On the semiring variety generated by $B^0, (B^0)^*, N_2, T_2, Z_2, W_2$

Aifa Wang
School of Science
Chongqing University of Technology
Chongqing, 400054
P.R. China
wangaf1980@163.com

Lili Wang*
School of Science
Chongqing University of Technology
Chongqing, 400054
P.R. China
wllaf@163.com

Abstract. In this paper, we study the semiring variety generated by $B^0, (B^0)^*, N_2, T_2, Z_2, W_2$. We prove that this variety is finitely based and prove that the lattice of subvarieties of this variety is a distributive lattice of order 1014. Moreover, we deduce this variety is hereditarily finite based.

Keywords: semiring, variety, lattice, identity, hereditarily finite based.

1. Introduction

A semiring is an algebra with two associative binary operations $+, \cdot$, in which $+$ is commutative and $\cdot$ distributive over $+$ from the left and right. Such an algebra is a common generalization of both rings and distributive lattice. It has broad applications in information science and theoretical computer science (see [5], [6]). In this paper, we shall investigate some small-order semirings which will paly a crucial role in subsequent follows.

The semiring B with addition and multiplication table

+	a	b	c
a	a	b	c
b	b	b	b
c	c	b	c

·	a	b	c
a	a	a	a
b	b	b	b
c	a	b	c

Eight 2-element semirings with addition and multiplication table

*. Corresponding author

Semiring	+	·	Semiring	+	·
L_2	$\begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix}$	$\begin{matrix} 0 & 0 \\ 1 & 1 \end{matrix}$	R_2	$\begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix}$	$\begin{matrix} 0 & 1 \\ 0 & 1 \end{matrix}$
M_2	$\begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix}$	$\begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix}$	D_2	$\begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix}$	$\begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}$
N_2	$\begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix}$	$\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$	T_2	$\begin{matrix} 0 & 1 \\ 1 & 1 \end{matrix}$	$\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}$
Z_2	$\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$	W_2	$\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$	$\begin{matrix} 0 & 0 \\ 0 & 1 \end{matrix}$

For any semiring S, we denote by S^0 the semiring obtained from S by adding an extra element 0 and where $a = 0 + a = a + 0, 0 = 0a = a0$ for every $a \in S$. For any semiring S, S^* will denote the (multiplicative) left-right dual of S. Pastijn et al. [4, 10] studied the semiring variety generated by B^0 and $(B^0)^*$. They showed that the lattice of subvarieties of this variety is distributive and contains 78 varieties precisely. Moreover, each of these is finitely based. It is obvious that the variety generated by L_2, R_2, M_2, D_2 is properly contained in the variety generated by B^0 and $(B^0)^*$, that is, $\mathbf{HSP}(L_2, R_2, M_2, D_2) \subsetneqq \mathbf{HSP}(B^0, (B^0)^*)$. In 2016, Shao and Ren [14] studied the variety generated by $L_2, R_2, M_2, D_2, N_2, T_2$. They showed that the lattice of subvarieties of this variety is distributive and contains 64 varieties precisely. Moreover, each of these is finitely based. Recently, Ren and Zeng [13] studied the variety generated by $B^0, (B^0)^*, N_2, T_2$. They proved that the lattice of subvarieties of this variety is a distributive lattice of order 312 and that each subvarieties of its is finitely based. It is easy to check

$$\mathbf{HSP}(B^0, (B^0)^*) \subsetneqq \mathbf{HSP}(B^0, (B^0)^*, N_2, T_2) \subsetneqq \mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2).$$

So, the variety $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2)$ is a proper subvariety of the variety $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$. The main purpose of this paper is to study the variety $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$. We show that the lattice of subvarieties of this variety is a distributive lattice of order 1014. Moreover, we show this variety is hereditarily finitely based.

2. Preliminaries

Let $\mathbf{V}$ be a variety, $\mathcal{L}(\mathbf{V})$ denote the lattice of subvarieties of $\mathbf{V}$ and $\mathrm{Id}_{\mathbf{V}}(X)$ denote the set of all identities defining $\mathbf{V}$. If $\mathbf{V}$ can be defined by finitely many identities, then we say that $\mathbf{V}$ is *finitely based*. In other words, $\mathbf{V}$ is said to be finitely based if there exists a finite subset Σ of $\mathrm{Id}_{\mathbf{V}}(X)$ such that for any $p \approx q \in \mathrm{Id}_{\mathbf{V}}(X)$, $p \approx q$ can be derived from Σ, i.e., $\Sigma \vdash p \approx q$. Otherwise, we say that $\mathbf{V}$ is *nonfinitely based*. Recall that $\mathbf{V}$ is said to be *heredirarily finitely based* if all members of $\mathcal{L}(\mathbf{V})$ are finitely based. If a variety $\mathbf{V}$ is finitely based and $\mathcal{L}(\mathbf{V})$ is a finite lattice, then $\mathbf{V}$ is hereditarily finite based (see [13]).

The variety of all semirings is denoted by **SR**. A semiring is called an *additively idempotent semiring* (ai-semiring for short) if its additive reduct is a semilattice, i.e., a commutative idempotent semigroup. It is also called a *semilattice-order semigroup* (see [3], [8], [12]). The variety of all ai-semirings is denoted by **AI**. Let X denote a fixed countably infinite set of variables and X^+ the free semigroup on X. A *semiring identity* (**SR**-identity for short) is an expression of the form $u \approx v$, where u and v are terms with $u = u_1 + \cdots + u_k$, $v = v_1 + \cdots + v_\ell$, where $u_i, v_j \in X^+$ (An ai-semiring identity denoted by **AI**-identity). Let $\underline{k}$ denote the set $\{1, 2, \ldots, k\}$ for a positive integer k, Σ be a set of identities which include the identities determining **AI** and $u \approx v$ be an **AI**-identity. It is easy to check that the ai-semiring variety defined by $u \approx v$ coincides with the ai-semiring variety defined by the identities $u \approx u + v_j, v \approx v + u_i, i \in \underline{k}, j \in \underline{\ell}$. Thus, in order to show that $u \approx v$ is derivable from Σ, we only need to show that $u \approx u + v_j, v \approx v + u_i, i \in \underline{k}, j \in \underline{\ell}$ can be derived from Σ.

To solve the word problem for the variety $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$, the following notions and notations are needed. Let q be an element of X^+. Then

- the *head* of q, denoted by $h(q)$, is the first variable occurring in q;
- the *tail* of q, denoted by $t(q)$, is the last variable occurring in q;
- the *content* of q, denoted by $c(q)$, is the set of variables occurring in q;
- the *length* of q, denoted by $|q|$, is the number of variables occurring in q counting multiplicities;
- the *initial part* of q, denoted by $i(q)$, is the word obtained from q by retaining only the first occurrence of each variable;
- the *final part* of q, denoted by $f(q)$, is the word obtained from q by retaining only the last occurrence of each variable.

The basis for each one of N_2, T_2, Z_2, W_2 can be found from [11] (See Table 1).

Table 1. Bases for N_2, T_2, Z_2, W_2

Semiring	Basis
N_2	$xy \approx zt, x + x^2 \approx x$
T_2	$xy \approx zt, x + x^2 \approx x^2$
Z_2	$x + y \approx z + u, xy \approx x + y$
W_2	$x + y \approx z + u, x^2 \approx x, xy \approx yx$

By [14, Lemma 1.1] and the bases for Z_2, W_2 in the above Table 1, we have

Lemma 2.1. *Let $u \approx v$ be a nontrivial* **SR**-*identity, where $u = u_1+u_2+\cdots+u_m$, $v = v_1+ v_2+\cdots+v_n$, $u_i, v_j \in X^+$, $i \in \underline{m}, j \in \underline{n}$. Then:*

(*i*) *N_2 satisfies $u \approx v$ if and only if $\{u_i \in u \,|\, |u_i| = 1\} = \{v_i \in v \,|\, |v_i| = 1\}$;*

(*ii*) *T_2 satisfies $u \approx v$ if and only if $\{u_i \in u \,|\, |u_i| \geq 2\} \neq \phi$, $\{v_i \in v \,|\, |v_i| \geq 2\} \neq \phi$;*

(*iii*) *Z_2 satisfies $u \approx v$ if and only if $(\forall x \in X) \,|\, u \neq x$, $v \neq x$;*

(*iv*) *W_2 satisfies $u \approx v$ if and only if $m = n = 1, c(u_1) = c(v_1)$ or $m, n \geq 2$.*

Suppose that $u = u_1 + \cdots + u_m, u_i \in X^+, i \in \underline{m}$. Let 1 be a symbol which is not in X and Y an arbitray subset of $\bigcup_{i=1}^{i=m} c(u_1)$. For any u_i in u, if $c(u_i) \subseteq Y$, put $h_Y(u_i) = 1$. Otherwise, we shall denote by $h_Y(u_i)$ the first variable occurring in the word obtained from u_i by deleting all variables in Y. The set $\{h_Y(u_i)|u_i \in u\}$ is written $H_Y(u)$. Dually, we have the notations $t_Y(u_i)$ and $T_Y(u_i)$. In particular, if $Y = \emptyset$, then $h_Y(u_i) = h(u_i)$ and $t_Y(u_i) = t(u_i)$. Moreover, if $c(u_i) \cap Y \neq \emptyset$ for every u_i in u, then we write $D_Y(u) = \emptyset$. Otherwise, $D_Y(u)$ is the sum of all terms u_i in u such that $c(u_i) \cap Y = \emptyset$. By [4, Lemma 2.4 and its dual, Lemma 2.5 and 2.6], we have

Lemma 2.2. *Let $u \approx u + q$ be an* ***AI***-*identity, where $u = u_1 + \cdots + u_m, u_i, q \in X^+, i \in \underline{m}$. If $u \approx u + q$ holds in* **HSP**$(B^0,(B^0)^*)$, *then $c(q) \subseteq \bigcup_{i=1}^{i=m} c(u_i)$ and for the set $Z = \bigcup_{i=1}^{i=m} c(u_i) \setminus c(q)$ and for any subset Y of Z, $H_Y(D_Z(u)) = H_Y(D_Z(u) + q)$ and $T_Y(D_Z(u)) = T_Y(D_Z(u) + q)$.*

For other notations and terminology used in this paper, the read is referred to [1, 4, 7].

3. Equational basis of HSP$(B^0,(B^0)^*,N_2,T_2,Z_2,W_2)$

In [13], Ren and Zeng studied the join **W** of semiring variety **HSP**$(B^0,(B^0)^*)$ and semiring variety **HSP**(N_2,T_2) and obtained the following result.

Lemma 3.1 ([13])**.** *$\mathcal{L}(\mathbf{W})$ is a 312-element distributive lattice and* **W** *is determined by*

$$2x \approx x; \tag{1}$$
$$x^2y \approx xy; \tag{2}$$
$$xy^2 \approx xy; \tag{3}$$
$$(xy)^2 \approx xy; \tag{4}$$
$$xyzt \approx xzyt; \tag{5}$$
$$x+yz \approx x+yz+x^2; \tag{6}$$
$$x+yz \approx x+yz+xyz; \tag{7}$$
$$x+yz \approx x+yz+yzx; \tag{8}$$
$$x+yz \approx x+yz+yxz. \tag{9}$$

In the following Theorem, we shall give an Equational basis of $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$. From Lemma 2.1, Z_2 and W_2 dose not satisfy the identity $2x \approx x$, that is, Z_2 and W_2 are not ai-semirings. In deed, we have

Theorem 3.1. *The semiring variety* $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$ *is determined by the identities (2)-(9) and the following identity*

$$x + y \quad \approx \quad x + 2y. \tag{10}$$

Proof of Theorem 3.1. From [4, 10] and Lemma 2.1, both $\mathbf{W}$ and $\mathbf{HSP}(Z_2, W_2)$ satisfy the identities (2)-(10) and so dose $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$.

Next, we shall show that every identity that holds in $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$ can be derived from (2)-(10) and the identities determining $\mathbf{SR}$. Let $u \approx v$ be such an identity, where $u = u_1 + u_2 + \cdots + u_m$, $v = v_1 + v_2 + \cdots + v_n$, $u_i, v_j \in X^+$, $1 \leq i \leq m, 1 \leq j \leq n$. By Lemma 2.1 (iv), we only need to consider the following two cases:

Case 1 $m = n = 1$ and $C(u_1) = C(v_1)$. Now that $L_2, R_2, T_2, Z_2 \models u_1 \approx v_1$, it follows that $H(u_1) = H(v_1)$, $T(u_1) = T(v_1)$, $|u_1| \geq 2$ and $|v_1| \geq 2$. Hence $u_1 \overset{(2),(3),(5)}{\approx} v_1$.

Case 2 $m, n \geq 2$. It is easy to verify that $u \approx v$ and the identity (10) can imply the identities $u \approx u + v_j$, $v \approx v + u_i$ for all i, j such that $1 \leq i \leq m, 1 \leq j \leq n$. Conversely, the latter $m + n$ identities can imply $u \approx u + v \approx v$. Thus, to show that $u \approx v$ is derivable from (2)-(10) and the identities determining $\mathbf{SR}$, we only need to show that the simpler identities $u \approx u + v_j$, $v \approx v + u_i$ for all i, j such that $1 \leq i \leq m, 1 \leq j \leq n$. Hence, we need to consider the following two cases:

Case 2.1 $u \approx u + q$, where $|q| = 1$. Since $N_2 \models u \approx u + q$, there exists $u_s = q$. Thus $u + q \approx u' + u_s + q \approx u' + u_s + u_s \overset{(10)}{\approx} u' + u_s \approx u$.

Case 2.2 $u \approx u + q$, where $|q| \geq 2$. By (2), (3) and (5), we have

$$q \approx i(q)q \approx i(q)qf(q) \approx i(q)f(q)$$

and so

$$q \approx i(q)f(q). \tag{11}$$

Note that $c(q) = c(i(q)) = c(f(q))$. Since $u \approx u + q$ holds in T_2, it follows from Lemma 2.1 (ii) that there exists u_i in u such that $|u_i| \geq 2$. Put $Z = (\bigcup_{i=1}^{i=m} c(u_i)) \setminus c(q)$. Assume that $D_Z(u) = u_1 + \cdots + u_k$. Then $\bigcup_{i=1}^{i=k} c(u_i) = c(q)$. Moreover, we have

$$\begin{aligned} u &\approx u + u_i + D_Z(u) && \text{(by (10))} \\ &\approx u + u_i + D_Z(u) + u_1^2 && \text{(by (6))} \\ &\approx u + u_i + D_Z(u) + u_1^2 + u_1^2 u_2 \cdots u_k. && \text{(by (8))} \end{aligned}$$

Denote p for $u_1^2 u_2 \cdots u_k$. Thus $c(p) = c(q)$ and we have derived the identity

$$u \approx u + p. \tag{12}$$

Now that $|p| > 1$, by (4), we have

$$p^2 \approx p. \tag{13}$$

Suppose that $i(q) = x_1x_2\cdots x_\ell$. We shall show by induction on j that for every $1 \le j \le \ell$, $u \approx u + x_1x_2\cdots x_\ell p$ is derivable from (2)-(10) and the identities defining **SR**.

From Lemma 2.2, there exists u_{i_1} in $D_Z(u)$ with $c(u_{i_1}) \subseteq c(q)$ such that $h(u_{i_1}) = h(q) = x_1$. Furthermore,

$$\begin{aligned}
u &\approx u + u_{i_1} + p && \text{(by (12))}\\
&\approx u + u_{i_1} + p + u_{i_1}p && \text{(by (7))}\\
&\approx u + u_{i_1} + p + x_1u_{i_1}p && \text{(by (2))}\\
&\approx u + u_{i_1} + p + x_1u_{i_1}p + x_1pu_{i_i}p && \text{(by (9))}\\
&\approx u + u_{i_1} + p + x_1u_{i_1}p + x_1p. && (by(5),(13))
\end{aligned}$$

Therefore,

$$u \approx u + x_1p. \tag{14}$$

Assume that, for some $1 < j \le \ell$,

$$u \approx u + x_1x_2\cdots x_{j-1}p \tag{15}$$

is derivable from (2)-(10) and the identities defining **SR**. By Lemma 2.2, there exists u_i in $D_Z(u)$ with $c(u_i) \subseteq c(q)$ such that $u_i = u_{i_1}x_ju_{i_2}$ and $c(u_{i_1}) \subseteq \{x_1, x_2, \ldots, x_{j-1}\}$. It follows that

$$\begin{aligned}
u &\approx u + u_i + p\\
&\approx u + u_i + p + u_ip && \text{(by (7))}\\
&\approx u + u_i + p + u_{i_1}x_ju_{i_2}p\\
&\approx u + u_i + p + u_{i_1}x_ju_{i_2}p + u_{i_1}x_jpu_{i_2}p && \text{(by (9))}\\
&\approx u + u_i + p + u_{i_1}x_ju_{i_2}p + u_{i_1}x_jp. && \text{(by (5),(13))}
\end{aligned}$$

Consequently

$$u \approx u + u_{i_1}x_jp. \tag{16}$$

Moreover, we have

$$\begin{aligned}
u &\approx u + x_1x_2\cdots x_{j-1}p + u_{i_1}x_jp && \text{(by (15),(16))}\\
&\approx u + x_1x_2\cdots x_{j-1}p + u_{i_1}x_jp + x_1x_2\cdots x_{j-1}u_{i_1}x_jpp && \text{(by (9))}\\
&\approx u + x_1x_2\cdots x_{j-1}p + u_{i_1}x_jp + x_1x_2\cdots x_{j-1}x_jp. && \text{(by (2),(5),(13))}
\end{aligned}$$

Hence $u \approx u + x_1x_2\cdots x_{j-1}x_jp$. Using induction we have

$$u \approx u + i(q)p. \tag{17}$$

Dually,

$$u \approx u + pf(q). \tag{18}$$

Thus

$$\begin{aligned} u &\approx u + p + i(q)p + pf(q) && \text{(by (12),(17),(18))} \\ &\approx u + p + i(q)p + pf(q) + i(q)ppf(q) && \text{(by (7))} \\ &\approx u + p + i(q)p + pf(q) + i(q)f(q) && \text{(by (2),(3),(4),(5))} \\ &\approx u + p + i(q)p + pf(q) + q. && \text{(by (11))} \end{aligned}$$

It follows that $u \approx u + q$.

4. The lattice $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2))$

In this section we characterize the lattice $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2))$. Throughout this section, $t(x_1, \ldots, x_n)$ denotes the term t which contains no other variables than $x_1, \ldots, x_n$ (but not necessarily all of them). Let $S \in \mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$ and $E^+(S)$ denote the set $\{a \in S \mid 2a = a\}$, where the elements of $E^+(S)$ is said to be *additive idempotent* of $(S, +)$. Notice that $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$ satisfies the identities

$$2(x + y) \approx 2x + 2y; \tag{19}$$

$$2xy \approx (x + x)(y + y). \tag{20}$$

By (19) and (20), it is easy to verify that $E^+(S) = \{2a \mid a \in S\}$ forms a subsemiring of S. To characterize the lattice $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2))$, we need to consider the following

$$\varphi : \mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)) \to \mathcal{L}(\mathbf{W}), \mathbf{V} \mapsto \mathbf{V} \cap \mathbf{W}. \tag{21}$$

It is easy to prove that $\varphi(\mathbf{V}) = \{E^+(S) \mid S \in \mathbf{V}\}$ for each member $\mathbf{V}$ of $\mathcal{L}(\mathbf{W})$. If $\mathbf{V}$ is the subvariety of $\mathbf{W}$ determined by the identities

$$u_i(x_{i_1}, \ldots, x_{i_n}) \approx v_i(x_{i_1}, \ldots, x_{i_n}), i \in \underline{k},$$

then $\widehat{\mathbf{V}}$ denotes the subvariety of $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$ determined by the identities

$$u_i(2x_{i_1}, \ldots, 2x_{i_n}) \approx v_i(2x_{i_1}, \ldots, 2x_{i_n}), i \in \underline{k}. \tag{22}$$

Lemma 4.1. *Let $\mathbf{V}$ be a member of $\mathcal{L}(\mathbf{W})$. Then, $\widehat{\mathbf{V}} = \mathbf{V} \vee \mathbf{HSP}(Z_2, W_2)$.*

Proof of Lemma 4.1. Since $\mathbf{V}$ satisfies the identities (22), it follows that $\mathbf{V}$ is a subvariety of $\widehat{\mathbf{V}}$. And both Z_2 and W_2 are members of $\widehat{\mathbf{V}}$ and so the join $\mathbf{V} \vee \mathbf{HSP}(Z_2, W_2) \subseteq \widehat{\mathbf{V}}$. To show the converse inclusion, it suffices to show that every identity that is satisfied by $\mathbf{V} \vee \mathbf{HSP}(Z_2, W_2)$ can be derived by the identities holding in $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$ and $u_i(2x_{i_1}, \ldots, 2x_{i_n}) \approx v_i(2x_{i_1}, \ldots, 2x_{i_n}), i \in \underline{k}$, if $\mathbf{V}$ is the subvariety of $\mathbf{W}$ determined by $u_i(x_{i_1}, \ldots, x_{i_n}) \approx v_i(x_{i_1}, \ldots, x_{i_n}), i \in \underline{k}$. Let $u \approx v$ be such an identity, where $u = u_1 + u_2 + \cdots + u_m, v = v_1 + v_2 + \cdots + v_n, u_i, v_j \in X^+, 1 \leq i \leq m, 1 \leq j \leq n$. By Lemma 2.1 (8), we only need to consider the following two cases.

Case 1 $m, n \geq 2$. By identity (10), $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$ satisfies the identities

$$2u \approx u; \tag{23}$$

$$2v \approx v. \tag{24}$$

Since $u \approx v$ holds in $\mathbf{W}$, we have that it is derivable from the collection Σ of $u_i \approx v_i, i \in \underline{k}$ and the identities determining $\mathbf{W}$. From [1, Exercise II.14.11], it follows that there exist $t_1, t_2, \ldots, t_\ell \in P_f(X^+)$ such that

- $t_1 = u, t_\ell = v$;
- For any $i = 1, 2, \ldots, \ell - 1$, there exist $p_i, q_i, r_i \in P_f(X^+)$ (where p_i, q_i and r_i may be empty words), a semiring substitution φ_i and an identity $u'_i \approx v'_i \in \Sigma$ such that

$$t_i = p_i\varphi_i(w_i)q_i + r_i, t_{i+1} = p_i\varphi_i(s_i)q_i + r_i,$$
$$\text{where either } w_i = u'_i, s_i = v'_i \text{ or } w_i = v'_i, s_i = u'_i.$$

Let Σ' denote the set $\{2u \approx 2v \,|\, u \approx v \in \Sigma\}$. For any $i = 1, 2, \ldots, \ell - 1$, we shall show that $2t_i \approx 2t_{i+1}$ is derivable from Σ' and the identities holding in $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$. In deed, we have

$$\begin{aligned}
2t_i &= 2(p_i\varphi_i(w_i)q_i + r_i) \\
&\approx 2(p_i\varphi_i(w_i)q_i) + 2r_i \\
&\approx p_i(\varphi_i(2w_i))q_i + 2r_i \\
&\approx p_i(\varphi_i(2s_i))q_i + 2r_i \\
&\quad\; (\text{since } 2w_i \approx 2s_i \in \Sigma' \text{ or } 2s_i \approx 2w_i \in \Sigma') \\
&\approx 2(p_i\varphi_i(s_i)q_i) + 2r_i \\
&\approx 2(p_i\varphi_i(s_i)q_i + r_i) \\
&= 2t_{i+1}.
\end{aligned}$$

Further,

$$2u = 2t_1 \approx 2t_2 \approx \cdots \approx 2t_\ell = 2v.$$

This implies the identity

$$2u \approx 2v. \tag{25}$$

We now have

$$u \overset{(24)}{\approx} 2u \overset{(25)}{\approx} 2v \overset{(24)}{\approx} v. \tag{26}$$

Case 2 $m = n = 1$ and $C(u) = C(v)$. Since $Z_2 \models u_1 \approx v_1$, $u_1 \neq x, v_1 \neq x$, for every $x \in X$. Since $u_1 \approx v_1$ holds in $\mathbf{W}$, we have that it is derivable from the collection Σ of $u_i \approx v_i, i \in \underline{k}$ and the identities definging $\mathbf{W}$. From [1, Exercise II.14.11], it follows that there exist $t_1, t_2, \ldots, t_\ell \in P_f(X^+)$ such that

- $t_1 = u_1, t_\ell = v_1$;
- For any $i = 1, 2, \ldots, \ell-1$, there exist $p_i, q_i \in P_f(X^+)$ (where p_i and q_i may be empty words), a semiring substitution φ_i and an identity $u_i' \approx v_i' \in \Sigma$ (where u_i' and v_i' are words) such that

$$\begin{aligned}&t_i = p_i\varphi_i(w_i)q_i, t_{i+1} = p_i\varphi_i(s_i)q_i,\\ &\text{where either } w_i = u_i', s_i = v_i' \text{ or } w_i = v_i', s_i = u_i'.\end{aligned}$$

By Lemma 3.1, $u_1 \approx v_1$ can be derived from (2), (3), (4) and (5), moreover, by Lemma 3.1, it can be derived from monomial identities holding in $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$. This completes the proof. $\square$

Lemma 4.2. *The following equality holds:*

$$\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)) = \bigcup_{\mathbf{V} \in \mathcal{L}(\mathbf{W})} [\mathbf{V}, \widehat{\mathbf{V}}]. \tag{27}$$

There are 312 *intervals in* (27)*, and each interval is a congruence class of the kernel of the complete epimorphism* φ *in* (21)*.*

Proof of Lemma 4.2. First, we shall show that equality (27) holds. It is easy to see that

$$\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)) = \bigcup_{\mathbf{V} \in \mathcal{L}(\mathbf{W})} \varphi^{-1}(\mathbf{V}).$$

So it suffices to show that

$$\varphi^{-1}(\mathbf{V}) = [\mathbf{V}, \widehat{\mathbf{V}}], \tag{28}$$

for each member $\mathbf{V}$ of $\mathcal{L}(\mathbf{W})$. If $\mathbf{V}_1$ is a member of $[\mathbf{V}, \widehat{\mathbf{V}}]$, then it is routine to verity that $\mathbf{V} \subseteq \{E^+(S) \mid S \in \mathbf{V}_1\} \subseteq \mathbf{V}$. This implies that $\{E^+(S) \mid S \in \mathbf{V}_1\} = \mathbf{V}$ and so $\varphi(\mathbf{V}_1) = \mathbf{V}$. Hence, $\mathbf{V}_1$ is a member of $\varphi^{-1}(\mathbf{V})$ and so

$[\mathbf{V}, \widehat{\mathbf{V}}] \subseteq \varphi^{-1}(\mathbf{V})$. Conversely, if $\mathbf{V}_1$ is a member of $\varphi^{-1}(\mathbf{V})$, then $\mathbf{V} = \varphi(\mathbf{V}_1) = \{E^+(S) \mid S \in \mathbf{V}_1\}$ and so $\varphi^{-1}(\mathbf{V}) \subseteq [\mathbf{V}, \widehat{\mathbf{V}}]$. This shows that (27) holds.

From Lemma 3.1, we know that $\mathcal{L}(\mathbf{W})$ is a lattice of order 312. So there are 312 intervals in (27). Next, we show that φ a complete epimorphism. On the one hand, it is easy to see that φ is a complete $\wedge$-epimorphism. On the other hand, let $(\mathbf{V}_i)_{i\in I}$ be a family of members of $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2))$. Then, by (21), we have that $\varphi(\mathbf{V}_i) \subseteq \mathbf{V}_i \subseteq \widehat{\varphi(\mathbf{V}_i)}$ for each $i \in I$. Further,

$$\bigvee_{i\in I} \varphi(\mathbf{V}_i) \subseteq \bigvee_{i\in I} \mathbf{V}_i \subseteq \bigvee_{i\in I} \widehat{\varphi(\mathbf{V}_i)} \subseteq \widehat{\bigvee_{i\in I} \varphi(\mathbf{V}_i)}.$$

This implies that $\varphi(\bigvee_{i\in I} \mathbf{V}_i) = \bigvee_{i\in I} \varphi(\mathbf{V}_i)$. Thus, φ is a complete $\vee$-homomorphism and so φ is a complete epimorphism. By (28), we deduce that each interval in (21) is a congruence class of the kernel of the complete epimorphism φ. □

In order to characterize the lattice $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2))$, by Lemma 4.2, we only need to describe the interval $[\mathbf{V}, \widehat{\mathbf{V}}]$ for each member $\mathbf{V}$ of $\mathcal{L}(\mathbf{W})$. Next, we have

Lemma 4.3. *Let $\mathbf{V}$ be a member of $\mathcal{L}(\mathbf{W})$. Then, $\mathbf{V} \vee \mathbf{HSP}(Z_2)$ is the subvariety of $\widehat{\mathbf{V}}$ determined by the identity*

$$xy \approx 2xy. \tag{29}$$

Proof of Lemma 4.3. It is easy to see that both $\mathbf{V}$ and $\mathbf{HSP}(Z_2)$ satisfy the identity (29) and so does $\mathbf{V} \vee \mathbf{HSP}(Z_2)$. In the following we prove that every identity that is satisfied by $\mathbf{V} \vee \mathbf{HSP}(Z_2)$ is derivable from (29) and the identities holding in $\widehat{\mathbf{V}}$. Let $u \approx v$ be such an identity, where $u = u_1 + u_2 + \cdots + u_m, v = v_1 + v_2 + \cdots + v_n, u_i, v_j \in X^+, 1 \leq i \leq m, 1 \leq j \leq n$. We only need to consider the following cases.

Case 1. $m = n = 1$. Since Z_2 satisfies $u_1 \approx v_1$, it follows that $|u_1| \neq 1$ and $|v_1| \neq 1$. By Lemma 4.1, $\widehat{\mathbf{V}}$ satisfies the identity $2u_1 \approx 2v_1$. Hence $u_1 \overset{(29)}{\approx} 2u_1 \approx 2v_1 \overset{(29)}{\approx} v_1$.

Case 2. $m = 1$, $n \geq 2$. Since Z_2 satisfies $u_1 \approx v$, it follows that $|u_1| \neq 1$. By Lemma 4.1, $\widehat{\mathbf{V}}$ satisfies the identity $2u_1 \approx 2v$. Hence $u_1 \overset{(29)}{\approx} 2u_1 \approx 2v \overset{(10)}{\approx} v$.

Case 3. $m \geq 2$, $n = 1$. Similar to case 2.

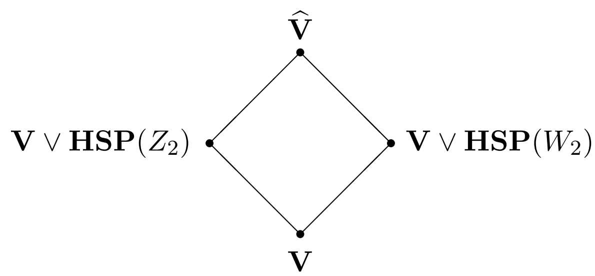
Case 4. $m, n \geq 2$. By Lemma 4.1, $\widehat{\mathbf{V}}$ satisfies the identity $2u \approx 2v$. Hence $u \overset{(10)}{\approx} 2u \approx 2v \overset{(10)}{\approx} v$. □

Lemma 4.4. *Let $\mathbf{V}$ be a member of $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*))$. Then $\mathbf{V} \vee \mathbf{HSP}(W_2)$ is the subvariety of $\widehat{\mathbf{V}}$ determined by the identity*

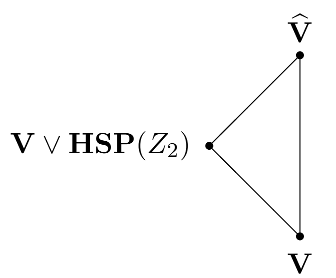
$$x^2 \approx x. \tag{30}$$

Proof of Lemma 4.4. It is easy to see that both $\mathbf{V}$ and $\mathbf{HSP}(W_2)$ satisfy the identity (30) and so does $\mathbf{V} \vee \mathbf{HSP}(W_2)$. So it suffices to show that every identity that is satisfied by $\mathbf{V} \vee \mathbf{HSP}(W_2)$ is derivable from (30) and the identities holding in $\widehat{\mathbf{V}}$. Let $u \approx v$ be such an identity, where $u = u_1 + u_2 + \cdots + u_m, v = v_1 + v_2 + \cdots + v_n, u_i, v_j \in X^+, 1 \leq i \leq m, 1 \leq j \leq n$. By Lemma 4.1, $\widehat{\mathbf{V}}$ satisfies the identity $u^2 \approx v^2$. Hence, $u \overset{(30)}{\approx} u^2 \approx v^2 \overset{(30)}{\approx} v$. $\square$

Lemma 4.5. *Let $\mathbf{V} \in \mathcal{L}(\mathbf{W})$. Then the interval $[\mathbf{V}, \widehat{\mathbf{V}}]$ of $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2))$ is given in Fig.1*



Case. 1 $N_2, T_2 \notin \mathbf{V}$



Case. 2 $N_2 \in \mathbf{V}$ *or* $T_2 \in \mathbf{V}$

Fig.1 *The interval* $[\mathbf{V}, \widehat{\mathbf{V}}]$

Proof of Lemma 4.5. Suppose that $\mathbf{V}_1$ is a member of $[\mathbf{V}, \widehat{\mathbf{V}}]$ such that $\mathbf{V}_1 \neq \widehat{\mathbf{V}}$ and $\mathbf{V}_1 \neq \mathbf{V}$. Then, there exists a nontrivial identity $u \approx v$ holding in $\mathbf{V}_1$ such that it is not satisfied by $\widehat{\mathbf{V}}$. Also, we have that $\mathbf{V}_1$ dose not satisfy the identity $2x \approx x$. By Lemma 4.1, we only need to consider the following two cases.

Case 1 $\mathbf{HSP}(Z_2) \models u \approx v, \mathbf{HSP}(W_2) \not\models u \approx v$. Then, $u \approx v$ satisfies one of the following three cases:

- $m = n = 1$, $c(u_1) \neq c(v_1)$, $|u_1| \neq 1$ and $|v_1| \neq 1$;
- $m = 1, n > 1$ and $|u_1| \neq 1$;

- $m > 1, n = 1$ and $|v_1| \neq 1$.

It is easy to see that, in each of the above cases, $u \approx v$ can imply the identity $xy \approx 2xy$. By Lemma 4.3, we have that $\mathbf{V}_1$ is a subvariety of $\mathbf{V} \vee \mathbf{HSP}(Z_2)$. On the other hand, since $\mathbf{V}_1 \models xy \approx 2xy$ and $\mathbf{V}_1 \not\models 2x \approx x$, it follows that Z_2 is a member of $\mathbf{V}_1$ and so $\mathbf{V} \vee \mathbf{HSP}(Z_2)$ is a subvariety of $\mathbf{V}_1$. Thus, $\mathbf{V}_1 = \mathbf{V} \vee \mathbf{HSP}(Z_2)$.
Case 2 $\mathbf{HSP}(Z_2) \not\models u \approx v, \mathbf{HSP}(W_2) \models u \approx v$. Then, $u \approx v$ satisfies one of the following two cases:

- $m = n = 1$, $c(u_1) = c(v_1)$ and $|u_1| = 1$;
- $m = n = 1$, $c(u_1) = c(v_1)$ and $|v_1| = 1$.

If $N_2, T_2 \notin \mathbf{V}$, then, in each of the above cases, $u \approx v$ can imply the identity $x \approx x^2$. By Lemma 4.4, $\mathbf{V}_1$ is a subvariety of $\mathbf{V} \vee \mathbf{HSP}(W_2)$. On the other hand, since $\mathbf{V}_1 \models x \approx x^2$ and $\mathbf{V}_1 \not\models x \approx 2x$, it follows that W_2 is a member of $\mathbf{V}_1$ and so $\mathbf{V} \vee \mathbf{HSP}(W_2)$ is a subvariety of $\mathbf{V}_1$. Thus, $\mathbf{V}_1 = \mathbf{V} \vee \mathbf{HSP}(W_2)$.

If $N_2 \in \mathbf{V}$, then, by Lemma 2.1 (i), $|u_1| = |v_1| = 1$, a contradiction. Thus, $\mathbf{V}_1 = \widehat{\mathbf{V}}$.

If $T_2 \in \mathbf{V}$, then, by Lemma 2.1 (ii), $|u_1| \geq 2, |v_1| \geq 2$, a contradiction. Thus, $\mathbf{V}_1 = \widehat{\mathbf{V}}$. □

Theorem 4.1. $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2))$ *is a distributive lattice of order* 1014.

Proof of Theorem 4.1. By (27) and Lemma 4.5, we can show that $\mathcal{L}(\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2))$ has exactly 1014 elements. Suppose that $\mathbf{W}_1, \mathbf{W}_2$ and $\mathbf{W}_3$ are members of $\mathcal{L}(\mathbf{W})$ such that $\mathbf{W}_1 \vee \mathbf{W}_2 = \mathbf{W}_1 \vee \mathbf{W}_3$ and $\mathbf{W}_1 \wedge \mathbf{W}_2 = \mathbf{W}_1 \wedge \mathbf{W}_3$. Then, by Lemma 4.2

$$\varphi(\mathbf{W}_1) \vee \varphi(\mathbf{W}_2) = \varphi(\mathbf{W}_1) \vee \varphi(\mathbf{W}_3)$$

and

$$\varphi(\mathbf{W}_1) \wedge \varphi(\mathbf{W}_2) = \varphi(\mathbf{W}_1) \wedge \varphi(\mathbf{W}_3).$$

Since $\mathcal{L}(\mathbf{W})$ is distributive, it follows that $\varphi(\mathbf{W}_2) = \varphi(\mathbf{W}_3)$. Write $\mathbf{V}$ for $\varphi(\mathbf{W}_2)$. Then both $\mathbf{W}_2, \mathbf{W}_3$ are members of $[\mathbf{V}, \widehat{\mathbf{V}}]$. By Fig.1, we deduce that $\mathbf{W}_2 = \mathbf{W}_3$. □

By Theorem 3.1, 4.1 and [13, Corollary 1.2], we now immediately deduce

Corollary 4.1. $\mathbf{HSP}(B^0, (B^0)^*, N_2, T_2, Z_2, W_2)$ *is hereditarily finitely based.*

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