomorphism among all groups. However, in many cases this notion is too strong. For instance, in the case of finite groups one would like to consider abelian groups being classified as a one family.

P. Hall in 1940 introduced the concept of isoclinism [3]. This is an equivalence relation on the class of all groups, which is weaker than isomorphism and such that all abelian groups fall into one equivalence class, namely they are equivalent to the trivial group. Roughly speaking two groups are isoclinic if and only if there exists an isomorphism between their central quotients, which induces an isomorphism between their commutator subgroups.

In [2], the second and third authors introduced and studied the concept of *right* and *left α-commutator*, as follows:

Definition 1.1. *For arbitrary elements x and y in a given group G and $\alpha \in \mathrm{Aut}(G)$, we say x and y commute under the automorphism α whenever $yx = xy^\alpha$ or $y^{\phi_x} = y^\alpha$, where ϕ_x is the inner automorphism induced by x.*

Moreover, $[x, y]_\alpha = x^{-1}y^{-1}xy^\alpha$ is called right α-commutator of x and y. Also, ${}_\alpha[x, y] = (x^{-1})^\alpha y^{-1}xy$ is called left α-commutator of x and y.

For $n \geqslant 3$, we may define inductively right and left α-commutator of weight n as follows:

$$[x_1, x_2, \cdots, x_n]_\alpha = [[x_1, x_2, \cdots, x_{n-1}]_\alpha, x_n]_\alpha,$$
$${}_\alpha[x_1, x_2, \cdots, x_n] = {}_\alpha[{}_\alpha[x_1, x_2, \cdots, x_{n-1}], x_n],$$

for all $x_i \in G$ and $1 \leqslant i \leqslant n$. It is clear that, if α is the identity automorphism of G or x_i's are in $C_G(\alpha)$ then we have ordinary commutator $[x_1, x_2, \cdots, x_n]$ of weight n, where

$$C_G(\alpha) = \{x \in G \mid [x, \alpha] = x^{-1}x^\alpha = x^{-1}\alpha(x) = 1\},$$

is the centralizer of α in G.

For a given group G and automorphisms α and β in $\mathrm{Aut}(G)$ we consider, $\alpha^\beta = \beta^{-1}\alpha\beta$. The following lemma is very useful in our further investigations.

Lemma 1.1. *Let x, y and z be elements of a group G and $\alpha, \beta \in \mathrm{Aut}(G)$. Then the following identities hold:*

(i) $[x, y]_\alpha = [x, y][y, \alpha]$;

(ii) $[x, x]_\alpha = [x, \alpha]$;

(iii) $([x, y]_\alpha)^\alpha = [x^\alpha, y^\alpha]_\alpha$;

(iv) $[x, y^{-1}]_\alpha = [x, y]_\alpha^{-(y^\alpha)^{-1}}$;

(v) $([x, y]_{\alpha^\beta})^\beta = [x^\beta, y^\beta]_\alpha$;

(*vi*) $[xy, z]_\alpha = ([x, z]_\alpha)^y [y, z^\alpha]$;

(*vii*) $[x, yz]_\alpha = [x, z]_\alpha ([x, y]_\alpha)^{z^\alpha}$;

(*viii*) $([[x, y^{-1}]_\alpha, z]_\alpha)^{y^\alpha} = [x, y^{-1}, z]^y [z^y, \alpha]$.

Proof. All parts follow using the definition of right α-commutator and the above notation. □

One can easily see that $[x, y]_\alpha^{-1} = {}_\alpha[y, x]$, hence we may state similar relations, as the above lemma, for left α-commutator. Here we work with right α-commutators in the rest of article.

Remark 1. For an automorphism α of a group G, the action $\psi : G \times G \to G$ given by $\psi(x, y) = y^{-1}xy^\alpha$, partitions the group G into α-conjugacy classes, which we denote it by x_α^G, i.e.

$$x_\alpha^G = \{y^{-1}xy^\alpha \mid y \in G\}.$$

Note that the number of α-conjugacy classes is equal with the number of ordinary conjugacy classes, which are invariant under α and it is also equal to the number of irreducible characters which are invariant under α (see [7, 9] for more details).

Now, we recall that the following subgroup is called *α-centre* of the group G

$$Z^\alpha(G) = \bigcap_{x \in G} C_G^\alpha(x) = \{y \in G \mid [x, y]_\alpha = 1, \forall x \in G\},$$

where $C_G^\alpha(x) = \{y \in G \mid [x, y]_\alpha = 1\}$ and $|x_\alpha^G| = [G : C_G^\alpha(x)]$ (see [1, 9] for more information). One can easily check that $Z^\alpha(G) = Z(G) \cap C_G(\alpha)$ and so $Z^\alpha(G) \trianglelefteq G$. Also, $L(G) = \bigcap_{\alpha \in \mathrm{Aut}(G)} Z^\alpha(G)$, and hence

$$L(G) \subseteq Z^\alpha(G) \subsetneqq Z(G),$$

as $[x, y]_\alpha = [x, y][y, \alpha] = 1$, for all $x \in G$ and $y \in Z^\alpha(G)$, while $[y, x]_\alpha = [y, x][x, \alpha] \neq 1$.

Now, one may define *α-commutator* subgroup of G as follows

$$K^\alpha(G) = \langle [x, y]_\alpha \mid x, y \in G \rangle.$$

Clearly, Lemma 1.2 (i) and (ii) imply that $G' \subseteq K^\alpha(G) \subseteq K(G)$, where $K(G)$ is the autocommutator subgroup of G (see [4]). Note that, Lemma 1.2 (iii) implies that $K^\alpha(G)$ is an α-invariant subgroup of G.

Let α be an automorphism of the group G and for any $x \in G$, then α is called *class preserving* if $x^\alpha \in x^G$. Clearly, if α is class preserving automorphism of a group G then $x^\alpha = x^g$ for some $g \in G$, and hence $[g, x]_\alpha = 1$. This topic has been studied by many authors (see [5, 6, 10], for more details).

2. Main results

Clearly α-commutator subgroup $K^{\alpha}(G)$ of an abelian group G is always normal in G, for any automorphism $\alpha \in \mathrm{Aut}(G)$. In the following, we show that $K^{\alpha}(G)$ is a normal subgroup in a non abelian group G, for any automorphism α of G.

One may define the action of a group G on $\mathrm{Aut}(G)$ given by $\alpha^g = \alpha^{\varphi_g} = \varphi_{g^{-1}} \circ \alpha \circ \varphi_g$ and the action of $\mathrm{Aut}(G)$ on G given by $g^{\alpha} = \alpha(g)$, for all $g \in G$, $\alpha \in \mathrm{Aut}(G)$ and $\varphi_g \in \mathrm{Inn}(G)$ (see also [8]).

Theorem 2.1. *Let α be any automorphism of a given group G, then $K^{\alpha}(G)$ is always a normal subgroup of G.*

Proof. Take α to be any automorphism of the group G and for any $x, y, g \in G$, Lemma 1.2 (i) implies that

$$\begin{aligned} [x,y]_{\alpha}^g = [x,y]^g[y,\alpha]^g &= [x^g, y^g][y,\alpha]^g \\ &= [x^g, y^g]g^{-1}y^{-1}\alpha(y)g \\ &= [x^g, y^g][g,y]_{\alpha}[\alpha(y), g] \in K^{\alpha}(G). \end{aligned}$$

Hence, $K^{\alpha}(G) \trianglelefteq G$. □

Here the notion of (θ_1, θ_2)-isoclinism between two groups is introduced and we study some of its properties.

Definition 2.1. *Let G_1 and G_2 be two groups, θ_1 and θ_2 be suitable automorphisms of G_1 and G_2, respectively, such that there exist $\alpha : \frac{G_1}{Z^{\theta_1}(G_1)} \to \frac{G_2}{Z^{\theta_2}(G_2)}$ and $\beta : K^{\theta_1}(G_1) \to K^{\theta_2}(G_2)$ so that the following diagram is commutative*

$$\frac{G_1}{Z^{\theta_1}(G_1)} \times \frac{G_1}{Z^{\theta_1}(G_1)} \xrightarrow{\alpha\times\alpha} \frac{G_2}{Z^{\theta_2}(G_2)} \times \frac{G_2}{Z^{\theta_2}(G_2)}$$

$$\begin{array}{ccc} (g_1Z^{\theta_1}(G_1), g_2Z^{\theta_1}(G_1)) & \longmapsto & (g_1'Z^{\theta_2}(G_2), g_2'Z^{\theta_2}(G_2)) \\ \rho\downarrow & & \downarrow\psi \\ [g_1, g_2]_{\theta_1} & \longrightarrow & [g_1', g_2']_{\theta_2} \end{array}$$

$$K^{\theta_1}(G_1) \xrightarrow{\beta} K^{\theta_2}(G_2)$$

where $\alpha(g_iZ^{\theta_1}(G_1)) = g_i'Z^{\theta_2}(G_2)$ and $g_i' \in \alpha(g_i)Z^{\theta_2}(G_2)$ for every $g_i \in G_1$ and $g_i' \in G_2$ $(i = 1, 2)$. Moreover, $\beta([g_1, g_2]_{\theta_1}) = [g_1', g_2']_{\theta_2}$, i.e. the commutative diagram is compatible.

Then the pair (α, β) is called (θ_1, θ_2)-isoclinism from G_1 to G_2 and denoted by $G_1 \overset{(\theta_1,\theta_2)}{\sim} G_2$. In this case, G_1 and G_2 are called (θ_1, θ_2)-isoclinic.

Observe that the above notion generalizes the concept of isoclinism (see [3]). In fact, if θ_1 and θ_2 are identities, then the above definition is the concept of ordinary isoclinism of groups.

Example 2.2. (i) There are no automorphisms θ_1 and θ_2 of the groups $\mathbb{Z}_4$ and $\mathbb{Z}_6$, respectively, such that $\mathbb{Z}_4 \overset{(\theta_1,\theta_2)}{\sim} \mathbb{Z}_6$. As, for any automorphisms θ_1 and θ_2 of $\mathbb{Z}_4$ and $\mathbb{Z}_6$, we have $|\frac{\mathbb{Z}_4}{Z^{\theta_1}(\mathbb{Z}_4)}| = 2$ and $|\frac{\mathbb{Z}_6}{Z^{\theta_2}(\mathbb{Z}_6)}| = 3$.

(ii) Consider the cyclic groups $\mathbb{Z}_4(x)$ and $\mathbb{Z}_8(y)$ of orders 4 and 8 with generators x and y, and take the automorphisms $\theta_1 : x \mapsto x^3$ and $\theta_2 : y \mapsto y^5$. Then one can easily check that $Z^{\theta_1}(\mathbb{Z}_4(x)) = \{1, x^2\}$, $K^{\theta_1}(\mathbb{Z}_4(x)) = \{1, x^2\}$. Also, $Z^{\theta_2}(\mathbb{Z}_8(y)) = \{1, y^2, y^4, y^6\}$ and $K^{\theta_2}(\mathbb{Z}_8(y)) = \{1, y^4\}$. Now, it is easy to verify that $\frac{\mathbb{Z}_4(x)}{Z^{\theta_1}(\mathbb{Z}_4(x))} \cong \frac{\mathbb{Z}_8(y)}{Z^{\theta_2}(\mathbb{Z}_8(y))}$ and $K^{\theta_1}(\mathbb{Z}_4(x)) \cong K^{\theta_2}(\mathbb{Z}_8(y))$, hence $\mathbb{Z}_4(x) \overset{(\theta_1,\theta_2)}{\sim} \mathbb{Z}_8(y)$.

(iii) Assume $D_8 = \langle x, y : x^4 = y^2 = 1, x^y = x^{-1}\rangle$ and $Q_8 = \langle x, y : x^4 = 1, x^2 = y^2, x^y = x^{-1}\rangle$ are Dihedral and Quaternion groups of orders 8. Also, take the automorphisms θ_1 and θ_2 both given by: $x \mapsto x^3, y \mapsto x^2y$ of D_8 and Q_8, respectively. One can calculate that $Z^{\theta_1}(D_8) \cong Z^{\theta_2}(Q_8) = \{1, x^2\}$ and $K^{\theta_1}(D_8) \cong K^{\theta_2}(Q_8) = \{1, x^2\}$. Hence, $D_8 \overset{(\theta_1,\theta_2)}{\sim} Q_8$.

Now, the question arises that; "In what cases, there exist some suitable automorphisms θ_1 and θ_2 in arbitrary finite cyclic groups, which force them to be (θ_1, θ_2)-isoclinic?"

In the following, we give a complete answer to the above question, for finite cyclic groups.

Remark 2. (i) Let $\mathbb{Z}_m(x_1)$ and $\mathbb{Z}_n(x_2)$ be cyclic groups with a common divisor p^r of m and n, where p is a prime number and $r \geqslant 2$.

Assume $m=p^r p_2^{r_2} \cdots p_s^{r_s}$ and $n=p^r q_2^{r'_2} \cdots q_t^{r'_t}$. Clearly $\theta_1 : x_1 \mapsto x_1^{p^{r-1}p_2^{r_2}\cdots p_s^{r_s}+1}$ and $\theta_2 : x_2 \mapsto x_2^{p^{r-1}q_2^{r'_2}\cdots q_t^{r'_t}+1}$ are automorphisms of cyclic groups of orders m and n, respectively.

As m and $\frac{m}{p} + 1$ are co-prime, then $K^{\theta_1}(\mathbb{Z}_m) = \langle x_1^{\frac{m}{p}} \rangle$ and

$$Z^{\theta_1}(\mathbb{Z}_m) = \{x_1^p, x_1^{2p}, \cdots, x_1^{\frac{m}{p}p} = 1\}.$$

The same argument implies that $|K^{\theta_1}(\mathbb{Z}_m)|=|\frac{\mathbb{Z}_m}{Z^{\theta_1}(\mathbb{Z}_m)}|=|K^{\theta_2}(\mathbb{Z}_n)| = |\frac{\mathbb{Z}_n}{Z^{\theta_2}(\mathbb{Z}_n)}| = p$, and hence $\mathbb{Z}_m(x_1) \overset{(\theta_1,\theta_2)}{\sim} \mathbb{Z}_n(x_2)$. Such as $\mathbb{Z}_{12}$ and $\mathbb{Z}_{20}$.

(ii) If the orders of cyclic groups are with different prime decomposition factors, then they can not be (θ_1, θ_2)-isoclinic, for any automorphisms θ_1 and θ_2. Such as $\mathbb{Z}_6$ and $\mathbb{Z}_{35}$.

(iii) Consider the cyclic groups $\mathbb{Z}_{m_1}(x_1)$ and $\mathbb{Z}_{m_2}(x_2)$ with $(m_1, m_2) = p$. Clearly, if $(\frac{k_i m_i}{p}, m_i) = 1$, for $i = 1, 2$ and $1 \leq k_i < p$, then

$$\theta : x_i \mapsto x_i^{\frac{k_i m_i}{p}+1},$$

is an automorphism of the cyclic group $\mathbb{Z}_{m_i}(x_i)$. Now,

$$K^{\theta_i}(\mathbb{Z}_{m_i}(x_i)) = \langle [x_i,\theta_i] = x_i^{-1}x^{\theta_i}\rangle = \langle x^{\frac{k_i m_i}{p}}\rangle,$$

which is a cyclic group of order p, for $i=1,2$; i.e. $K^{\theta_1}(\mathbb{Z}_{m_1}(x_1)) \cong K^{\theta_2}(\mathbb{Z}_{m_2}(x_2))$.

On the other hand, we have

$$Z^{\theta_i}(\mathbb{Z}_{m_i}(x_i)) = \{x_i^r \mid [x_i, x_i^r]_{\theta_i} = [x_i^r,\theta_i] = x_i^{\frac{rk_i m_i}{p}} = 1\}.$$

Hence, $p|r$ and $|Z^{\theta_i}(\mathbb{Z}_{m_i}(x_i))| = \frac{m_i}{p}$, which implies that

$$\frac{\mathbb{Z}_{m_1}(x_1)}{Z^{\theta_1}(\mathbb{Z}_{m_1}(x_1))} \cong \frac{\mathbb{Z}_{m_2}(x_2)}{Z^{\theta_2}(\mathbb{Z}_{m_2}(x_2))}$$

and so $\mathbb{Z}_{m_1}(x_1) \overset{(\theta_1,\theta_2)}{\sim} \mathbb{Z}_{m_2}(x_2)$.

Using the technique of Remark 2 (*iii*), we have the following examples.

Example 2.3. (*i*) Consider $\mathbb{Z}_{15}(x_1)$ and $\mathbb{Z}_{21}(x_2)$. One notes that $(\frac{15}{3}+1,15)\neq 1$, while $(\frac{30}{3}+1,15)=1$. Also, $(\frac{21}{3}+1,21)=1$. Hence, $\theta_1 : x_1 \mapsto x_1^{11}$ and $\theta_2 : x_2 \mapsto x_2^{8}$ are automorphisms of $\mathbb{Z}_{15}(x_1)$ and $\mathbb{Z}_{21}(x_2)$, respectively. These automorphisms guaranty that $\mathbb{Z}_{15}(x_1) \overset{(\theta_1,\theta_2)}{\sim} \mathbb{Z}_{21}(x_2)$.

(*ii*) Consider $\mathbb{Z}_6(x_1)$ and $\mathbb{Z}_{15}(x_2)$. we observe that $(\frac{6}{3}+1,6)\neq 1$ and $(\frac{15}{3}+1,15)\neq 1$, while $(\frac{12}{3}+1,6)=1$ and $(\frac{30}{3}+1,15)=1$. Hence, the automorphisms $\theta_1 : x_1 \mapsto x_1^5$ and $\theta_2 : x_2 \mapsto x_2^{11}$ will do the job and so $\mathbb{Z}_6(x_1) \overset{(\theta_1,\theta_2)}{\sim} \mathbb{Z}_{15}(x_2)$.

(*iii*) $\mathbb{Z}_6 \overset{(\theta_1,\theta_2)}{\nsim} \mathbb{Z}_{10}$, since there are no suitable automorphisms, as the above.

In case of 1-isoclinism, P. Hall [3] showed that in every family there exists a group S with the property that $Z(S) \subseteq \gamma_2(S)$. Such a group is called *stemgroup*. In the case of finite groups, the stemgroups in a given family are characterized by the fact that they are just the groups of smallest order in that family. They play an essential role in classification problem.

Clearly, (θ_i,θ_j)-isoclinism forms an equivalence relation on the pair of groups. Hence, such relation partitions the group into equivalence classes, or family of (θ_i,θ_j)-isoclinism of groups.

Here, we introduce α-stemgroup in the case of (θ_i,θ_j)-isoclinism of groups.

Definition 2.2. *Let $\mathcal{C}$ be a family of (θ_i,θ_j)-isoclinism of groups. If there exists a group S with the property that $G_r \overset{(\theta_i,\theta_j)}{\sim} S$ and $Z^\alpha(S) \subseteq K^\alpha(S)$, where $G_r \in \mathcal{C}$ and α is an automorphism of the group S. Then such a group S is said to be α-stem group. In finite case, the α-stem group S has the least possible order among all other groups in the family.*

Example 2.4. Consider a famiy of finite cyclic groups, which their orders have a common prime divisor p^r, where $r \geqslant 2$. Then it is clear that $\alpha : x \mapsto x^{p+1}$ is an automorphism of $\mathbb{Z}_{p^2}(x)$ and $K^\alpha(\mathbb{Z}_{p^2}(x)) = Z^\alpha(\mathbb{Z}_{p^2}(x)) = \langle x^p \rangle$. Therefore Remark 2 and Definition 2.5 imply that $\mathbb{Z}_{p^2}(x)$ is the α-stem group.

The above example shows that in a family of finite cyclic groups, for which p^r, $(r \geqslant 2)$, is a common divisor of their orders, the cyclic group $\mathbb{Z}_{p^2}$ is α-stem group with smallest order in such family of groups.

Our final result gives a useful criterion for two groups to be (θ_1, θ_2)-isoclinic.

Proposition 2.1. *Let $A \leq Z^{\theta_1}(G)$ and $B \leq Z^{\theta_2}(H)$. Also, assume $\alpha : G/A \to H/B$ and $\beta : K^{\theta_1}(G) \to K^{\theta_2}(H)$ are isomorphisms so that $\alpha(gZ^{\theta_1}(G)) = hZ^{\theta_2}(H)$ and $\beta([g, g']_{\theta_1}) = [h, h']_{\theta_2}$, for all $g, g' \in G$ and $h, h' \in H$. Then G and H are (θ_1, θ_2)-isoclinic.*

Proof. We must show that α induces an isomorphism from $G/Z^{\theta_1}(G)$ onto $H/Z^{\theta_2}(H)$.

Since $G/Z^{\theta_1}(G) \simeq (G/A)/(Z^{\theta_1}(G)/A)$ and $H/Z^{\theta_1}(H) \simeq (H/B)/(Z^{\theta_2}(H)/B)$, it is sufficient to show that $\alpha(Z^{\theta_1}(G)/A) = Z^{\theta_2}(H)/B$. So for any $g \in Z^{\theta_1}(G)$, we have $[g', g]_{\theta_1} = 1$ for all g' in G. Then there exists h in H such that $[h', h]_{\theta_2} = 1$ for all $h' \in H$, as β is an isomorphism. Thus $h \in Z^{\theta_2}(H)$ and $\alpha(Z^{\theta_1}(G)/A) \leq Z^{\theta_2}(H)/B$.

On the other hand, if $h_0 \in Z^{\theta_2}(H)$ is an arbitrary element, then there exists an element $g_0 \in G$ such that $\alpha(g_0A) = h_0B$, as α is surjective. Now, $\beta([g, g_0]_{\theta_1}) = [h, h_0]_{\theta_2}$ and hence $g_0 \in Z^{\theta_1}(G)$, as β is isomorphism. Therefore $\alpha(Z^{\theta_1}(G)/A) \geq Z^{\theta_2}(H)/B$, which completes the proof. □

The following corollary is obtained by replacing $G = H_1$, $H = H_2$, $A = Z^{\theta_1}(G_1)$ and $B = Z^{\theta_2}(G_2)$ in the above proposition.

Corollary 2.1. *Let (α, β) be (θ_1, θ_2)-isoclinism between two groups G_1 and G_2 and H_i be a characteristic subgroup of G_i for $i = 1, 2$. If $Z^{\theta_1}(G_1) \leq H_1 \leq G_1$ and $\alpha(H_1/Z^{\theta_1}(G_1)) = H_2/Z^{\theta_2}(G_2)$, then H_1 and H_2 are also (θ_1, θ_2)-isoclinic, where $Z^{\theta_2}(G_2) \leq H_2 \leq G_2$.*

Acknowledgements

The authors would like to thank the referee for his or her helpful suggestions, which made some improvements of the article.

References

[1] P.N. Achar, *Generalized conjugacy classes*, Mathematical Sciences Technical Reports (MSTR), 1997, 71-01.

[2] M. Haghparast, M. R. R. Moghaddam, M. A. Rostamyari, *Some results of left and right α-commutators of groups*, Asian-Eur. J. Math., Vol. 14, 2050148 (2021), doi.org/10.1142/S179355712050148X.

[3] P. Hall, *The classification of prime power groups*, J. Reine Angew. Math., 182 (1940), 130-141.

[4] P.V. Hegarty, *Autocommutator subgroups of finite groups*, J. Algebra, 190 (1997), 556-562.

[5] E.I. Khukhro, *On finite soluble groups with almost fixed-point-free automorphisms of non-coprime order*, Siberian Mathematical Journal, 56 (2015), 541-548.

[6] E.I. Khukhro, *Automorphisms of finite p-groups admitting a partition*, Algebra Logika, 51 (2012), 392-411.

[7] E. Jabara, *Automorphisms with finite reidemeister number in residually finite groups*, J. Algebra, 320 (2008), 3671-3679.

[8] M.R.R. Moghadam, M.J. Sadeghifard, *Non-abelian tensor analogues of 2-auto Engel groups*, Bull. Korean Math. Soc., 52 (2015), 1097-1105.

[9] R. Ree, *On generalized conjugate classes in a finite groups*, Illinois J. Math., 3 (1959), 440-444.

[10] M.K. Yadav, *Class preserving automorphisms of finite p-groups*, J. Lond. Math. Soc., 75 (2007), 755-772.

Accepted: September 21, 2021