Common fixed point theorems for four self maps satisfying generalized (ψ, ϕ)-weak contraction in metric space

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Abstract. In this manuscript, we shall prove a common fixed point theorem for four weakly compatible self-maps P, Q, R and S on a metric space (M, d^*) satisfying the following generalized (ψ, ϕ)-weak contraction:

$$\psi(d^*(Ru, Sv)) \leq \psi(\triangle\,(u, v)) - \phi(\triangle\,(u, v)),$$

where

$$\triangle\,(u, v) = \max\Big\{d^*(Ru, Sv), d^*(Ru, Pu), d^*(Sv, Qv), \frac{1}{2}[d^*(Pu, Sv) + d^*(Qv, Ru)], \frac{d^*(Pu, Ru)d^*(Qv, Sv)}{1 + d^*(Ru, Sv)}, \frac{d^*(Pu, Sv)d^*(Qv, Ru)}{1 + d^*(Ru, Sv)}, d^*(Ru, Pu)[\frac{1 + d^*(Ru, Qv) + d^*(Sv, Pu)}{1 + d^*(Ru, Pu) + d^*(Sv, Qv)}]\Big\}.$$

Also, we have proved common fixed point theorems for the above mentioned contraction using weakly compatible self-maps along with E.A. property and (CLR) property. An illustrative example is also provided to support our results.

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1. Introduction

Definition 1.1. *A coincidence point of a pair of self-maps $P, Q : M \to M$ is a point $u \in M$ for which $Pu = Qu$.*

A common fixed point of a pair of self-maps $P, Q : M \to M$ is a point $u \in M$ for which $Pu = Qu = u$.

In 1996, Jungck [2] introduced the concept of weakly compatible maps to study common fixed point theorems:

Definition 1.2. *Let (M, d^*) be a metric space. A pair of self-maps $P, Q : M \to M$ is weakly compatible if they commute at their coincidence points, that is, if there exists $u \in M$ such that $PQu = QPu$, where u is coincidence point of P and Q.*

In 2002, Aamri and Moutawakil [1] introduced the notion of E.A. property as follows:

Definition 1.3. *Let (M, d^*) be a metric space. Two self-maps P and Q on M are said to satisfy the E.A. property, if there exists a sequence $\{u_n\}$ in M such that* $\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Qu_n = t$, *for some $t \in M$.*

In 2011, Sintunavarat *et al.* [5] introduced the notion of (CLR) property as follows:

Definition 1.4. *Let (M, d^*) be a metric space. Two self-maps P and Q on M are said to satisfy the (CLR_P) property, if there exists a sequence $\{u_n\}$ in M such that* $\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Qu_n = Pt$, *for some $t \in M$.*

2. Main results

In this section, we prove some common fixed point theorems for weakly compatible four self maps along with (E.A.) property and (CLR) property.

Theorem 2.1. *Let (M, d^*) be a metric space and let P, Q, R and S be self-maps on M satisfying the followings:*

$$RM \subseteq QM, SM \subseteq PM, \tag{1}$$

For all $u, v \in M$, there exist right continuous functions $\psi, \phi : \mathbb{R}^+ \to \mathbb{R}^+$, with $\psi(0) = 0 = \phi(0)$ and $\psi(a) < a$ for $a > 0$ such that:

$$\psi(d^*(Ru, Sv) \leq \psi(\triangle\,(u, v)) - \phi(\triangle\,(u, v)), \tag{2}$$

where

$$\begin{aligned}\triangle\,(u,v) = \max\{& d^*(Ru,Sv), d^*(Ru,Pu), d^*(Sv,Qv),\\ & \frac{1}{2}[d^*(Pu,Sv)+d^*(Qv,Ru)],\\ & \frac{d^*(Pu,Ru)d^*(Qv,Sv)}{1+d^*(Ru,Sv)}, \frac{d^*(Pu,Sv)d^*(Qv,Ru)}{1+d^*(Ru,Sv)},\\ & d^*(Ru,Pu)[\frac{1+d^*(Ru,Qv)+d^*(Sv,Pu)}{1+d^*(Ru,Pu)+d^*(Sv,Qv)}]\}.\end{aligned}$$

If one of PM,QM,RM or SM is complete subspace of M, then the pair (P,R) or (Q,S) have a coincidence point. Moreover, if the pairs (P,R) and (Q,S) are weakly compatible, then P,Q,R and S have a unique common fixed point.

Proof. Let $u_0 \in M$ be an arbitrary point of M. From (2), we can construct a sequence $\{v_n\}$ in M as follows:

(3) $$v_{2n+1} = Ru_{2n} = Qu_{2n+1}, v_{2n+2} = Su_{2n+1} = Pu_{2n+2},$$

for all $n = 0,1,2,....$ Now, we define $d^*_n = d^*(v_n, v_{n+1})$. If $d^*_{2n} = 0$ for some n, then $d^*(v_{2n}, v_{2n+2}) = 0$. Then $v_{2n} = v_{2n+1}$, that is, $Su_{2n-1} = Pu_{2n} = Ru_{2n} = Qu_{2n+1}$ and P and R have a coincidence point. Similarly, if $d^*_{2n+1} = 0$, then Q and S have a coincidence point. Assume that $d^*_n \neq 0$ for each n.

On putting $u = u_{2n}$ and $v = u_{2n+1}$ in (2), we get

(4) $$\psi(d^*(Ru_{2n}, Su_{2n+1})) \leq \psi(\triangle\,(u_{2n}, u_{2n+1})) - \phi(\triangle\,(u_{2n}, u_{2n+1})),$$

where

$$\begin{aligned}\triangle\,(u_{2n},u_{2n+1}) &= \max\{d^*(Ru_{2n},Su_{2n+1}), d^*(Ru_{2n},Pu_{2n}), d^*(Su_{2n+1},Qu_{2n+1}),\\ &\quad \frac{1}{2}[d^*(Pu_{2n},Su_{2n+1})+d^*(Qu_{2n+1},Ru_{2n})],\\ &\quad \frac{d^*(Pu_{2n},Ru_{2n})\cdot d^*(Qu_{2n+1},Su_{2n+1})}{1+d^*(Ru_{2n},Su_{2n+1})},\\ &\quad \frac{d^*(Pu_{2n},Su_{2n+1})\cdot d^*(Qu_{2n+1},Ru_{2n})}{1+d^*(Ru_{2n},Su_{2n+1})},\\ &\quad d^*(Ru_{2n},Pu_{2n})\frac{1+d^*(Ru_{2n},Qu_{2n+1})+d^*(Su_{2n+1},Pu_{2n})}{1+d^*(Ru_{2n},Pu_{2n})+d^*(Su_{2n+1},Qu_{2n+1})}\}\\ &= \max\{d^*(v_{2n+1},v_{2n+2}), d^*(v_{2n+1},v_{2n}), d^*(v_{2n},v_{2n+1}),\\ &\quad \frac{1}{2}[d^*(v_{2n},v_{2n+2})+d^*(v_{2n+1},v_{2n+1})],\\ &\quad \frac{d^*(v_{2n},v_{2n+1})\cdot d^*(v_{2n+1},v_{2n+2})}{1+d^*(v_{2n+1},v_{2n+2})},\\ &\quad \frac{d^*(v_{2n},v_{2n+2})\cdot d^*(v_{2n+1},v_{2n+1})}{1+d^*v_{2n+1},v_{2n+2})},\\ &\quad d^*(v_{2n+1},v_{2n})\frac{1+d^*(v_{2n+1},v_{2n+1})+d^*(v_{2n+2},v_{2n})}{1+d^*(v_{2n+1},v_{2n})+d^*(v_{2n+2},v_{2n+1})}\}\end{aligned}$$

$$= \max\{d^*_{2n+1}, d^*_{2n}, d^*_{2n+1}, \frac{1}{2}[d^*_{2n} + d^*_{2n+1} + 0], \frac{d^*_{2n} \cdot d^*_{2n+1}}{1 + d^*_{2n+1}},$$
$$0, d^*_{2n} \frac{1 + d^*_{2n} + d^*_{2n+1}}{1 + d^*_{2n} + d^*_{2n+1}}\},$$

that is

$$\triangle\ (u_{2n}, u_{2n+1}) = \max\{d^*_{2n}, d^*_{2n+1}\}. \tag{5}$$

Now, from (4), we have

$$\psi(d^*(v_{2n+1}, v_{2n+2})) \le \psi(\max\{d^*_{2n}, d^*_{2n+1}\}) - \phi(\max\{d^*_{2n}, d^*_{2n+1}\}), \tag{6}$$

Now, if $d^*_{2n+1} \ge d^*_{2n}$ for some n, then from (6), we get

$$\begin{aligned} \psi(d^*_{2n+1}) &\le \psi(d^*_{2n+1}) - \phi(d^*_{2n+1}) \\ &< \psi(d^*_{2n+1}), \end{aligned} \tag{7}$$

which is a contradiction. Thus, $d^*_{2n} > d^*_{2n+1}$ for all n, and so, from (6), we have

$$\psi(d^*_{2n+1}) \quad \le \quad \psi(d^*_{2n}) - \phi(d^*_{2n})\, for\, all\, n \in N. \tag{8}$$

Similarly,

$$\begin{aligned} \psi(d^*_{2n}) &\le \psi(d^*_{2n-1}) - \phi(d^*_{2n-1}), \\ \psi(d^*_{2n-1}) &\le \psi(d^*_{2n-2}) - \phi(d^*_{2n-2}). \end{aligned}$$

In general, we have for all $n = 1, 2, 3. \dots$

$$\begin{aligned} \psi(d^*_n) &\le \psi(d^*_{n-1}) - \phi(d^*_{n-1}) \\ &< \psi(d^*_{n-1}). \end{aligned} \tag{9}$$

Hence, sequence $\{\psi(d^*_n)\}$ is monotonically decreasing and bounded below. Thus, there exists $s \ge 0$, such that

$$\lim_{n\to\infty} \psi(d^*_n) = s. \tag{10}$$

From (9), we deduce that

$$0 \le \phi(d^*_{n-1}) \le \psi(d^*_{n-1}) - \psi(d^*_n)$$

Taking limit as $n \to \infty$ and using (10), we get

$$\lim_{n\to\infty} \phi(d^*_{n-1}) = 0,$$

this implies that

$$\lim_{n\to\infty} \phi(d^*_{n-1}) = \lim_{n\to\infty} \phi(d^*(v_{n-1}, v_n)) = 0. \tag{11}$$

$$\lim_{n\to\infty} d_n^* = \lim_{n\to\infty} d^*(v_n, v_{n+1}) = 0. \tag{12}$$

Now, we claim that $\{v_n\}$ is a Cauchy sequence. For this, it is sufficient to show that $\{v_{2n}\}$ is a Cauchy sequence. Let, if possible, $\{v_{2n}\}$ is not a Cauchy sequence. Then there exists an $\epsilon > 0$, such that for each even integer $2a$ there exists even integers $2m(a) > 2n(a) > 2a$ such that

$$d^*(v_{2n(a)}, v_{2m(a)}) \geq \epsilon. \tag{13}$$

for every even integer $2a$, suppose that $2m(a)$ be the least positive integer exceeding $2n(a)$ satisfying (13), such that

$$d^*(v_{2n(a)}, v_{2m(a)-2}) < \epsilon. \tag{14}$$

From (13), we get

$$\begin{aligned}\epsilon &\leq d^*(v_{2n(a)}, v_{2m(a)}) \\ &\leq d^*(v_{2n(a)}, v_{2m(a)-2}) + d^*(v_{2m(a)-2}, v_{2m(a)-1}) + d^*(v_{2m(a)-1}, v_{2m(a)}).\end{aligned}$$

Using (12) and (14) in the above inequality, we get

$$\lim_{n\to\infty} d^*(v_{2n(a)}, v_{2m(a)}) = \epsilon. \tag{15}$$

Also, by the triangular inequality,

$$|d^*(v_{2n(a)+1}, v_{2m(a)-1}) + d^*(v_{2n(a)}, v_{2m(a)})| \leq d^*_{2m(a)-1} + d^*_{2m(a)}. \tag{16}$$

Using (12), we have

$$\lim_{n\to\infty} d^*(v_{2n(a)}, v_{2m(a)-1}) = \lim_{n\to\infty} d^*(v_{2n(a)+1}, v_{2m(a)-1}) = \epsilon. \tag{17}$$

Now, from (2), we have

$$\begin{aligned}\psi(d^*(Ru_{2n(a)}, Su_{2m(a)-1})) \leq \psi(\triangle\ (u_{2n(a)}, u_{2m(a)-1})) \\ - \phi(\triangle\ (u_{2n(a)}, u_{2m(a)-1})),\end{aligned} \tag{18}$$

where

$$\begin{aligned}&\triangle\ (u_{2n(a)}, u_{2m(a)-1}) \\ &= \max\{d^*(Ru_{2n(a)}, Su_{2m(a)-1}), d^*(Ru_{2n(a)}, Pu_{2n(a)}), \\ &d^*(Su_{2m(a)-1}, Qu_{2m(a)-1}), \\ &\frac{1}{2}[d^*(Pu_{2n(a)}, Su_{2m(a)-1}) + d^*(Qu_{2m(a)-1}, Ru_{2m(a)})],\end{aligned}$$

$$\frac{d^*(Pu_{2n(a)}, Ru_{2n(a)}) \cdot d^*(Qu_{2m(a)-1}, Su_{2m(a)-1})}{1 + d^*(Ru_{2m(a)}, Su_{2m(a)-1})},$$
$$\frac{d^*(Pu_{2m(a)}, Su_{2m(a)-1}) \cdot d^*(Qu_{2m(a)-1}, Ru_{2n(a)})}{1 + d^*(Ru_{2m(a)}, Su_{2m(a)-1})},$$
$$d^*(Ru_{2n(a)}, Pu_{2n(a)})\frac{1 + d^*(Ru_{2n(a)}, Qu_{2m(a)-1}) + d^*(Su_{2m(a)-1}, Pu_{2n(a)})}{1 + d^*(Ru_{2n(a)}, Pu_{2n(a)}) + d^*(Su_{2m(a)-1}, Qu_{2m(a)-1})}\}$$
$$= \max\{d^*(v_{2n(a)+1}, v_{2m(a)}), d^*(v_{2n(a)+1}, v_{2m(a)}), d^*(v_{2m(a)}, v_{2m(a)-1}),$$
$$\frac{1}{2}[d^*(v_{2n(a)}, v_{2m(a)}) + d^*(v_{2m(a)-1}, v_{2m(a)+1})],$$
$$\frac{d^*(v_{2n(a)}, v_{2n(a)+1}) \cdot d^*(v_{2m(a)-1}, v_{2m(a)})}{1 + d^*(v_{2n(a)+1}, v_{2m(a)})},$$
$$\frac{d^*(v_{2n(a)}, v_{2n(a)+1}) \cdot d^*(v_{2m(a)-1}, v_{2m(a)})}{1 + d^*(v_{2n(a)+1}, v_{2m(a)})},$$
$$d^*(v_{2n(a)+1}, v_{2n(a)})\frac{1 + d^*(v_{2n(a)+1}, v_{2m(a)-1}) + d^*(v_{2m(a)}, v_{2n(a)})}{1 + d^*(v_{2n(a)+1}, v_{2n(a)}) + d^*(v_{2m(a)}, v_{2m(a)-1})}\}.$$

Now, taking limit as $a \to \infty$ and using equations (12), (14), (15) and (17), we get $\triangle\ (u_{2n(a)}, u_{2m(a)-1}) = \max\{\epsilon, 0, 0, \frac{1}{2}(\epsilon + \epsilon), 0, \frac{\epsilon \cdot \epsilon}{1+\epsilon}, 0\}$, that is

$$\triangle\ (u_{2n(a)}, u_{2m(a)-1}) = \epsilon.$$

Now, by (18), we have

$$\psi(\epsilon) \leq \psi(\epsilon) - \phi(\epsilon),$$

which is a contradiction, since $\epsilon > 0$. Thus, $\{v_{2n}\}$ is a Cauchy sequence. So, $\{v_n\}$ is a Cauchy sequence. Now, suppose that PM is complete. Since $\{v_{2n}\}$ is contained in PM and has limit in PM say p, that is, $\lim_{n\to\infty} v_{2n} = p$. Let $q \in P^{-1}(p)$ then $Pq = p$.

Now, we shall prove that $Rq = p$.

Let, if possible, $Rq \neq p$ that is, $d^*(Rq, p) = k > 0$. On putting $u = q$,$v = u_{2n-1}$ in (2), we have

$$\psi(d^*(Rq, Su_{2n-1})) \leq \psi(\triangle\ (q, u_{2n-1})) - \phi(\triangle\ (q, u_{2n-1})), \tag{19}$$

where

$$\triangle\ (q, u_{2n-1}) = \max\{d^*(Rq, Su_{2n-1}), d^*(Rq, Pq), d^*(Su_{2n-1}, Qu_{2n-1}),$$
$$\frac{1}{2}[d^*(Pq, Su_{2n-1}){+}d^*(Qu_{2n-1}, Rq)], \frac{d^*(Pq, Rq) \cdot d^*(Qu_{2n-1}, Su_{2n-1})}{1{+}d^*(Rq, Su_{2n-1})},$$
$$\frac{(Pq, Su_{2n-1}) \cdot d^*(Qu_{2n-1}, Rq)}{1 + d^*(Rq, Su_{2n-1})},$$
$$d^*(Rq, Pq)\frac{1 + d^*(Rq, Qu_{2n-1}) + d^*(Su_{2n-1}, Pq)}{1 + d^*(Rq, Pq) + d^*(Su_{2n-1}, Qu_{2n-1})}\}.$$

Taking limit as $n \to \infty$, we get

$$\lim_{n\to\infty} \triangle\,(q, u_{2n-1}) = \lim_{n\to\infty} \max\{d^*(Rq, Su_{2n-1}), d^*(Rq, Pq), d^*(Su_{2n-1}, Qu_{2n-1}),$$
$$\frac{1}{2}[d^*(Pq, Su_{2n-1}) + d^*(Qu_{2n-1}, Rq)], \frac{d^*(Pq, Rq)\cdot d^*(Qu_{2n-1}, Su_{2n-1})}{1 + d^*(Rq, Su_{2n-1})},$$
$$\frac{d^*(Pq, Su_{2n-1})\cdot d^*(Qu_{2n-1}, Rq)}{1 + d^*(Rq, Su_{2n-1})},$$
$$d^*(Rq, Pq)\frac{1 + d^*(Rq, Qu_{2n-1}) + d^*(Su_{2n-1}, Pq)}{1 + d^*(Rq, Pq) + d^*(Su_{2n-1}, Qu_{2n-1})}\}$$
$$= \max\{d^*(Rq, p), d^*(Rq, p), d^*(p, p), \frac{1}{2}[d^*(Pq, p) + d^*(p, Rq)],$$
$$\frac{d^*(p, Rq)\cdot d^*(p, p)}{1 + d^*(Rq, p)}, \frac{d^*(p, p)\cdot d^*(p, Rq)}{1 + d^*(Rq, p)}, d^*(Rq, p)\frac{1 + d^*(Rq, p) + d^*(p, p)}{1 + d^*(Rq, p) + d^*(p, p)}\}.$$
$$\lim_{n\to\infty} \triangle\,(q, u_{2n-1}) = d^*(p, Rq) = k.$$

Thus, from (19),we have $\psi(d^*(Rq, p)) \le \psi(k) - \phi(k)$, $\psi(k) \le \psi(k) - \phi(k)$, which is a contradiction, since $k > 0$. Thus, $Rq = Pq = p$. Hence, q is coincidence point of the pair (P, R). Since $RM \subseteq QM$, $Rq = p$ implies that, $p \in QM$. Let $w \in B^{-1}p$. Then $Bw = p$. By using the same arguments as above, we can easily verify that S $w = p = Qw$, that is, w is the coincidence point of the pair (Q, S). Similarly, we can prove the result if QM is complete subspace of M instead of PM. Now, if SM is complete then by (1), $p \in SM \subseteq PM$. In the same manner if RM is complete then $p \in RM \subseteq QM$. Now, since the pair (P, R) and (Q, S) are weakly compatible, so

$$\begin{aligned} p &= Rq = Pq = Sw = Qw,\\ Pp &= PRq = RPq = Rp,\\ Qp &= QSw = SQw = Sp. \end{aligned} \tag{20}$$

Now, we shall prove that $Sp = p$. Let, if possible, $Sp \ne p$. From (2), we have

$$\psi(d^*(p, Sp)) = \psi(d^*(Rq, Sp)) \le \psi(\triangle\,(q, p)) - \phi(\triangle\,(q, p)),$$

where

$$\triangle\,(q, p) = \max\{d^*(Rq, Sp), d^*(Rq, Pq), d^*(Sp, Qp), \frac{1}{2}[d^*(Pq, Sp) + d^*(Qp, Rq)],$$
$$\frac{d^*(Pq, Rq)\cdot d^*(Qp, Sp)}{1 + d^*(Rq, Sp)}, \frac{d^*(Pq, Sp)\cdot d^*(Qp, Rq)}{1 + d^*(Rq, Sp)},$$
$$d^*(Rq, Pq)\frac{1 + d^*(Rq, Qp) + d^*(Sp, Pq)}{1 + d^*(Rq, Pq) + d^*(Sp, Qp)}\}.$$

Using (20), we have

$$\triangle\,(q, p) = \max\{d^*(p, Sp), 0, 0, \frac{1}{2}[d^*(p, Sp) + d^*(Sp, p)], 0, \frac{d^*(Pq, Sp)\cdot d^*(Qp, Rq)}{1 + d^*(Rq, Sp)}, 0\}$$
$$\triangle\,(q, p) = d^*(p, Sp).$$

Thus, we have

$$\psi(d^*(p,Sp)) \leq \psi(d^*(p,Sp)) - \phi(d^*(p,Sp)) < \psi(d^*(p,Sp)),$$

which is a contradiction. So, $Sp = p$. Similarly, $Rp = p$. Thus, we get $Pp = Rp = Qp = Sp = p$. Hence, p is the common fixed point of P, Q, R and S. For the uniqueness, let t be another common fixed point of P, Q, R and S.

Now, we claim that $t = p$. Let, if possible $t \neq p$. From (2), we have

$$\begin{aligned}\psi(d^*(p,t)) = \psi(d^*(Rp,St)) &\leq \psi(\triangle\,(p,t)) - \phi(\triangle\,(p,t))\\ &= \psi(d^*(p,t)) - \phi(d^*(p,t))\, since \triangle\,(p,t) = d^*(p,t) < \psi(d^*(p,t)),\end{aligned}$$

which is a contradiction. Thus, $t = p$, and hence the uniqueness follows. This completes the proof of the theorem. □

Theorem 2.2. *Let (M, d^*) be a metric space and P, Q, R and S be self-maps on M satisfying (1) and (2) and the followings:*

$$\text{(21)} \qquad Pairs\,(P,R)\,and\,(Q,S)\,are\,weakly\,compatible.$$

$$\text{(22)} \qquad Pair\,(P,R)\,or\,(Q,S)\,satisfy\,the\,E.A.\,property.$$

If any one of PM, QM, RM or SM is a complete subspace of M, then P, Q, R and S have a unique common fixed point.

Proof. Suppose that the pair (P, R) satisfies the E.A. property. Then, there exists a sequence $\{u_n\}$ in M, such that $\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Ru_n = p$, for some p in M. Since $RM \subseteq QM$, there exists a sequence $\{v_n\}$ in M such that $R\{u_n\} = Q\{v_n\}$. Hence, $\lim_{n\to\infty} Qv_n = p$.

We shall show that $\lim_{n\to\infty} Sv_n = p$.

Let, if possible, $Sv_n = q \neq p$. From (2), we have

$$\psi(d^*(Ru_n, Sv_n)) \leq \psi(\triangle\,(u_n, v_n)) - \phi(\triangle\,(u_n, v_n)).$$

Now, taking limit as $n \to \infty$, we get

$$\text{(23)} \qquad \lim_{n\to\infty} \psi(d^*(Ru_n, Sv_n)) \leq \lim_{n\to\infty} \psi(\triangle\,(u_n, v_n)) - \lim_{n\to\infty} \phi(\triangle\,(u_n, v_n)),$$

where

$$\begin{aligned}\lim_{n\to\infty} \triangle\,(u_n, v_n) = \lim_{n\to\infty} \max\{&d^*(Ru_n, Sv_n), d^*(Ru_n, Pu_n), d^*(Sv_n, Qv_n),\\ &\frac{1}{2}[d^*(Pu_n, Sv_n) + d^*(Qv_n, Ru_n)],\\ &\frac{d^*(Pu_n, Ru_n)\cdot d^*(Qv_n, Sv_n)}{1 + d^*(Ru_n, Sv_n)},\end{aligned}$$

$$
\begin{aligned}
&\frac{d^*(Pu_n, Sv_n)\cdot d^*(Qv_n, Ru_n)}{1+d^*(Ru_n, Sv_n)},\\
&d^*(Ru_n, Pu_n)\frac{1+d^*(Ru_n, Qv_n)+d^*(Sv_n, Pu_n)}{1+d^*(Ru_n, Pu_n)+d^*(Sv_n, Qv_n)}\}\\
=&\max\{d^*(p,q), d^*(p,p), d^*(q,p), \frac{1}{2}[d^*(p,q)+d^*(p,p)],\\
&\frac{d^*(p,p)\cdot d^*(p,q)}{1+d^*(p,q)}, \frac{d^*(p,p)\cdot d^*(p,q)}{1+d^*(p,q)},\\
&d^*(p,p)[\frac{1+d^*(p,p)+d^*(q,p)}{1+d^*(p,p)+d^*(q,p)}]\}\\
=&\, d^*(p,q).
\end{aligned}
$$

From (23), we have

$$\psi(d^*(p,q)) \le \psi(d^*(p,q)) - \phi(d^*(p,q)) < \psi(d^*(p,q)),$$

which is a contradiction. Therefore, $p = q$, that is $\lim_{n\to\infty} Sv_n = p$. Suppose that QM is a complete subspace of M. Then $p = Qa$ for some $a \in M$. Subsequently, we have $\lim_{n\to\infty} Sv_n = \lim_{n\to\infty} Ru_n = \lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Qv_n = p = Qa$. Now, we shall show that $Sa = Qa$. Let, if possible $Sa \ne Qa$.

From (2), we have

$$\psi(d^*(Ru_n, Sa)) \le \psi(\triangle\,(u_n, a)) - \phi(\triangle\,(u_n, a)).$$

Taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty}\psi(d^*(Ru_n, Sa)) \le \lim_{n\to\infty}\psi(\triangle\,(u_n, a)) - \lim_{n\to\infty}\phi(\triangle\,(u_n, a)), \tag{24}$$

where

$$
\begin{aligned}
\lim_{n\to\infty}\triangle\,(u_n, a) =& \lim_{n\to\infty}\max\{d^*(Ru_n, Sa), d^*(Ru_n, Pu_n), d^*(Sa, Qa),\\
&\frac{1}{2}[d^*(Pu_n, Sa)+d^*(Qa, Ru_n)],\\
&\frac{d^*(Pu_n, Ru_n)\cdot d^*(Qa, Sa)}{1+d^*(Ru_n, Sa)},\\
&\frac{d^*(Pu_n, Sa)\cdot d^*(Qa, Ru_n)}{1+d^*(Rq, Sa)},\\
&d^*(Ru_n, Pu_n)\frac{1+d^*(Ru_n, Qa)+d^*(Sa, Pu_n)}{1+d^*(Ru_n, Pu_n)+d^*(Sa, Qa)}\}\\
=&\max\{d^*(p, Sa), d^*(p,p), d^*(Sa, p), \frac{1}{2}[d^*(p, Sa)+d^*(p,p)],\\
&\frac{d^*(p,p)\cdot d^*(p, Sa)}{1+d^*(p, Sa)}, \frac{d^*(p,p)\cdot d^*(p, Sa)}{1+d^*(p, Sa)},\\
&d^*(p,p)[\frac{1+d^*(p,p)+d^*(Sa,p)}{1+d^*(p,p)+d^*(Sa,p)}]\}\\
=&\, d^*(Sa, p).
\end{aligned}
$$

Thus, from (24), we have

$$\psi(d^*(p,Sa)) \leq \psi(d^*(p,Sa)) - \phi(d^*(p,Sa)) < \psi(d^*(p,Sa)),$$

which is a contradiction. Therefore, $Sa = p = Qa$. Since Q and S are weakly compatible, therefore, $QSa = SQa$, implies that, $SSa = SQa = QSa = QQa$. Since $SM \subseteq PM$, there exists $b \in M$, such that, $Sa = Pb$.

Now, we claim that $Pb = Rb$. Let, if possible, $Pb \neq Rb$. From (2), we have

$$\psi(d^*(Rb,Sa)) \leq \psi(\triangle\,(b,a)) - \phi(\triangle\,(b,a)), \tag{25}$$

where

$$\begin{aligned}\triangle\,(b,a) = \max\{&d^*(Rb,Sa), d^*(Rb,Pb), d^*(Sa,Qa),\\ &\frac{1}{2}[d^*(Pb,Sa) + d^*(Qa,Rb)],\\ &\frac{d^*(Pb,Rb)\cdot d^*(Qa,Sa)}{1+d^*(Rb,Sa)},\\ &\frac{d^*(Pb,Sa)\cdot d^*(Qa,Rb)}{1+d^*(Rb,Sa)},\\ &d^*(Rb,Pb)\frac{1+d^*(Rb,Qa)+d^*(Sa,Pb)}{1+d^*(Rb,Pb)+d^*(Sa,Qa)}\}\\ = d^*(Rb,Sa).&\end{aligned}$$

From (25), we have

$$\psi(d^*(Rb,Sa)) \leq \psi(d^*(Rb,Sa)) - \phi(d^*(Rb,Sa)) < \psi(d^*(Rb,Sa)),$$

which is a contradiction. Therefore, $Rb = Sa = Pb$. Now, since (P,R) is weakly compatible. This implies that $PRb = RPb = RRb = PPb$.

Now, we claim that Sa is common fixed point of P, Q, R and S. Let, if possible, $SSa \neq Sa$. From (2), we have

$$\psi(d^*(Sa,SSa)) = \psi(d^*(Rb,SSa)) \leq \psi(\triangle\,(b,Sa)) - \phi(\triangle\,(b,Sa)), \tag{26}$$

where

$$\begin{aligned}\triangle\,(b,Sa) = \max\{&d^*(Rb,SSa), d^*(Rb,Pb), d^*(SSa,QSa),\\ &\frac{1}{2}[d^*(Pb,SSa) + d^*(QSa,Rb)],\\ &\frac{d^*(Pb,Rb)\cdot d^*(QSa,SSa)}{1+d^*(Rb,SSa)},\\ &\frac{d^*(Pb,SSa)\cdot d^*(QSa,Rb)}{1+d^*(Rb,SSa)},\\ &d^*(Rb,Pb)\frac{1+d^*(Rb,QSa)+d^*(SSa,Pb)}{1+d^*(Rb,Pb)+d^*(SSa,QSa)}\}\\ = d^*(Sa,SSa).&\end{aligned}$$

Thus, from (26), we have

$$\psi(d^*(Sa, SSa)) \leq \psi(d^*(Sa, SSa)) - \phi(d^*(Sa, SSa)) < \psi(d^*(Sa, SSa)),$$

which is a contradiction. Therefore, $Sa = SSa = QSa$. Hence, Sa is the common fixed point of Q and S. Similarly, we can prove that Rb is common fixed point of R and P. Since $Sa = Rb$, Sa is the common fixed point of P, Q, R and S. If we assume RM is complete subspace of M, the proof is similar. Similarly we can prove the theorem for cases when PM or QM is a complete subspace of M. Since $SM \subseteq PM$ and $RM \subseteq QM$.

Now, we shall prove the uniqueness of common fixed point. If possible, let c and d be two common fixed points of P, Q, R and S, such that $c \neq d$. From (2), we have

$$\psi(d^*(c,d)) = \psi(d^*(Rc, Sd)) \leq \psi(\triangle\,(c,d)) - \phi(\triangle\,(c,d)), \tag{27}$$

where

$$\begin{aligned}\triangle\,(c,d) = \max\{&d^*(Rc, Sd), d^*(Rc, Pc), d^*(Sd, Qd),\\ &\frac{1}{2}[d^*(Pc, Sd) + d^*(Qd, Rc)],\\ &\frac{d^*(Pc, Rd)\cdot d^*(Qd, Rc)}{1 + d^*(Rc, Sd)},\\ &\frac{d^*(Pc, Sd)\cdot d^*(Qd, Rc)}{1 + d^*(Rc, Sd)},\\ &d^*(Rc, Pc)\frac{1 + d^*(Rc, Qd) + d^*(Sd, Pc)}{1 + d^*(Rc, Pc) + d^*(Sd, Qd)}\}\\ = {}& d^*(c,d).\end{aligned}$$

From (27), we have

$$\psi(d^*(c,d)) \leq \psi(d^*(c,d)) - \phi(d^*(c,d)) < \psi(d^*(c,d)),$$

which is a contradiction. Therefore, $c = d$ and this follows the uniqueness and completes the proof of the theorem. □

Theorem 2.3. *Let (M, d^*) be a metric space. Let P, Q, R and S be self maps on M satisfying (1), (2), (21) and the followings:*

$$\begin{aligned}&RM \subseteq QM \, and \, the \, pair \, (P,R) \, satisfies \, (CLR_P) \, property,\\ &SM \subseteq PM \, and \, the \, pair \, (Q,S) \, satisfies \, (CLR_Q) \, property.\end{aligned} \tag{28}$$

Then P, Q, R and S have unique common fixed point.

Proof. Without loss of generality, assume that $RM \subseteq QM$ and the pair (P, R) satisfies the (CLR_P) property. Then, there exists a sequence $\{u_n\}$ in M such that $\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Ru_n = Pp$, for some p in M.

Since $RM \subseteq QM$, there exists a sequence $\{v_n\}$ in M such that $R\{u_n\} = Q\{v_n\}$.

Hence, $\lim_{n\to\infty} Qv_n = Pp$. Now, we shall show that $\lim_{n\to\infty} Sv_n = Pp$. Let if possible, $\lim_{n\to\infty} Sv_n = q \neq Pp$. From (2), we have

$$\psi(d^*(Ru_n, Sv_n)) \leq \psi(\triangle\ (u_n, v_n)) - \phi(\triangle\ (u_n, v_n)).$$

Now, taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \psi(d^*(Ru_n, Sv_n)) \leq \lim_{n\to\infty} \psi(\triangle\ (u_n, v_n)) - \lim_{n\to\infty} \phi(\triangle\ (u_n, v_n)), \tag{29}$$

where

$$\begin{aligned}\lim_{n\to\infty} \triangle\ (u_n, v_n) = \lim_{n\to\infty} \max\{& d^*(Ru_n, Sv_n), d^*(Ru_n, Pu_n), d^*(Sv_n, Qv_n),\\ & \frac{1}{2}[d^*(Pu_n, Sv_n) + d^*(Qv_n, Ru_n)],\\ & \frac{d^*(Pu_n, Ru_n) \cdot d^*(QV_n, Sv_n)}{1 + d^*(Ru_n, Sv_n)},\\ & \frac{d^*(Pu_n, Sv_n) \cdot d^*(Qv_n, Ru_n)}{1 + d^*(Ru_n, Sv_n)},\\ & d^*(Ru_n, Pu_n)\frac{1 + d^*(Ru_n, Qv_n)+d^*(Sv_n, Pu_n)}{1+d^*(Ru_n, Pu_n)+d^*(Sv_n, Qv_n)}\}\\ = \max\{& d^*(Pq, q), d^*(Pp, Pp), d^*(q, Pp), \frac{1}{2}[d^*(Pp, q)+d^*(Pp, Pp)],\\ & \frac{d^*(Pq, q) \cdot d^*(Pp, Pp)}{1 + d^*(Pp, q)}, \frac{d^*(Pp, Pp) \cdot d^*(Pp, q)}{1 + d^*(Pp, q)},\\ & d^*(Pp, Pp)[\frac{1 + d^*(Pp, Pp) + d^*(q, Pp)}{1 + d^*(Pp, Pp) + d^*(q, Pp)}]\}\\ = d^*(&Pp, q).\end{aligned}$$

From (29), we have

$$\psi(d^*(Pp, q)) \leq \psi(d^*(Pp, q)) - \phi(d^*(Pp, q)) < \psi(d^*(Pp, q)),$$

which is a contradiction. Therefore, $Pp = q$, that is, $\lim_{n\to\infty} Sv_n = Pp = q$.

Subsequently, we have $\lim_{n\to\infty} Sv_n = \lim_{n\to\infty} Ru_n = \lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Qv_n = Pp = q$. Now, we shall show that $Rp = q$. Let, if possible, $Rp \neq q$. From (2), we have

$$\psi(d^*(Rp, Sv_n)) \leq \psi(\triangle\ (p, v_n)) - \phi(\triangle\ (p, v_n)).$$

Now, taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \psi(d^*(Rp, Sv_n)) \leq \lim_{n\to\infty} \psi(\triangle\ (p, v_n)) - \lim_{n\to\infty} \phi(\triangle\ (p, v_n)), \tag{30}$$

where

$$\begin{aligned}\lim_{n\to\infty} \triangle\,(p, v_n) = \lim_{n\to\infty} &\max\{d^*(Rp, Sv_n), d^*(Rp, Pp), d^*(Sv_n, Qv_n),\\ &\frac{1}{2}[d^*(Pp, Sv_n) + d^*(Qv_n, Rp)], \frac{d^*(Pp, Rp)\cdot d^*(QV_n, Sv_n)}{1 + d^*(Rp, Sv_n)},\\ &\frac{d^*(Pp, Sv_n)\cdot d^*(Qv_n, Rp)}{1 + d^*(Rp, Sv_n)},\\ &d^*(Rp, Pp)\frac{1 + d^*(Rp, Qv_n) + d^*(Sv_n, Pp)}{1 + d^*(Rp, Pp) + d^*(Sv_n, Qv_n)}\}\\ =& \max\{d^*(Rp, q), d^*(Rp, q), d^*(q, q), \frac{1}{2}[d^*(q, q) + d^*(q, Rp)],\\ &\frac{d^*(q, Rp)\cdot d^*(q, q)}{1 + d^*(Rp, q)}, \frac{d^*(q, q)\cdot d^*(q, Rp)}{1 + d^*(Rp, q)},\\ &d^*(Rp, q)[\frac{1 + d^*(Rp, q) + d^*(q, q)}{1 + d^*(Rp, q) + d^*(q, q)}]\}\\ =& d^*(Rp, q).\end{aligned}$$

Thus, from (30), we get

$$\psi(d^*(Rp, q)) \le \psi(d^*(Rp, q)) - \phi(d^*(Rp, q)) < \psi(d^*(Rp, q)),$$

which is a contradiction. Therefore, $Rp = q = Pp$. Since the pair (P, R) is weakly compatible, it follows that $Pq = Rq$. Also, since $RM \subseteq QM$, there exists some r in M, such that, $Rp = Qr$, that is, $Qr = q$. Now, we show that $Sr = q$. Let, if possible $Sr \neq q$. From (2), we have

$$\psi(d^*(Ru_n, Sr)) \le \psi(\triangle\,(u_n, r)) - \phi(\triangle\,(u_n, r)).$$

Now, taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \psi(d^*(Ru_n, Sr)) \le \lim_{n\to\infty} \psi(\triangle\,(u_n, r)) - \lim_{n\to\infty} \phi(\triangle\,(u_n, r)), \tag{31}$$

where

$$\begin{aligned}\lim_{n\to\infty} \triangle\,(u_n, r) = \lim_{n\to\infty} &\max\{d^*(Ru_n, Sr), d^*(Ru_n, Pu_n), d^*(Sr, Qr),\\ &\frac{1}{2}[d^*(Pu_n, Sr) + d^*(Qr, Ru_n)],\\ &\frac{d^*(Pu_n, Ru_n)\cdot d^*(Qr, Sr)}{1 + d^*(Ru_n, Sr)}, \frac{d^*(Pu_n, Sr)\cdot d^*(Qr, Ru_n)}{1 + d^*(Ru_n, Sr)},\\ &d^*(Ru_n, Pu_n)\frac{1 + d^*(Ru_n, Qr) + d^*(Sr, Pu_n)}{1 + d^*(Ru_n, Pu_n) + d^*(Sr, Qr)}\}\\ =& \max\{d^*(q, Sr), d^*(q, q), d^*(Sr, q), \frac{1}{2}[d^*(q, Sr) + d^*(q, q)],\end{aligned}$$

$$\frac{d^*(q,q)\cdot d^*(q,Sr)}{1+d^*(q,Sr)}, \frac{d^*(q,Sr)\cdot d^*(q,q)}{1+d^*(q,Sr)},$$
$$d^*(q,q)[\frac{1+d^*(q,q)+d^*(Sr,q)}{1+d^*(q,q)+d^*(Sr,q)}]\}$$
$$= d^*(Sr,q).$$

Thus, from (31), we get

$$\psi(d^*(q,Sr)) \leq \psi(d^*(q,Sr)) - \phi(d^*(q,Sr)) < \psi(d^*(q,Sr)),$$

which is a contradiction. Therefore, $Sr = q = Qr$. Since the pair (Q,S) is weakly compatible, it follows that $Sq = Qq$. Now, we claim that $Rq = Sq$. Let, if possible, $Rq \neq Sq$. From (2), we have

$$\psi(d^*(Rq,Sq)) \leq \psi(\triangle\,(q,q)) - \phi(\triangle\,(\triangle\,(q,q)), \tag{32}$$

where

$$\triangle\,(q,q) = \max\{d^*(Rq,Sq), d^*(Rq,Pq), d^*(Sq,Qq), \frac{1}{2}[d^*(Pq,Sq)+d^*(Qq,Rq)],$$
$$\frac{d^*(Pq,Rq)\cdot d^*(Qq,Sq)}{1+d^*(Rq,Sq)}, \frac{d^*(Pq,Sq)\cdot d^*(Qq,Rq)}{1+d^*(Rq,Sq)},$$
$$d^*(Rq,Pq)\frac{1+d^*(Rq,Qq)+d^*(Sq,Pq)}{1+d^*(Rq,Pq)+d^*(Sq,Qq)}\}$$
$$= d^*(Sq,Rq).$$

From (32), we have

$$\psi(d^*(Rq,Sq)) \leq \psi(d^*(Rq,Sq)) - \phi(d^*(Rq,Sq)) < \psi(d^*(Rq,Sq)),$$

which is a contradiction. Thus, $Rq = Sq$, that is, $Pq = Rq = Sq = Qq$. Now, we shall show that $q = Sq$. Let, if possible, $q \neq Sq$. From (2), we have

$$\psi(d^*(Rp,Sq)) \leq \psi(\triangle\,(p,q)) - \phi(\triangle\,(p,q)), \tag{33}$$

where

$$\triangle\,(p,q) = \max\{d^*(Rp,Sq), d^*(Rp,Pp), d^*(Sq,Qq), \frac{1}{2}[d^*(Pp,Sq)+d^*(Qq,Rp)],$$
$$\frac{d^*(Pp,Rp)\cdot d^*(Qq,Sq)}{1+d^*(Rp,Sq)}, \frac{d^*(Pp,Sq)\cdot d^*(Qq,Rp)}{1+d^*(Rp,Sq)},$$
$$d^*(Rp,Pp)\frac{1+d^*(Rp,Qq)+d^*(Sq,Pp)}{1+d^*(Rp,Pp)+d^*(Sq,Qq)}\}$$
$$= d^*(Sq,Rp).$$

From (33), we have

$$\psi(d^*(Rp,Sq)) \leq \psi(d^*(Rp,Sq)) - \phi(d^*(Rp,Sq)) < \psi(d^*(Rp,Sq)),$$

which is a contradiction. Therefore, $q = Sq = Qq = Pq = Rq$. Hence, q is the common fixed point of P, Q, R and S.

Now, we shall prove the uniqueness of common fixed point. Let c and d be two common fixed point of P, Q, R and S. Let, if possible, $c \neq d$. From (2), we have

$$\begin{aligned}\psi(d^*(c,d)) = \psi(d^*(Rc,Sd)) \leq \psi(\triangle\ (c,d)) - \phi(\triangle\ (c,d)) = \psi(d^*(c,d)) - \phi(d^*(c,d))\\ < \psi(d^*(c,d)),\end{aligned}$$

which is a contradiction. Therefore, $c = d$. This proves the uniqueness of common fixed point. □

Example 2.1. Let $M = [0,1]$ be endowed with the Euclidean metric $d^*(u,v) = |u - v|$. Let the self maps P, Q, R and S be defined by

$$Ru = \frac{u}{9}, Qu = \frac{u}{6}, Su = \frac{u}{3}, Pu = u.$$

Clearly, $RM = [0, \frac{1}{9}] \subseteq [0, \frac{1}{6}] = QM$, $SM = [0, \frac{1}{3}] \subseteq [0,1] = PM$. Also, PM is complete subspace of M and pair (P,R), (Q,S) are weakly compatible.

Now,

$$\begin{aligned}
&d^*(Ru,Sv) = |\frac{u}{9} - \frac{v}{3}| = \frac{1}{9}|u - 3v|,\\
&d^*(Pu,Qv) = |u - \frac{v}{6}| = \frac{1}{6}|6u - v|,\\
&d^*(Ru,Pu) = |\frac{u}{9} - u| = \frac{8u}{9},\\
&d^*(Qv,Sv) = |\frac{v}{6} - \frac{v}{3}| = \frac{v}{6},\\
&d^*(Ru,Su) = |\frac{u}{9} - \frac{u}{3}| = \frac{2u}{9},\\
&d^*(Pu,Su) = |u - \frac{u}{3}| = \frac{2u}{3},\\
&d^*(Qu,Rv) = |\frac{u}{6} - \frac{v}{9}| = \frac{1}{18}|3u - 2v|,\\
&\frac{1}{2}[d^*(Pu,Su) + d^*(Qu,Rv)] = \frac{1}{2}[\frac{2u}{3} + \frac{1}{18}|3u - 2v|] = \frac{1}{36}|15u - 2v|,\\
&\frac{(d^*(Pv,Ru) \cdot d^*(Qu,Su)}{(1 + d^*(Rv,Su)} = \frac{\frac{8v}{9} \cdot \frac{u}{6}}{1 + \frac{1}{9}|3v - u|} = \frac{4uv}{3(9 + 3v - u)},\\
&\frac{1 + d^*(Rv,Qu) + d^*(Su,Pv)}{1 + d^*(Rv,Pv) + d^*(Su,Qu)} = \frac{1 + \frac{1}{18}(3u - 2v) + \frac{1}{9}(u - 9v)}{1 + \frac{8v}{9} + \frac{u}{6}} = \frac{|18 + 5u - 20v|}{|18 + 16v + 3u|}.
\end{aligned}$$

Let $\psi(a) = \frac{a}{2}$ and $\phi(a) = \frac{a}{4}$. Thus, we have

$$\psi(d^*(Ru,Sv)) = \psi(\frac{u}{9} - \frac{v}{3}) = \frac{1}{2}|\frac{u}{9} - \frac{v}{3}| = \frac{1}{18}|u - 3v|,$$

$$\begin{aligned}\triangle\ (u,v) = \max\{&d^*(Ru,Sv), d^*(Ru,Pu), d^*(Sv,Qv),\\ &\frac{1}{2}[d^*(Pu,Sv)+d^*(Qv,Ru)],\\ &\frac{d^*(Pu,Ru)\cdot d^*(Qv,Sv)}{1+d^*(Ru,Sv)}, \frac{d^*(Pu,Sv)\cdot d^*(Qv,Ru)}{1+d^*(Ru,Sv)},\\ &d^*(Ru,Pu)\frac{1+d^*(Ru,Qv)+d^*(Sv,Pu)}{1+d^*(Ru,Pu)+d^*(Sv,Qv)}\} = d^*(Ru,Pu),\end{aligned}$$

$$\psi(\triangle\ (u,v)) = \psi(d^*(Ru,Pu)) = \psi(\frac{8u}{9}) = \frac{1}{2}\cdot\frac{8u}{9} = \frac{4u}{9},$$

$$\phi(\triangle\ (u,v)) = \phi(d^*(Ru,Pu)) = \phi(\frac{8u}{9}) = \frac{1}{4}\cdot\frac{8u}{9} = \frac{2u}{9}.$$

Thus, we have

$$\psi(\triangle\ (u,v)) - \phi(\triangle\ (u,v)) = \frac{4u}{9} - \frac{2u}{9} = \frac{2u}{9}.$$

Hence

$$\psi(d^*(Ru,Sv)) \leq \psi(\triangle\ (u,v)) - \phi(\triangle\ (u,v)).$$

This satisfies (2). If we consider the sequence $\{u_n\} = \{\frac{1}{2n}\}$, then

$$\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} u_n = \lim_{n\to\infty}\frac{1}{2n} = 0, \lim_{n\to\infty} Ru_n \quad = \lim_{n\to\infty}\frac{u_n}{9} = \lim_{n\to\infty}\frac{1}{2n\times 9} = 0.$$

Therefore,

$$\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Ru_n = 0, \text{ where } 0\in M.$$

So, the pair (P,R) satisfied the E.A. property. Also,

$$\lim_{n\to\infty} Pu_n = \lim_{n\to\infty} Ru_n = 0 = P(0).$$

So, the pair (P,R) satisfies the (CLR_P) property. Hence, all the conditions of above Theorems are satisfied. Therefore, P,Q,R and S must have unique common fixed point. Here 0 is the unique common fixed point of P,Q,R and S.

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