Non-cancellation group of a direct product

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Abstract. The non-cancellation set of a group G, denoted by $\chi(G)$, is defined to be the set of all isomorphism classes of groups H such that $G\times\mathbb{Z}\cong H\times\mathbb{Z}$. While investigating when $\mathbb{Z}$ can be cancelled in this direct product, $\chi(G)$ has become the focus of many studies. For the semidirect product $G_i=\mathbb{Z}_{n_i}\rtimes_{\omega_i}\mathbb{Z}$, $i=1,2$, methods for computation of the non-cancellation groups $\chi(G_1\times G_2)$, $\chi(G_i^k)$, $k\in\mathbb{N}$ and $\chi(G_i,h_i)$ have been developed. We present in this study, a general method of computing $\chi(G_1\times G_2,h)$, where $h:F\hookrightarrow G_1\subseteq G_1\times G_2$ and F a finite group.

Keywords: localization, non cancellation, restricted genus, groups under a finite group.

1. Introduction

The theory of π-localization of groups, where π is a family of primes, appears to have been first discussed in [7, 8] by Mal'cev and Lazard. In the 1970s, Hilton and Mislin became interested, through their work on the localization of nilpotent spaces, in the localization of nilpotent groups. Mislin [11] define the genus $\mathcal{G}(N)$ of a finitely generated nilpotent group N to be the set of isomorphism classes of finitely generated nilpotent groups M such that the localizations M_p and N_p are isomorphic at every prime p. This version of genus became known as the *Mislin genus*, and other very useful variations of this concept came into being.

In [3] Hilton and Mislin define an abelian group structure on the genus set $\mathcal{G}(N)$ of a finitely generated nilpotent group N with finite commutator subgroup. Throughout this study, finitely generated group with finite commutator subgroup will be called χ_0-group.

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For nilpotent groups which belong to class $\mathcal{K}$ (of semidirect products of the form $T \rtimes \mathbb{Z}^k$, where T is a finite abelian group and k is a positive integer), many computations of the genus groups appear in the literature. Indeed, the groups considered in [1, 4, 5, 14] all belong to this class. The groups used in the computation method developed in this paper are χ_o-groups belonging to $\mathcal{K}$ and will be called $\mathcal{K}_0$-groups.

For a non-nilpotent χ_o-groups, the kernel of the localizing homomorphism maybe be bigger than what it is required. So, for such χ_0-groups, the idea of genus can be generalized through non-cancellation.

For a χ_o-group G, the non-cancellation set denoted by $\chi(G)$ is defined to be the set of all isomorphism classes of groups H such that $G\times\mathbb{Z} \cong H\times\mathbb{Z}$. Scevenels and Witbooi in [15], gave an alternate description of the non-cancellation group of $\mathcal{K}_0$-groups. This enables them to make some computations. Warfield [17] proved that, if N is a nilpotent χ_o-group, then $\mathcal{G}(N) = \chi(N)$. In [18], the author showed that for a χ_o-group G the non- cancellation set $\chi(G)$ has a group structure similar to the group structure on the Mislin genus of a nilpotent χ_o-group. For any two χ_o- groups H and G, O'Sullivan in [12] proved that $H \times \mathbb{Z} \cong G \times \mathbb{Z}$ if and only if for every finite set π of primes, we have $H_\pi \cong G_\pi$ (π-localizations are isomorphic). To illuminate our understanding of genera of groups, the restricted genus of a χ_0-group under a finite group F was introduced in [10]. More precisely, for a fixed morphism $h : F \to G$, the restricted genus $\chi(G, h)$[1] is the set of isomorphism classes of morphisms $F \to H$, which are π-equivalent to h at every finite set of primes π. For a well-defined integer n depending on G, in [10] an epimorphism $\zeta : (\mathbb{Z}/n)^*/\pm 1 \to \chi(G, h)$ is established and it is shown that there exist natural epimorphisms $\chi(G, h) \to \chi(G/h(F))$ (provided $h(F)$ is normal in G) and $\chi(G, h) \to \chi(G, h \circ i))$ (provided $i : F_0 \to F$ is a morphism), which are compatible with the various involved maps ζ.

Having such homomorphisms is not always given. In [10], computation methods of $\chi(G, h)$ in the special case G is a semidirect product $T \rtimes_\omega \mathbb{Z}^k$ are used in a very particular example to provide a concrete computation of $\chi(G, h)$, where T is a finite abelian group. We extend this result to compute the restricted genus $\chi(G_1 \times G_2, h)$ of the direct product $G_1 \times G_2$, where $G_i = \mathbb{Z}_{m_i} \rtimes_{\omega_i} \mathbb{Z}$ and $h : F \hookrightarrow G_1 \times G_2$ a monomorphism, with F a finite group.

The rest of the paper is organized as follows: Section 2 is on preliminaries, Section 3 presents the group structure on the restricted genus $\chi(G, h)$ and Section 4 is on the computation method for $\chi(G_1 \times G_2, h)$.

2. Preliminaries

2.1 Definitions and notations

An interesting topic in the theory of nilpotent groups is the extraction of roots. A group G is said to be a *rational group* if n-th roots exist in G, for all positive

1. For the special case F is trivial ($F = *$), $\chi(G, * \to G) = \chi(G)$

integers n. A group which has unique extraction of roots is *torsion-free*. Roots are unique in torsion-free nilpotent groups. However, extraction of roots is not usually possible in such groups. For example, extraction of roots is not possible in the additive group of integers $\mathbb{Z}$. However, this group can be embedded in the rational group $\mathbb{Q}$. Mal'cev [9] generalized this by showing that any torsion-free nilpotent group G can be embedded in a rational nilpotent group G_0. The extraction of roots is unique in G_0 and every element of G_0 has a positive power in G. Moreover, G_0 is unique up to isomorphism.

Given any set of primes π, let π' be the set of natural numbers which are relatively prime to elements of π. Let G_π denote the subgroup of G_0 generated by G and its m-th roots whenever the prime divisors of m are in π'.

A group G is said to be π-*local* if for each $n \in \pi'$, the function $g \mapsto g^n$ of G into itself is a bijection.

The group G_π is called the π-*localization* of G and has the universal property that given any homomorphism $\phi : G \to H$, where H is a π-local group, there exists a unique $\phi_\pi : G_\pi \to H$ such that $\phi = \phi_\pi \varphi_\pi$ where φ_π is the π-*localizing homomorphism* $G \to G_\pi$. If $\pi = \{p\}$, then G_π is simply denoted by G_p.

The **genus** of a finitely generated nilpotent group G denoted by $\mathcal{G}(G)$ (known as *Mislin genus*, [11]), is the set of all isomorphism classes of finitely generated nilpotent groups H such that $G_p \cong H_p$ for every prime number p.

The set $\tau_f(G)$ of all isomorphism classes of finitely generated group H such that $G_\pi \cong H_\pi$ for every finite set of primes π is called the **restricted genus** of G.

When localizing non-nilpotent groups, it may happen that the kernel of the localizing homomorphism is bigger than what we would require. For a non-nilpotent finitely generated group G with finite commutator subgroup, the idea of the genus is generalized through *non-cancellation*, rather than considering localizations.

For groups, we know that cancellation holds in the category of finitely generated abelian groups. If G is a finitely generated abelian group, then for any abelian groups H and K, $G \oplus H \cong G \oplus K$ implies $H \cong K$. Thus, finitely generated abelian group is cancellable in the category of all abelian groups. The abelian group $\mathbb{Z}$ is cancellable in the category of abelian groups. However, it is known that $\mathbb{Z}$ is not cancellable in the category of groups in general. This was shown by an example of William Scott, which was included in [16]. Another example was given independently by Hirshon [6]. Our study of cancellation property of a group G is examined through the isomorphism of direct products $G \times \mathbb{Z} \cong H \times \mathbb{Z}$. For a group G, the non-cancellation set $\chi(G)$ measures to what extend $\mathbb{Z}$ can be cancelled in $G \times \mathbb{Z} \cong H \times \mathbb{Z}$ for some group H. For some type of groups G, the computation of $\chi(G)$ have been the object of many studies. For $\mathcal{K}_o$-groups, methods for computation of the non-cancellation groups $\chi(G_1 \times G_2)$ and $\chi(G_i^k)$, $k \in \mathbb{N}$ were developed in [19] and [2] respectively. In these construction, the integer n(G) described below play a central role.

Given a $\mathcal{X}_0$-group G. Let n_1 be the exponent of T_G, n_2 the exponent of the group $\mathrm{Aut}(T_G)$ and n_3 the exponent of the torsion subgroup of the centre of G. Consider $n(G) = n_1n_2n_3$. The integer $n = n(G)$ has the property that the subgroup $G^{(n)} = \langle g^n \ : \ g \in G\rangle$ of G belongs to the centre of G and $G/G^{(n)}$ is a finite group.

Aspects of localization as in groups and related categories have been studied in a unified way in a categorical setting, see [13] for instance. The following subsection give a more specific presentation.

2.2 Category of groups under a finite group F

Fix a finite group F and let $h : F \longrightarrow G$ be a monomorphism. We denote by $\mathrm{Grp_F}$ the category of groups under F as in [10]. For the category $\mathrm{Grp_F}$, the objects denoted by (G_1, h_1), (G_2, h_2) are group homomorphisms $h_1 : F \longrightarrow G_2$ and $h_2 : F \longrightarrow G_2$ and a morphism in $\mathrm{Grp_F}$ is a group homomorphism $\beta : G_1 \longrightarrow G_2$ such that $\beta \circ h_1 = h_2$.

The π-localization of an object $h : F \longrightarrow G$ is the object $h_\pi : F \longrightarrow G_\pi$ where π is a set of primes. Denote by $\mathcal{X}_F$ the full subcategory of χ_o- groups under F. Then, the restricted genus $\chi(G, h)$ is the set of isomorphism classes k such that k_π is isomorphic to h_π for $k \in \mathcal{X}_F$. If F is a trivial group, then $\mathcal{X}_F$ is identified with χ_o-groups.

The restricted genus $\chi(G, h)$ has been computed in [10] and has been shown that $\chi(G, h)$ coincides with $\chi(G)$ if F is a trivial group.

Let $\mathcal{K}$ be the class of groups of the form $T \rtimes_\omega F$ where F is a finite rank free abelian group and T a finite abelian group. For a pair of relatively prime natural numbers m, u, the symbol $G(m, u)$ denotes the group $H = \mathbb{Z}_m \rtimes_\nu \mathbb{Z}$. H is a $\mathcal{K}$- group and $\mathcal{K}_\mathrm{F}$ determines a full subcategory of $\mathrm{Grp_F}$, see [10].

Let $G_i = \mathbb{Z}_{m_i} \rtimes_{\omega_i} \mathbb{Z}$ and let $h : F \hookrightarrow G_1 \times G_2$ where h is a monomorphism and F is a finite group.

In this paper we develop a general method for computing $\chi(G_1 \times G_2, h)$.

3. Group structure on the restricted genus

Recall from ([18], Section 2), to a χ_o-group G assign a natural number $n(G) = n_1n_2n_3$ where n_1 is the exponent of the torsion subgroup T_G, n_2 the exponent of $\mathrm{Aut}(T_G)$ and n_3 the exponent of the torsion of the center T_{Z_G}. It was shown in [18] that for a χ_o-group G whose subgroups H are of finite index with $T_G = T_H$, the non-cancellation set $\chi(G)$ has a group structure and is given by $\chi(G) = \mathbb{Z}_n^*/\{1, -1\}$. For a pair of relatively prime natural numbers m, u, let $H = \mathbb{Z}_m \rtimes_\nu \mathbb{Z}$, where $\nu : \mathbb{Z} \longrightarrow \mathrm{Aut}(\mathbb{Z}_m)$ is the automorphism of $\mathbb{Z}_m$ defined by $\nu(1)(t) = ut$. Methods of computing $\chi(H)$ and $\chi(H^r)$ for r a natural number were developed in [2], [15] and [19]. It is shown in [15] that $\chi(H) = \mathbb{Z}_d^*/\{1, -1\}$, where d is the multiplicative order of u modulo m. For a direct product H^r which can be considered to be $\mathbb{Z}_m^r \rtimes_\omega \mathbb{Z}^r$, the authors in [2] showed that there is a well defined surjective homomorphism $\Gamma : \chi(H) \longrightarrow \chi(H^r)$ given by $K \mapsto K \times H^{r-1}$ where K is a group such that $K \times \mathbb{Z} \simeq H \times \mathbb{Z}$. Thus, in order to compute the group $\chi(H^r)$, one needs only to compute the kernel of the homomorphism Γ.

Let $G_i = G(m_i, u_i)$ for some $m_i,\, u_i \in \mathbb{N}$ with $\gcd(m_i, u_i) = 1$ and let d_i be the multiplicative order of u_i modulo m_i. Consider the direct product $G = G_1 \times G_2 = (\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}) \rtimes_\omega \mathbb{Z}^2$ and $h : F \hookrightarrow G_1 \times G_2$ be the inclusion map. Let $t = d(G)$ be the smallest invariant factors of $\operatorname{Im}\omega$. Note that, if $\gcd(d_1, d_2) = 1$ then $t = d_1 d_2$ and if $\gcd(d_1, d_2) \neq 1$ then $t = \gcd(d_1, d_2)$. For an object $(G_1 \times G_2,\, h)$ in $\mathcal{K}_F$, we obtain an epimorphism $\Upsilon : \mathbb{Z}_t^* \longrightarrow \chi(G_1 \times G_2,\, h)$, where d depends exclusively on $\operatorname{Im}\omega$ [10]. Thus, in order to find $\chi(G_1 \times G_2,\, h)$, one only needs to find the kernel of Υ. Note that t divides the exponent of $\operatorname{Aut}(\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2})$, and so t divides n(G).

Following ([10], Lemma 3.2), we have the following Lemma:

Lemma 3.1. *Given objects (G, h) and (K, k) in $\mathcal{K}_F$ with $G = (\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}) \rtimes_\omega \mathbb{Z}^2$ and $K = (\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}) \rtimes_\nu \mathbb{Z}^2$, then a morphism $\alpha : (G, h) \longrightarrow (K, k)$ is an isomorphism if and only if there exist group isomorphisms $\theta : (\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}) \rtimes_\omega \mathbb{Z}^2 \longrightarrow (\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}) \rtimes_\nu \mathbb{Z}^2$ and $\beta : \mathbb{Z}^2 \longrightarrow \mathbb{Z}^2$ with $\theta \circ h = k$ such that for any $z \in \mathbb{Z}^2$ and $t \in (\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2})$, $(\nu \circ \beta(z))(\theta(t)) = \theta(\omega(z)(t))$.*

Notation 3.1. We will follow the notation introduced in [10]. Let G be a χ_o-group, $n = n(G)$ and let $X(n) = \{u \in \mathbb{N} : (u, n) = 1\}$. Let $Y(G, h)$ be the set of all $u \in X(n)$ for which there exists a subgroup K of G with $[G : K] = u$ and the object $(K,\, h_K)$ is a member in $\chi(G, h)$. Let G_u be a subgroup of G such that $T_G \subseteq G_u$ and $[G : G_u] = u$ for each $u \in Y(G, h)$. Define the induced homomorphism $h_u : x \mapsto h(x)$ and a morphism $\varsigma : Y(G, h) \longrightarrow \chi(G, h)$. Let $Y^*(G, h)$ be the image of $Y(G, h)$ in $\mathbb{Z}_n^*$. From ([10], Theorem 2.5) we have $Y^*(G, h)$ is a subgroup of $\mathbb{Z}_n^*$ and $Y^*(G, h)/\pm 1 \cong \chi(G, h)$. Now for any object (G, h) of $\mathcal{K}_F$ denote by $V(G, h)$ the set of all $u \in X(t)$ for which there exist a subgroup K of G with $[G : K] = u$ and (K, h_K) is a member of $\chi(G, h)$. Let $V^*(G, h)$ be the image of $V(G, h)$ in $\mathbb{Z}_t^*$. Choose a subgroup K of G such that (K, h_K) represents a member in $\chi(G, h)$ and $[G : K] = u$. We obtain a function $V(G, h) \longrightarrow \chi(G, h)$ given by $u \mapsto [K, h_K]$. Since $t \,|\, n$ we have the following ([10], Proposition 3.4)

Proposition 3.1. *Let $n = n(G)$ and $\rho : Y(G, h) \longrightarrow \chi(G, h)$ be the epimorphism that takes a residue* $\operatorname{mod} \mathrm{n}$ *and reduces it* $\operatorname{mod}\ \mathrm{t}$. *The epimorphism $\zeta : Y^*(G, h) \longrightarrow \chi(G, h)$ factorises through the epimorphism ξ':*

$$\begin{array}{ccc} Y^*(G,h) & \xrightarrow{\ \zeta\ } & \chi(G,h) \\ {\scriptstyle\rho}\big\downarrow & \nearrow_{\xi'} & \\ V^*(G,h) & & \end{array}$$

The kernel of $V^*(G, h) \longrightarrow \chi(G, h)$ is caculated through the following theorem ([10], Theorem 3.5)

Theorem 3.1. *For $m \in V(G,h)$, the following conditions are equivalent:*

(*a*) $\overline{m} \in \ker[V^*(G,h) \longrightarrow \chi(G,h)]$.

(*b*) *There exists $\alpha \in \operatorname{Aut}(T)$ with $v \circ h = h$ such that $\alpha \in N_{\operatorname{Aut} T} \operatorname{Im}\omega$ and for an automorphism $\bigwedge : \operatorname{Im}\omega \longrightarrow \operatorname{Im}\omega$ defined by $v \mapsto \alpha v \alpha^{-1}$, we have $\det(\bigwedge) = \pm\overline{m}^{-1} \in V^*(G,h)$.*

4. Computation

1. For a pair of relatively prime natural numbers m, u, define a group $G(m,\, u) =< a,\, b\, :\, a^m = 1,\, bab^{-1} = a^u >$. The group $G(m,\, u)$ can be considered to be the semidirect product $\mathbb{Z}_m \rtimes_\omega \mathbb{Z}$ where $\omega : \mathbb{Z} \longrightarrow \operatorname{Aut}(\mathbb{Z}_m)$ is such that $\omega(1)$ is the automorphism of $\mathbb{Z}_m$ given by $\omega(1)(t) = ut$.

2. Let q be the product of all distinct prime divisors of m and assume that q^2 divides m.

3. Let $G_i = G(m_i, u_i)$ for some $m_i,\, u_i \in \mathbb{N}$ with $\gcd(m_1, m_2) = 1, \gcd(m_i, u_j) = 1,\, i,\, j = 1,\, 2$ and let d_i be the multiplicative order of u_i modulo m_i. Let $d = \operatorname{lcm}(d_1, d_2)$ and $m = \operatorname{lcm}(m_1, m_2)$. If $\gcd(d_1, d_2) = 1$ let $t = d_1 d_2$ and if $\gcd(d_1, d_2) \neq 1$ let $t = \gcd(d_1, d_2)$.

4. Consider the direct product $G_1 \times G_2 = (\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}) \rtimes_\omega \mathbb{Z}^2$ where $\omega : \mathbb{Z}^2 \longrightarrow \operatorname{Aut}(\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2})$ is such that $\omega(\epsilon_i) = \omega_i : (t_1, t_2) \mapsto (u_1^{\delta(i,1)} t_1, u_2^{\delta(i,2)} t_2)$ where $\delta(i,j)$ is the Kronecker function and $\{\epsilon_1, \epsilon_2\}$ is the standard basis of $\mathbb{Z}^2$.

5. Write $\omega(\epsilon_1) = \omega_1$ and $\omega(\epsilon_2) = \omega_2$. Each automorphism ω_i is of order d_i and $\operatorname{Im}(\omega)$ is the direct product of the cyclic subgroups $< \omega_i >$ of $\operatorname{Aut}(\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2})$. By [10], there is an epimorphism $\mathbb{Z}_t^* \longrightarrow \chi(G_1 \times G_2)$ where t is the smallest of the invariant factors of $\operatorname{Im}\omega$.

6. Let J be the subgroup of $\operatorname{Aut}(T_{G_1 \times G_2})$ generated by $\{\omega_1,\, \omega_2\}$ that is, $J = \operatorname{Im}\omega =< \omega_1, \omega_2 >$. Note that:

   - $J = \operatorname{Im}\omega =< \omega_1, \omega_2 >$ is a free $\mathbb{Z}_d$-module.
   - The determinant of an endomorphism of J is defined and is an element of $\mathbb{Z}_d$.

   Consider $J_* = N_{\operatorname{Aut}(T_{G_1 \times G_2})} J$. For any $\alpha \in \operatorname{Aut}(T_{G_1 \times G_2})$, let $\bigwedge_\alpha$ be the inner automorphism such that $\bigwedge_\alpha : v \mapsto \alpha\, v\, \alpha^{-1}$.

7. Let q_2 be a multiple of q such that $q_2 q$ divides m. Let $e_1 = (1,0)$, $e_2 = (0,1)$ be elements of $T_{G_1 \times G_2}$ and let $F = \{a q_2 e_2 \,:\, a \in \mathbb{Z}\}$ be a subgroup of $T_{G_1 \times G_2}$ and $h : F \hookrightarrow G_1 \times G_2$ be the inclusion map.

4.1 Inner automorphism of $\mathrm{Aut}\,(T_{G_1\times G_2})$

Fix $\alpha \in J^*$ such that $\alpha(x) = x$ for all $x \in F$. There exists a 2×2 matrix (α_{ij}) of integers such that $\alpha(e_i) = \sum_{j=1}^{2} \alpha_{ji}e_j$. Suppose that $\bigwedge$ is the inner automorphism of J determined by α.

Proposition 4.1. *For the matrix (α_{ij}), α_{ii} is a unit modulo m for $i = 1, 2$.*

Proof. We note that $\alpha(0, q_2) = (0, q_2)$ since $(0, q_2) \in F$. Also $\alpha(0, q_2) = (q_2\alpha_{12}, q_2\alpha_{22})$. Thus $(0, q_2) = (q_2\alpha_{12}, q_2\alpha_{22})$ and we have that m divides α_{12}. In particular q divides α_{12} while α_{22} is a unit modulo m. Therefore the matrix of α is of the form $\begin{pmatrix} \alpha_{11} & tq \\ \alpha_{21} & u \end{pmatrix}$ where $t, u \in \mathbb{Z}$. Now $\det(\alpha) = \alpha_{11}u - \alpha_{21}tq$.

We claim that α_{11} is a unit modulo m. Suppose that α_{11} is not a unit modulo m then let p be a common prime divisor of α_{11} and m. Then p divides q and p divides $\det(\alpha)$ which is a contradiction since $\det(\alpha)$ is a unit modulo m (α is an automorphism). Thus α_{11} is a unit modulo m. □

Proposition 4.2. *The inner automorphism $\bigwedge$ of J coincides with the identity automorphism of J.*

Proof. For the inner automorphism $\bigwedge$ induced by α, there exists a matrix $(\bigwedge_{ij})$ of integers such that $\bigwedge\omega_i = \omega_1^{\bigwedge_{1i}}\omega_2^{\bigwedge_{2i}}$ for each i. Let $\bigwedge\omega_i = v_i$. Then $\bigwedge\omega_i = \alpha\omega_i\alpha^{-1} = v_i$ and $\alpha\omega_i = v_i\alpha$. On one hand $\alpha\omega_i(e_i) = \alpha(u_ie_i) = \sum_{j=1}^{2} u_i^{\delta(i,j)}\alpha_{ji}e_j$ and on the other hand $v_i\alpha(e_i) = v_i\left(\sum_{j=1}^{2}\alpha_{ji}e_j\right) = \sum_{j=1}^{2}\alpha_{ji}u_i^{\delta(i,j)\bigwedge_{ji}}e_j$. That is, $\sum_{j=1}^{2} u_i^{\delta(i,j)}\alpha_{ji}e_j = \sum_{j=1}^{2}\alpha_{ji}u_i^{\delta(i,j)\bigwedge_{ji}}e_j$.

For $j = i$, we have from , that α_{ii} is a unit modulo m, therefore $u_i^{\bigwedge_{ii}} \equiv u_i \bmod m$. Thus $\bigwedge_{ii} \equiv 1 \bmod d$ and consequently $\bigwedge_{ii} \equiv 1 \bmod t$.

For the case $j \neq i$, we have $\alpha\omega_i(e_j) = \alpha(e_j) = \sum_{k=1}^{2}\alpha_{kj}e_k$ and $v_i\alpha(e_j) = \sum_{k=1}^{2}\alpha_{kj}u_k^{\bigwedge_{ki}}e_k$. Therefore $\sum_{k=1}^{2}\alpha_{kj}e_k = \sum_{k=1}^{2}\alpha_{kj}u_k^{\bigwedge_{ki}}e_k$. For the case $k = j$. Since α_{jj} is a unit modulo m then $u_j^{\bigwedge_{ji}} \equiv 1 \bmod m$, that is, $\bigwedge_{ji} \cong 0 \bmod d$. Consequently $\bigwedge_{ji} \equiv 0 \bmod t$. Thus $\det(\bigwedge) = 1$ and $\bigwedge$ is coincides with the identity automorphism on J. □

Proposition 4.3. *Let $G_i = G(m_i, u_i)$, $i = 1, 2$ and $(G_1 \times G_2, h)$ be an object of $\mathcal{K}_F$. Let d_i be the multiplicative order of u_i modulo m_i. Then, $\chi(G_1 \times G_2, h) \cong \mathbb{Z}_t^*/\pm 1$, where ($t = d_1d_2$, if $(d_1, d_2) = 1$) or ($t = (d_1, d_2)$, otherwise).*

Proof. The Proposition follows from Theorem 3.1 and Proposition 4.2. □

This construction can be generalized to compute $\chi(G_1 \times \cdots \times G_n, h)$ and $\chi(G_i^k, l)$. This will be done in our future work.

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