Fixed point theorem for (ϕ, F)-contraction on C^*-algebra valued partial metric spaces

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Abstract. Recently, a new type of mapping called (ϕ, F)- contraction was introduced in the literature as a generalization of the concepts of contractive mappings. This present article extends the new notion in C^*-algebra valued partial metric spaces and establishing the existence and uniqueness of fixed point for them. Non-trivial examples are further provided to support the hypotheses of our results.

Keywords: fixed point, C^*-algebra valued partial metric spaces, C^*-algebra valued partial (ϕ, F)- contraction.

1. Introduction

Metric fixed point theory has its roots in methods from the late 19th century, when successive approximations were used to establish the existence and uniqueness of solutions to equations, and especially differential equations. This approach is particularly associated with the work of Picard, although it was Stefan Banach who in 1922 in [2] developed the ideas involved in an abstract setting.

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Banach's contraction principle is a fundamental result in fixed point theory. Due to its importance, several authors have obtained many interesting extensions and generalizations (see [1, 3, 5, 10, 12]).

Many generalizations of the concept of metric spaces are defined and some fixed point theorems were proved in these spaces. In particular, C^*-algebra valued metric spaces were introduced by Ma et al. [11] as a generalization of metric spaces they proved certain fixed point theorems, by giving the definition of C^*-algebra valued contractive mapping analogous to Banach contraction principle.

In this paper, inspired by the work done in [6, 9], we introduce the notion of C^*-algebra valued partial (ϕ, F)-contraction and establish some new fixed point theorems for mappings in the setting of complete C^*-algebra valued partial metric spaces. Moreover, an illustrative example is presented to support the obtained results.

2. Preliminaries

Throughout this paper, we denote $\mathbb{A}$ an unital C^*-algebra with linear involution $*$, such that for all $x, y \in \mathbb{A}$,

$$(xy)^* = y^* x^*, \ \ x^{**} = x.$$

We call an element $x \in \mathbb{A}$ a positive element, denote it by $x \succeq \theta$ if $x \in \mathbb{A}_h = \{x \in \mathbb{A} : x = x^*\}$ and $\sigma(x) \subset \mathbb{R}_+$, where $\sigma(x)$ is the spectrum of x.

Using positive element, we can define a partial ordering $\preceq$ on $\mathbb{A}_h$ as follows:

$$x \preceq y \text{ if and only if } y - x \succeq \theta,$$

where θ means the zero element in $\mathbb{A}$.

We denote the set $\{x \in \mathbb{A} : x \succeq \theta\}$ by $\mathbb{A}_+$ and $|x| = (x^* x)^{\frac{1}{2}}$.

Remark 2.1. When $\mathbb{A}$ is an unital C^*-algebra, then for any $x \in \mathbb{A}_+$ we have

$$x \preceq I \Longleftrightarrow \|x\| \leq 1.$$

Definition 2.2 ([8])**.** Let X be a non-empty set. A mapping $p : X \times X \to \mathbb{A}$ is called a C^*-algebra valued metric on X if the following conditions are satisfied:

(i) $\theta \preceq p(x, y)$ for all $x, y \in X$ and $p(x, x) = p(y, y) = p(x, y)$ if and only if $x = y$

(ii) $p(x, y) = p(y, x)$ for all $x, y \in X$;

(iii) $p(x, x) \preceq p(x, y)$ for all $x, y \in X$

(iv) $p(x, y) \preceq p(x, z) + p(z, y) - p(z, z)$ for all $x, y, z \in X$.

Then $(X, \mathbb{A}_+, p)$ is called a C^*-algebra valued partial metric space.

If we take $\mathbb{A} = \mathbb{R}$, then the new notion of C^*-algebra valued partial metric space becomes equivalent to the definition of the real partial metric space.

Example 2.3. Let$X = [0, 1]$ and $x \in \mathbb{A}$ be a nonzero element.

Define $p(s, t) = \max\{1 + s, 1 + t\}xx^*$. Then we can easily show that $p : X \times X \to \mathbb{A}$ is a C^*-algebra valued partial metric.

Example 2.4. Let $X = [0, 1]$ and $\mathbb{A} = \mathbb{R}^2$ with the usual norm is a real Banach space.

Let $p : X \times X \to \mathbb{R}^2$ be given as follows:

$$p(x, y) = (|x - y|, |x - y|).$$

Then, $(X, \mathbb{R}^2, p)$ is a complete C^*-algebra valued partial metric.

Definition 2.5 ([7])**.** Let $(X, \mathbb{A}, p)$ be a C^*-algebra valued partial metric space. Suppose that $\{x_n\} \subset X$ and $x \in X$.

(1) $\{x_n\} \subset X$ converges to x whenever for every $\varepsilon > 0$ there is a natural number N such that for all $n > N$,

$$\|p(x_n, x) - p(x, x)\| \leq \varepsilon.$$

We denote it by

$$\lim_{n\to\infty} p(x_n, x) - p(x, x) = \theta.$$

(2) $\{x_n\}$ is a partial Cauchy sequence respect to $\mathbb{A}$, whenever $\varepsilon > 0$ there is a natural number N such that

$$\begin{aligned}&(p(x_n, x_m) - \frac{1}{2}p(x_n, x_n) - \frac{1}{2}p(x_m, x_m))((p(x_n, x_m)\\&- \frac{1}{2}p(x_n, x_n) - \frac{1}{2}p(x_m, x_m))^* \preceq \varepsilon^2,\end{aligned}$$

for all $n, m > N$;

(3) $(X, \mathbb{A}_+, p)$ is said to be complete with respect to $\mathbb{A}$ if every partial Cauchy sequence with respect to $\mathbb{A}$ converges to a point x in X such that

$$\lim_{n\to\infty} (p(x_n, x) - \frac{1}{2}p(x_n, x_n) - \frac{1}{2}p(x, x)) = \theta.$$

From given C^*-algebra-valued partial metric, we can obtain a C^*- algebra-valued metric. Put

$$p^s(x, y) = 2p(x, y) - p(x, x) - p(y, y).$$

Then, p^s is a C^*- algebra -valued metric.

Lemma 2.6 ([7])**.** *Let $(X, \mathbb{A}, p)$ be a C^*- algebra- valued partial metric space.*

(1) *$\{x_n\}$ is a partial Cauchy sequence in $(X, \mathbb{A}, p)$ if and only if it is Cauchy in the C^*- algebra -valued metric $(X, \mathbb{A}, p^s)$.*

(2) *A C^*- algebra- valued partial metric space $(X, \mathbb{A}, p)$ is complete if and only if C^*- algebra- valued metric space $(X, \mathbb{A}, p^s)$ is complete. Furthermore,*

$$\lim_{n\to\infty} p^s(x_n, x) = \theta \Leftrightarrow \lim_{n\to\infty}(2p(x_n, x) - p(x_n, x_n) - p(x, x)) = \theta$$

or

$$\lim_{n\to\infty} p^s(x_n, x) = \theta \Leftrightarrow \lim_{n\to\infty} p(x_n, x) - p(x_n, x_n) = \theta,\ \lim_{n\to\infty} p(x_n, x) - p(x, x) = \theta.$$

Lemma 2.7 ([7])**.** *Assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in a C^*- algebra valued partial metric space $(X, \mathbb{A}, p)$. Then*

$$\lim_{n\to\infty}(p(x_n, y_n) - p(x_n, x_n)) = p(x, y) - p(x, x)$$

and

$$\lim_{n\to\infty}(p(x_n, y_n) - p(y_n, y_n)) = p(x, y) - p(y, y).$$

Definition 2.8 ([14])**.** Let the function $\phi : A^+ \to A^+$ be positive if having the following constraints:

(i) ϕ is continuous and nondecreasing;

(ii) $\phi(a) = \theta$ if and only if $a = \theta$;

(iii) $\lim_{n\to\infty} \phi^n(a) = \theta$.

Definition 2.9 ([14])**.** Suppose that A and B are C^*-algebra. A mapping $\phi : A \to B$ is said to be C^*- homomorphism if:

(i) $\phi(ax + by) = a\phi(x) + b\phi(y)$ for all $a, b \in \mathbb{C}$ and $x, y \in A$;

(ii) $\phi(xy) = \phi(x)\phi(y)$ for all $x, y \in A$;

(iii) $\phi(x^*) = \phi(x)^*$ for all $x \in A$;

(iv) ϕ maps the unit in A to the unit in B.

Definition 2.10 ([14])**.** Let A and B be C^*-algebra spaces and let $\phi : A \to B$ be a homomorphism, then ϕ is called an $*-$ homomorphism if it is one to one $*-$ homomorphism. A C^*-algebra A is $*-$isomorphic to a C^*-algebra B if there exists $*-$ isomorphism of A onto B.

Lemma 2.11 ([13])**.** *Let A and B be C^*-algebra spaces and $\phi : A \to B$ is a C^*- homomorphism for all $x \in A$ we have*

$$\sigma(\phi(x)) \subset \sigma(x),\ \ \|\phi(x)\| \leq \|\phi\|.$$

Corollary 2.12 ([14]). *Every C^*- homomorphism is bounded.*

Corollary 2.13 ([14]). *Suppose that ϕ is C^*- isomorphism from A to B, then $\sigma(\phi(x)) = \sigma(x)$ and $\|\phi(x)\| = \|\phi\|$ for all $x \in A$.*

Lemma 2.14 ([14]). *Every $*-$ homomorphism is positive.*

The following definition was given by D. Wardowski in [4].

Definition 2.15 ([10]). Let $F\colon \mathbb{R}_+ \to \mathbb{R}$ and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a mapping satisfying:

(i) F is strictly increasing, for $\alpha, \beta \in \mathbb{R}_+$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$.

(ii) For each sequence $\{x_n\}_{n\in\mathbb{N}}$ of positive numbers

$$\lim_{n\to 0} x_n = 0, \quad \text{if and only if} \quad \lim_{n\to\infty} F(x_n) = -\infty.$$

(iii) $\liminf_{s\to\alpha^+} \phi(s) > 0$, for all $s > 0$.

(iv) There exists $k \in]0, 1[$ such that $\lim_{x\to 0} x^k F(x) = 0$. A mapping $T : X \to X$ is said to be an (ϕ, F)-contraction in partial metric space if

$$\forall x, y \in X; p(Tx, Ty) \geq 0 \Rightarrow \phi(p(x, y)) + F(p(Tx, Ty) \leqslant F(p(x, y)).$$

Definition 2.16 ([10]). Let (X, p) be a complete partial metric space. A mapping $T : X \to X$ is called an $(\phi, F)-$ contraction on (X, p) if there exists F and ϕ defined in Definition 2.15 such that

$$(p(Tx, Ty) > 0 \Rightarrow F(p(Tx, Ty) + \phi(p(x, y)) \leqslant F(p(x, y)),$$

for all $x, y \in X$ for which $Tx \neq Ty$.

Theorem 2.17. *Let (X, p) be a complete partial metric space and $T : X \to X$ be an $(\phi, F)-$ contraction. Then T has a unique fixed point.*

3. Main result

Aspired by Wardowski in [10], we introduce the notion of $(\phi, F) - C^*$-valued partial contraction.

Definition 3.1. Let $F\colon \mathbb{A}_+ \to \mathbb{A}_+$ a function satisfying:

(i) F is continuous and nondecreasing.

(ii) $F(T) = \theta$ if and only if $T = \theta$.

1. A mapping $T : X \to X$ is said to be a (ϕ, F) C^* valued partial contraction of type (I) if there exists $\phi : \mathbb{A}_+ \to \mathbb{A}_+$ an $*-$ homomorphism such that

   (1) $\forall x, y \in X; (p(Tx, Ty) \succeq \theta \Rightarrow F(p(Tx, Ty)) + \phi(p(x, y)) \preceq F(p(x, y)).$

2. A mapping $T : X \to X$ is said to be a (ϕ, F) C^* valued partial contraction of type (II) if there exists $\phi : \mathbb{A}_+ \to \mathbb{A}_+$ an $*-$ homomorphism satisfying:
   (a) $\phi(a) \prec a$ for $a \in \mathbb{A}_+$.
   (b) Either $\phi(a) \preceq p(x, y)$ or $p(x, y) \preceq \phi(a)$, where $a \in \mathbb{A}_+$ and $x, y \in X$.
   (c) $F(a) \prec \phi(a)$. Such that
   $$(p(Tx, Ty) \succeq \theta \Rightarrow F(p(Tx, Ty) + \phi(p(x, y)) \preceq F(M(x, y)),$$
   where $M(x, y) = a_1 p(x, y) + a_2[p(Tx, y) + p(Ty, x)] + a_3[p(Tx, x) + p(Ty, y)]$, with $a_1, a_2, a_3 \geq 0$, $a_1 + 2a_2 + 2a_3 \leq 1$.
3. T is said to be (ϕ, F)- Kannan-type C^*- valued contraction if there exist ϕ satisfy (a), (b) and (c) such that $p(Tx, Ty) \succeq \theta$, we have
$$F(p(Tx, Ty) + \phi(p(x, y)) \preceq F(\frac{p(x, Tx) + p(y, Ty)}{2}).$$
4. T is said to be (ϕ, F)- Reich-type C^*- valued partial contraction if there exist ϕ satisfy (a), (b) and (c) such that $p(Tx, Ty) \succeq \theta$, we have
$$F(p(Tx, Ty) + \phi(p(x, y)) \preceq F(\frac{p(x, y) + p(x, Tx) + p(y, Ty)}{3}).$$

Example 3.2. Let $X = [0, 1]$ and $\mathbb{A} = \mathbb{R}^2$ Then $\mathbb{A}$ is a C^*- algebra with norm $\|.\| : \mathbb{A} \to \mathbb{R}$ defined by
$$\|(x, y)\| = (x^2 + y^2)^{\frac{1}{2}}.$$
Define a C^*- algebra valued partial metric $p : X \times X \to \mathbb{A}$ on X by $p(x, y) = (x + y, x + y)$, with ordering on $\mathbb{A}$ by
$$(a, b) \preceq (c, d) \ \Leftrightarrow a \leq c \ \text{ and } \ b \leq d.$$
A mapping $T : X \to X$ given by $Tx = x - \frac{1}{2}x^2$ is continuous with respect to $\mathbb{A}$. Let $F : \mathbb{A}_+ \to \mathbb{A}_+$. Defined by $F(x, y) = (x, y)$.

It is clear that F satisfies (i) and (ii).

We have $F(p(Tx, Ty)) = p(Tx, Ty) = (x - \frac{1}{2}x^2 + y - \frac{1}{2}y^2, x - \frac{1}{2}x^2 + y - \frac{1}{2}y^2)$ and $F(p(Tx, Ty)) - F(p(x, y)) \leq -(\frac{1}{4}(x + y)^2, \frac{1}{4}(x + y)^2)$. Therefore, T is a C^*-algebra valued partial F-contraction with $\phi(x, y) = (\frac{1}{4}(x + y)^2, \frac{1}{4}(x + y)^2)$.

Example 3.3. Let $X = [0, 1] \cup \{2, 3, 4, ...\}$ and $\mathbb{A} = \mathbb{C}$ with a norm $|| z ||=| z |$ be a C^*- algebra. We define $\mathbb{C}^+ = \{z = (x, y) \in \mathbb{C};\ x = Re(z) \geq 0,\ y = Im(z) \geq 0\}$.

The partial order $\leq$ with respect to the C^*- algebra $\mathbb{C}$ is the partial order in $\mathbb{C}$, $z_1 \leq z_2$ if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$ for any two elements z_1, z_2 in $\mathbb{C}$.

Let $p : X \times X \to \mathbb{C}$

$$p(x,y) = \begin{cases} (|x-y|\,, |x-y|), & \text{if } x, y \in [0,1], x \neq y, \\ (x+y, x+y), & \text{if at least one of } x \text{ or } y \notin [0,1] \text{ and } x \neq y, \\ (0,0), & \text{if } x = y. \end{cases}$$

Then, $(X, \mathbb{A}, p)$ be a complete C^*-algebra valued metric space.

Let $F : \mathbb{C}^+ \to \mathbb{C}$ be defined as

$$F(t) = \begin{cases} t, & \text{if } t \in [0,1], \\ t^2, & \text{if } t > 1. \end{cases}$$

It is clear that F satisfies (i) and (ii) Let $T : X \to X$ be defined as

$$T(x) = \begin{cases} x - \frac{1}{2}x^2, & \text{if } x \in [0,1], \\ x - 1, & \text{if } x \in \{2, 3, 4, ...\}. \end{cases}$$

Without loss of generality, we assume that $x > y$ and discuss the following cases:
Case 1. $(x \in [0;1])$. Then

$$\begin{aligned} F(p(Tx,Ty)) &= ((x - \frac{1}{2}x^2) - (y - \frac{1}{2}y^2), (x - \frac{1}{2}x^2) - (y - \frac{1}{2}y^2)) \\ &= ((x-y) - \frac{1}{2}(x-y)(x+y), (x-y) - \frac{1}{2}(x-y)(x+y)) \\ &\leq ((x-y) - \frac{1}{2}((x-y))^2, (x-y) - \frac{1}{2}((x-y))^2) \\ &= p(x,y) - \frac{1}{2}(p(x,y))^2 \\ &= F(p(x,y)) - \frac{1}{2}(p(x,y))^2. \end{aligned}$$

Then, there exists ϕ such $\phi(x,y) = \frac{1}{2}(p(x,y))^2$ and $\forall x, y \in X$, $p(Tx,Ty) \geq 0 \Rightarrow \phi(x,y) + F(p(Tx,Ty)) \leq F(p(x,y))$.
Case 2. $(x \in \{3, 4, ...\})$, then

$$p(Tx,Ty) = p(x-1, y - \frac{1}{2}y^2) \text{ if } y \in [0,1]$$

or

$$p(Tx,Ty) = (x - 1 + y - \frac{1}{2}y^2, x - 1 + y - \frac{1}{2}y^2) \leq (x + y - 1, x + y - 1),$$
$$p(Tx,Ty) = p(x-1, y-1) \text{ if } y \in \{2, 3, 4, ...\}$$

or

$$p(Tx,Ty) = (x + y - 2, x + y - 2) < (x + y - 1, x + y - 1).$$

Consequently,

$$\begin{aligned}F(p(Tx,Ty)) &= (p(Tx,Ty))^2 \leq ((x+y-1)^2,(x+y-1)^2)\\ &< ((x+y-1)(x+y+1),(x+y-1)(x+y+1))\\ &= ((x+y)^2-1,(x+y)^2-1) < ((x+y)^2-\frac{1}{2},(x+y)^2-\frac{1}{2})\\ &= F(p(x,y))-\frac{1}{2}.\end{aligned}$$

Case 3. $(x=2)$, then $y\in[0,1]$, $Tx=1$ and

$$p(Tx,Ty)=(1-(y-\frac{1}{2}y^2),1-(y-\frac{1}{2}y^2)).$$

So, we have $F(p(Tx,Ty))\leq F(1)=1$. Again, $p(x,y)=(2+y,2+y)$. So, $1=F(p(Tx,Ty))\leq F(p(x,y))-\frac{1}{2}$.

Theorem 3.4. *Let $(X,\mathbb{A},p)$ be a complete C^*-algebra valued partial metric space and let $T:X\to X$ be a (ϕ,F) C^*- valued partial contraction mapping of type (I). Then T has a unique fixed point $x*\in X$ and for every $x_0\in X$ a sequence $\{T^nx_0\}_{n\in\mathbb{N}}$ is convergent to $x*$.*

Proof. First, let us observe that T has at most one fixed point. Indeed if

$$x_1^*;x_2^*\in X,\ \ Tx_1^*=x_1^*\neq x_2^*=Tx_2^*$$

then, we get

$$\phi(p(x,y))\preceq F(p(x_1^*;x_2^*))-F(p(Tx_1^*;Tx_2^*))=\theta$$

which is a contradiction.

In order to show that thas a fixed point let $x_0\in X$ be arbitrary and fixed we define a sequence $\{x_n\}_{n\in\mathbb{N}}\subset X$; $x_{n+1}=Tx_n, n=0,1,2,\dots$ denote $p_n=p(x_{n+1};x_n), n=0,1,2,\dots$ if there exists $n_0\in\mathbb{N}$ for which $x_{n_0+1}=x_{n_0}$ then $Tx_{n_0}=x_{n_0}$ and the proof is finished.

Suppose now, that $x_{n+1}\neq x_n$, for every $n\in X$ then $p_n\succ\theta$, for all $n\in\mathbb{N}$ and using (1) the following holds, for every $n\in\mathbb{N}$

$$F(p_n)\preceq F(p_{n-1})-\phi(p_{n-1})\prec F(p_{n-1}). \tag{2}$$

Hence, F is non decreasing and so the sequence (p_n) is monotonically decreasing in $\mathbb{A}_+$. So, there exists $\theta\preceq t\in\mathbb{A}_+$ such that

$$p(x_n,x_{n+1})\to t \text{ as } n\to\infty.$$

From (2) we obtain $\lim_{n\to\infty}F(p_n)=\theta$ that together with (ii) gives

$$\lim_{n\to\infty}p_n=\theta. \tag{3}$$

Now, we shall show that $\{x_n\}$ is a Cauchy sequence in $(X, \mathbb{A}, p)$. By Lemma 2.6 it is sufficient To prove that $\{x_n\}$ is a Cauchy sequence in $(X, \mathbb{A}, p^s)$, we have proved $\lim_{n\to\infty} p_n = \theta$. Keeping in mind that $\theta \preceq p(x_n, x_n) \preceq p(x_n, x_{n+1})$, we get

$$\lim_{n\to\infty} p(x_n, x_n) = \theta. \tag{4}$$

Also, $\theta \preceq p(x_{n+1}, x_{n+1}) \preceq p(x_n, x_{n+1})$ this implies

$$\lim_{n\to\infty} p(x_{n+1}, x_{n+1}) = \theta. \tag{5}$$

Assume that $\{x_n\}$ is not a Cauchy sequence in $(X, \mathbb{A}, p^s)$. Then, exist $\varepsilon > 0$ and subsequences $\{x_{m_k}\}$ and $\{x_{n_k}\}$ with $n_k > m_k > k$ such that

$$\|p^s(x_{m_k}, x_{n_k})\| > \varepsilon.$$

Now, corresponding to m_k , we can choose n_k such that it is the smallest integer with $n_k > m_k$ and satisfying above inequality. Hence, $\|p^s(x_{m_k}, x_{n_k-1})\| \leq \varepsilon$. So, we have

$$\begin{aligned} \varepsilon \leq \|p^s(x_{m_k}, x_{n_k})\| &\leq \|p^s(x_{m_k}, x_{n_k-1} + p^s(x_{n_k-1}, x_{n_k}) - p^s(x_{n_k-1}, x_{n_k-1})\| \\ &\leq \|p^s(x_{m_k}, x_{n_k-1})\| + \|p^s(x_{n_k-1}, x_{n_k})\| \leq \varepsilon + \|p^s(x_{n_k-1}, x_{n_k})\|. \end{aligned} \tag{6}$$

We know that

$$p^s(x_{n_k-1}, x_{n_k}) = 2p(x_{n_k-1}, x_{n_k}) - p^s(x_{n_k-1}, x_{n_k-1}) - p^s(x_{n_k}, x_{n_k}). \tag{7}$$

Using (3), (4), (5) and (7) we have

$$\varepsilon \preceq \lim_{k\to\infty} \|p^s(x_{n_k-1}, x_{n_k})\| < \varepsilon + \theta.$$

This implies

$$\lim_{k\to\infty} \|p^s(x_{m_k}, x_{n_k})\| = \varepsilon. \tag{8}$$

Again,

$$\begin{aligned} \|p^s(x_{n_k}, x_{m_k})\| &\leq \|p^s(x_{n_k}, x_{n_k-1}) + p^s(x_{n_k-1}, x_{m_k}) - p^s(x_{n_k-1}, x_{n_k-1})\| \\ &\leq \|p^s(x_{n_k}, x_{n_k-1})\| + \|p^s(x_{n_k-1}, x_{m_k})\| \\ &\leq \|p^s(x_{n_k}, x_{n_k-1})\| + \|p^s(x_{n_k-1}, x_{m_k-1}) \\ &+ p^s(x_{m_k-1}, x_{m_k}) - p^s(x_{m_k-1}, x_{m_k-1})\| \\ &\leq \|p^s(x_{n_k}, x_{n_k-1})\| + \|p^s(x_{n_k-1}, x_{m_k-1})\| + \|p^s(x_{m_k-1}, x_{m_k})\|. \end{aligned} \tag{9}$$

Also,

$$\begin{aligned} \|p^s(x_{n_k-1}, x_{m_k-1})\| &\leq \|p^s(x_{n_k-1}, x_{n_k}) + p^s(x_{n_k}, x_{m_k-1}) - p^s(x_{n_k}, x_{m_k})\| \\ &\leq \|p^s(x_{n_k-1}, x_{n_k})\| + \|p^s(x_{n_k}, x_{m_k-1})\| \\ &\leq \|p^s(x_{n_k-1}, x_{nk})\| + \|p^s(x_{n_k}, x_{m_k}) \\ &+ p^s(x_{m_k}, x_{mk-1}) - p^s(x_{m_k}, x_{m_k})\| \\ &\leq \|p^s(x_{n_k-1}, x_{nk})\| + \|p^s(x_{n_k}, x_{m_k})\| + \|p^s(x_{m_k}, x_{m_k-1})\|. \end{aligned} \tag{10}$$

Letting $k \to \infty$ in (9) and (10) and using (4) and (8) we have

$$\lim_{k\to\infty} \|p^s(x_{n_k-1}, x_{m_k-1})\| = \varepsilon.$$

Thus,

$$\begin{aligned}\lim_{k\to\infty} \|p(x_{n_k-1}, x_{m_k-1})\| &= \frac{1}{2}\lim_{k\to\infty} \|2p^s(x_{n_k-1}, x_{m_k-1}) \\ &\quad - p^s(x_{n_k-1}, x_{n_k-1}) - p^s(x_{m_k-1}, x_{m_k-1})\| \\ &= \frac{1}{2}\lim_{k\to\infty} \|p^s(x_{n_k-1}, x_{m_k-1})\| = \frac{\varepsilon}{2}.\end{aligned}$$

Since $p(x_{n_k-1}, x_{m_k-1})$, $p(x_{n_k}, x_{m_k}) \in \mathbb{A}_+$ and

$$\lim_{k\to\infty} \|p(x_{n_k-1}, x_{m_k-1})\| = \lim_{k\to\infty} \|p(x_{n_k}, x_{m_k})\| = \frac{\varepsilon}{2}$$

there is exists $s \in \mathbb{A}_+$ with $\|s\| = \varepsilon$ such that

$$\lim_{k\to\infty} \|p(x_{n_k-1}, x_{m_k-1})\| = \lim_{k\to\infty} \|p(x_{n_k}, x_{m_k})\| = s \tag{11}$$

by (7) we have

$$F(s) = \lim_{k\to\infty} F(p(x_{n_k}, x_{m_k})) \preceq \lim_{k\to\infty} F(p(x_{n_{k-1}}, x_{m_{k-1}})).$$

Therefore, $F(s) \prec F(s)$. Thus, $F(s) = \theta$ and so $s = \theta$ which is a contradiction. Hence, $\{x_n\}$ is a Cauchy sequence in $(X, \mathbb{A}, p^s)$ and so $\{x_n\}$ is partially Cauchy in the complete C^*-algebra-valued partial metric space $(X, \mathbb{A}, p)$. Hence, there exist $z \in X$ such that $\lim_{n\to\infty} p(x_n, z) - p(x_n, x_n) = \theta$.

Using (4), we get $\lim_{n\to\infty} p(x_n, z) = \theta$ and thus $p(z, z) = \theta$

Now, we shall show that z is fixed point of T. Using (1), we get $\theta \preceq F(p(Tz, Tz)) \prec F(p(z, z)) = F(\theta) = \theta$. Thus, $F(p(Tz, Tz)) = \theta$ which implies $p(Tv, Tv) = \theta$. On the other hand, $F(p(x_n, Tz)) \prec F(p(x_{n-1}, z))$.

Letting $n \to \infty$ and using the concept of continuity of the function of T. We have $p(z, Tz) = \theta$. Hence, by Definition 2.2, we have $p(z, z) = p(Tz, Tz) = p(z, Tz) = \theta$, then $Tz = z$, which completes the proof. $\square$

Example 3.5. Considering all cases in Example 3.3, we conclude that inequality (1) remains valid for F and T constructed as above and consequently by an application of Theorem 3.3, T has a unique fixed point. It is seen that 0 is the unique fixed point of T.

Theorem 3.6. *Let $(X, \mathbb{A}, p)$ be a complete C^*-algebra valued partial metric space.*

Let $T : X \to X$ be a (ϕ, F) of type (II), i.e, there exist F and ϕ two $*-$homomorphisms such that for any $x, y \in X$ we have

$$p(Tx, Ty) \succeq \theta \Rightarrow F(p(Tx, Ty)) + \phi(p(x, y)) \preceq F(M(x, y)),$$

where $M(x, y) = a_1 p(x, y) + a_2[p(Tx, y) + p(Ty, x)] + a_3[p(Tx, x) + p(Ty, y)]$, with $a_1, a_2, a_3 \geq 0$,$a_1 + 2a_2 + 2a_3 \leq 1$. Then, T has a fixed point.

Proof. Let $x_0 \in X$ and define $x_1 = Tx_0, x_2 = Tx_1, ..., x_n = Tx_{n-1}$. We have

$$\begin{aligned} F(p(x_{n+2}, x_{n+1})) &= F(p(Tx_{n+1}, Tx_n)) \preceq F(M(x_{n+1}, x_n)) + \phi(p(x_{n+1}, x_n)) \\ &= F(a_1 p(x_{n+1}, x_n) + a_2[p(x_{n+2}, x_n) + p(x_{n+1}, x_{n+1})] \\ &\quad + a_3[p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n)]) - \phi(p(x_{n+1}, x_n)). \end{aligned}$$

Then, we have

$$\begin{aligned} F(p(x_{n+2}, x_{n+1})) \preceq\ & F(a_1 p(x_{n+1}, x_n) + a_2[p(x_{n+2}, x_n) + p(x_{n+1}, x_{n+1})] \\ & + a_3[p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n)]). \end{aligned}$$

Using the strongly monotone property of F, we have

$$\begin{aligned} p(x_{n+2}, x_{n+1}) \preceq\ & a_1 p(x_{n+1}, x_n) + a_2[p(x_{n+2}, x_n) + p(x_{n+1}, x_{n+1})] \\ & + a_3[p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n)]. \end{aligned}$$

That is

$$(1 - a_2 - a_3)p(Tx_{n+1}, Tx_n) \preceq (a_1 + a_2 + a_3)p(x_{n+1}, x_n).$$

Therefore,

$$p(x_{n+2}, x_{n+1}) \preceq \frac{a_1 + a_2 + a_3}{1 - a_2 - a_3} p(x_{n+1}, x_n).$$

Which implies that

$$p(x_{n+2}, x_{n+1}) \preceq p(x_{n+1}, x_n).$$

Since

$$\frac{a_1 + a_2 + a_3}{1 - a_2 - a_3} < 1.$$

Therefore, $\{p(x_{n+1}, x_n)\}$ is monotone decreasing sequence. There exists, $u \in \mathbb{A}_+$ such that $d(x_{n+1}, x_n) \to u$ as $n \to \infty$. Taking $n \to \infty$ in

$$\begin{aligned} F(p(x_{n+2}, x_{n+1})) \preceq\ & F(a_1 p(x_{n+1}, x_n) + a_2[p(x_{n+2}, x_n) + p(x_{n+1}, x_{n+1})] \\ & + a_3[p(x_{n+2}, x_{n+1}) + p(x_{n+1}, x_n)]). \end{aligned}$$

Using the continuities of F and ϕ, we have

$$F(u) \preceq F((a_1 + 2a_2 + 2a_3)u) - \phi(u)$$

which implies that $F(u) \preceq F(u) - \phi(u)$ since $a_1 + 2a_2 + 2a_3 \leq 1$ and F is strongly monotonic increasing wich is a contradiction unless $u = \theta$. Hence,

$$p(x_{n+1}, x_n) \to \theta \text{ as } n \to \infty. \tag{12}$$

Next, we show that $\{x_n\}$ is a Cauchy sequence.

If $\{x_n\}$ is not a Cauchy sequence then there exists $c \in \mathbb{A}$ such that $\forall n_0 \in \mathbb{N}$, $\exists n, m \in \mathbb{N}$ with $n > m \geq n_0, F(c) \preceq p(x_n, x_m)$. Therefore, there exists sequences $\{m_k\}$ and $\{n_k\}$ in $\mathbb{N}$ such that for all positive integers k, $n_k > m_k > k$ and $p(x_{n_{(k)}}, x_{m_{(k)}}) \succeq \phi(c)$ and $p(x_{n_{(k)}-1}, x_{m_{(k)}}) \preceq \phi(c)$ then

$$\phi(c) \preceq p(x_{n_{(k)}}, x_{m_{(k)}}) \preceq p(x_{n_{(k)}}, x_{n_{(k)}-1}) + p(x_{n_{(k)}-1}, x_{m_{(k)}})$$

that is $\phi(c) \preceq p(x_{n_{(k)}}, x_{m_{(k)}}) \preceq p(x_{n_{(k)}}, x_{n_{(k)}-1}) + \phi(c)$ letting $k \to \infty$, we have

$$\lim_{k\to\infty} p(x_{n_{(k)}}, x_{m_{(k)}}) = \phi(c) \tag{13}$$

again

$$p(x_{n_{(k)}}, x_{m_{(k)}}) \preceq [p(x_{n_{(k)}}, x_{n_{(k)}+1}) + p(x_{n_{(k)}+1}, x_{m_{(k)}}) - p(x_{n_{(k)}+1}, x_{n_{(k)}+1})]$$

and

$$p(x_{n_{(k)}+1}, x_{m_{(k)}+1}) \preceq [p(x_{n_{(k)}+1}, x_{n_{(k)}}) + p(x_{n_{(k)}}, x_{m_{(k)}+1}) - p(x_{n_{(k)}}, x_{n_{(k)}})]$$

letting $k \to \infty$ in above inequalities, we have

$$\lim_{k\to\infty} p(x_{n_{(k)+1}}, x_{m_{(k)+1}}) = \phi(c). \tag{14}$$

Again

$$p(x_{n_{(k)}}, x_{m_{(k)}+1}) \preceq [p(x_{n_{(k)}}, x_{m_{(k)}}) + p(x_{m_{(k)}}, x_{m_{(k)}+1})]$$

and

$$p(x_{n_{(k)}+1}, x_{m_{(k)}}) \preceq [p(x_{n_{(k)}+1}, x_{n_{(k)}}) + p(x_{n_{(k)}}, x_{m_{(k)}}) - p(x_{n_{(k)}}, x_{n_{(k)}})].$$

Further,

$$p(x_{n_{(k)}+1}, x_{m_{(k)}}) \preceq [p(x_{n_{(k)}+1}, x_{n_{(k)}}) + p(x_{n_{(k)}}, x_{m_{(k)}})]$$

and

$$p(x_{n_{(k)}}, x_{m_{(k)}}) \preceq [p(x_{n_{(k)}}, x_{n_{(k)}+1}) + p(x_{n_{(k)}+1}, x_{m_{(k)}})].$$

Letting $k \to \infty$ in the above four inequalities we have

$$\lim_{k\to\infty} p(x_{n_{(k)}}, x_{m_{(k)}+1}) = \phi(c), \tag{15}$$

$$\lim_{k\to\infty} p(x_{n_{(k)}+1}, x_{m_{(k)}}) = \phi(c). \tag{16}$$

Using (12), (13), (15) and (16) we have

$$\begin{aligned}\lim_{k\to\infty} M(x_{n_{(k)}}, x_{m_{(k)}}) = \lim_{k\to\infty} & a_1 p(x_{n_{(k)}}, x_{m_{(k)}}) + a_2[p(x_{n_{(k)}}, x_{m_{(k)}}) \\ & + p(x_{m_{(k)}}, x_{m_{(k)}+1})] + a_3[p(x_{n_{(k)}}, x_{m_{(k)}+1}) \\ & + p(x_{m_{(k)}}, x_{n_{(k)}+1})] = (a_1 + 2a_2)\phi(c).\end{aligned} \tag{17}$$

Clearly $x_{m_k} \preceq x_{n_k}$. Putting $x = x_{n_{(k)}}$,$y = x_{m_{(k)}}$

$$F(p(x_{n_{(k)}+1}, x_{m_{(k)}+1})) = F(p(Tx_{n_{(k)}}, Tx_{m_{(k)}})) \preceq F(M(x_{n_{(k)}}, x_{m_{(k)}})) - \phi(x_{n_{(k)}}, x_{m_{(k)}}).$$

Letting $k \to \infty$ in the above inequality using (13), (14) and (17) and the continuities of F and ϕ, we have $F(\phi(c)) \preceq F((a_1 + 2a_2)\phi(c)) - \phi(\phi(c))$ that is $F(\phi(c)) \preceq F(\phi(c)) - \phi(\phi(c))$, (since $(a_1 + 2a_2) < 1$) and F is strongly monotonic increasing. Which a contradiction by virtue of a proprety of ϕ. Hence $\{x_n\}$ is a Cauchy sequence. From the completeness of X, there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Since T is continuous and $Tx_n \to Tz$ as $n \to \infty$ that is $\lim_{n\to\infty} x_{n+1} = Tz$, that is $z = Tz$. Hence, z is a fixed point of T. □

Example 3.7. Let $X = [0, 1]$ and $\mathbb{A} = \mathbb{C}$ with a norm $\| z \| = | z |$ be a C^*- algebra.

We define $\mathbb{C}^+ = \{z = (x, y) \in \mathbb{C}; x = \mathcal{R}\mathrm{e}\,(z) \geq 0, y = \mathcal{I}\mathrm{m}\,(z) \geq 0\}$.

The partial order $\leq$ with respect to the C^*- algebra $\mathbb{C}$ is the partial order in $\mathbb{C}$, $z_1 \leq z_2$ if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$ for any two elements z_1, z_2 in $\mathbb{C}$.

Let $p : X \times X \to \mathbb{C}$. Suppose that $p(x, y) = (| x - y |, | x - y |)$ for $x, y \in X$. Then, $(X, \mathbb{C}, p)$ is a C^*- algebra valued metric space with the required properties of Theorem 3.6.

Let $F, \phi : \mathbb{C}^+ \to \mathbb{C}^+$ such that they can defined as follows: for $t = (x, y) \in \mathbb{C}^+$,

$$F(t) = \begin{cases} (x, y), & \text{if } x \leq 1, y \leq 1, \\ (x^2, y), & \text{if } x > 1, y \leq 1, \\ (x, y^2), & \text{if } x \leq 1, y > 1, \\ (x^2, y^2), & \text{if } x > 1,\ y > 1 \end{cases}$$

and for $s = (s_1, s_2) \in \mathbb{C}^+$ with $v = \min\{s_1, s_2\}$,

$$\phi = \begin{cases} \left(\dfrac{v^2}{2}, \dfrac{v^2}{2}\right), & \text{if } v \leq 1, \\ \left(\dfrac{1}{2}, \dfrac{1}{2}\right), & \text{if } v > 1. \end{cases}$$

Then, F and ϕ have the propreties mentioned in Definitions 2.8 and 2.9. Let $T : X \to X$ be defined as follows: $T(x) = \begin{cases} 0, & \text{if } 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{16}, & \text{if } \frac{1}{2} < x \leq 1. \end{cases}$ Then, T has the required properties mentioned in Theorem 3.6.

Let $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{8}$ and $a_3 = \frac{1}{8}$. It can be verified that $F(p(Tx, Ty)) \preceq F(M(x, y)) - \phi(p(x, y))$, for all $x, y \in X$ with $y \preceq x$ the conditions of Theorem 3.6 are satisfied. Here, it is seen that 0 is a fixed point of T.

Theorem 3.8. *Let $(X, \mathbb{A}, p)$ be a complete C^*-algebra valued partial metric space. Let $T : X \to X$ be a (ϕ, F)- Kannan-type C^*- valued partial contraction.Then T has a unique fixed point.*

Proof. Since T is a (ϕ, F)- Kannan-type C^*- valued partial contraction, then exist F and ϕ such that

$$F(p(Tx, Ty)) + \phi(p(x, y)) \preceq F(\frac{p(x, Tx) + p(y, Ty)}{2}) \preceq F(M(x, y)),$$

where $M(x, y) = a_1 p(x, y) + a_2[p(Tx, y) + p(Ty, x)] + a_3[p(Tx, x) + p(Ty, y)]$ with $a_1 = 0$, $a_2 = 0$ and $a_3 = \frac{1}{2}$. As in the proof of Theorem 3.6, T has a fixed point. □

Theorem 3.9. *Let $(X, \mathbb{A}, p)$ be a complete C^*-algebra valued partial metric space. Let $T : X \to X$ be a (ϕ, F)- Reich-type C^*- valued partial contraction. Then T has a unique fixed point.*

Proof. By taking $a_1 = \frac{1}{3}$, $a_2 = 0$ and $a_3 = \frac{1}{3}$, we have

$$F(p(Tx, Ty)) + \phi(p(x, y)) \preceq F(M(x, y)) = F(\frac{p(x, y) + p(x, Tx) + p(y, Ty)}{3}).$$

As in the proof of Theorem 3.6 T has a fixed point. □

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