Approximation properties of (p, q) bivariate Szász Beta type operators

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Abstract. In the present research article, we construct a new sequence of bivariate (p, q) hybrid type operators using $(p, q)-$ beta functions via Dunkl analogue. In the subsection sequence, we investigate the rate of convergence and the order of approximation for these sequences positive linear operators. Further, we study local approximation results in various class of functions. In the last section, we give the global approximation results using weight function.

Keywords: (p, q)-Bernstein operators; Rate of convergence; Order of approximation; (p, q)- beta operators; weighted spaces.

1. Introduction

The operator theory is an active research area for the last one century. Bernstein was the first who gave the first positive linear operator named as Bernstein operator to approximate the class of continuous functions over $[a, b]$. The motive of Bernstein was to give the elegant proof of Weierstrass approximation theorem using binomial distribution as follows.

$$(1) \qquad \mathcal{B}_n(f(x); x) = \sum_{k=0}^{n} b_{n,k}(x) f(x),$$

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where $b_{n,k}(x) = \binom{n}{k} x^k(1-x)^{n-k}$ and f is a bounded function defined in $C([0,1])$. To improve the rate of convergence of the operators defined by (1), the q-analogues of Bernstein operators were independently given by Lupaş [19] and Phillips [21] using quantum calculus. The (p,q)-analogue of Bernstein operators was given by Mursaleen *et al.* [29] which improves the Bezier curves and radius of convergence of the complex disk due to p-parametres (see Mursaleen and Khan [26]), Khan and Lobiyal [27]. Recently, a Dunkl type generalization [17] of Szász operators [24] via post-quantum calculus was studied by Alotaibi *et al.* [15]. For more details and research motivation in Dunkl type generalizations, we mention here some research articles [4, 11, 8, 12, 20, 22, 23, 28, 29, 30, 31].

Let $f \in C[0,1]$ denote the space of all continuous functions on $[0,1]$. For all $f \in C[0,1]$, $x \geqq 0$, $\tau > -\frac{1}{2}$ and $n \in \mathbb{N}$, the (p,q)-Dunkl analogue of Szász operators [15] (see also [11]) is defined as follows:

$$(2)\quad \mathcal{D}_n^{\mu}(h;u,p,q) = \frac{1}{e_{\mu,p,q}([n]_{p,q}u)} \sum_{k=0}^{\infty} \frac{([n]_{p,q}u)^k}{\gamma_{\mu,p,q}(k)} p^{\frac{k(k-1)}{2}} f\left(\frac{p^{k+2\mu\theta_k} - q^{k+2\mu\theta_k}}{p^{k-1}(p^n-q^n)}\right),$$

where $[n]_{p,q}$ is the (p,q)-integer defined as:

$$(3)\qquad [n]_{p,q} = p^{n-1} + qp^{n-3} + \ldots + q^{n-1} = \begin{cases} \dfrac{p^n-q^n}{p-q}, & (p \neq q \neq 1), \\ \dfrac{1-q^n}{1-q}, & (p=1), \\ n, & (p=q=1), \end{cases}$$

$$(au+bv)_{p,q}^n := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k}b^k u^{n-k} v^k,$$

$$(1-u)_{p,q}^n = (1-u)(p-qu)(p^2-q^2u)\ldots(p^{n-1}-q^{n-1}u),$$

$$(x-y)_{p,q}^n = \begin{cases} \prod_{j=0}^{n-1}(p^jx-q^jy), & \text{if } n \in \mathbb{N}, \\ 1, & \text{if } n = 0. \end{cases}$$

The (p,q)-power basis is explained as

$$(u \oplus v)_{p,q}^n = (u+v)(pu+qv)(p^2u+q^2v)\ldots(p^{n-1}u+q^{n-1}v).$$

Furthermore, the (p,q)-analogues of the exponential function are defined by

$$e_{p,q}(u) = \sum_{k=0}^{\infty} p^{\frac{k(k-1)}{2}} \frac{u^k}{[k]_{p,q}!}, \qquad E_{p,q}(u) = \sum_{k=0}^{\infty} q^{\frac{k(k-1)}{2}} \frac{u^k}{[k]_{p,q}!};$$

Moreover, the (p,q)-Dunkl analogue of the exponential function is defined by

$$e_{\mu,p,q}(u)=\sum_{k=0}^{\infty}p^{\frac{k(k-1)}{2}}\frac{u^k}{\gamma_{\mu,p,q}(k)}, \tag{4}$$

$$\gamma_{\mu,p,q}(k)=\frac{\prod_{i=0}^{[\frac{k+1}{2}]-1}p^{2\mu(-1)^{i+1}+1}((p^2)^i p^{2\mu+1}-(q^2)^i q^{2\mu+1})\prod_{j=0}^{[\frac{k}{2}]-1}p^{2\mu(-1)^j+1}((p^2)^j p^2-(q^2)^j q^2)}{(p-q)^k}, \tag{5}$$

$$\gamma_{\mu,p,q}(k+1)=\frac{p^{2\mu(-1)^{k+1}+1}(p^{2\mu\theta_{k+1}+k+1}-q^{2\mu\theta_{k+1}+k+1})}{(p-q)}\gamma_{\mu,p,q}(k), \tag{6}$$

$$\theta_k=\begin{cases}0, & \text{for } k=2m,\ m=0,1,2,\dots,\\ 1, & \text{for } k=2m+1,\ m=0,1,2,\dots.\end{cases} \tag{7}$$

For $m=0,1,2,\dots n$, the number $[\frac{m}{2}]$ denotes the greatest integer function.

In this section, we construct a class of (p,q)-Bivariate of Szász-beta operators of second kind generated by an exponential function via Dunkl generalization 1.1. This type of the construction of operators are a generalized version of the operators studied in [25].

Definition 1.1. *Let $f\in C([0,1])=\{f(t):\ f(t)=O(t^\rho),\quad t\to\infty,\quad f\in C[0,\infty)\}$ such as $x\in[0,\infty)$, $\rho>n,m$ and $n,m\in\mathbb{N}$. Then for all $0<q<p\leqq 1$, $\mu>-\frac{1}{2}$, $\nu>-\frac{1}{2}$ and $\theta_{\ell_1},\theta_{\ell_2}$ defined by (7), we define*

Let $I_1\times I_2=[0,D_n]\times[0,D_m]$ and $(x,y)\in I_1\times I_2$. Then, for a function $f\in C(I_1\times I_2)$, the (p,q)-Bivariate of Szász-beta operators of second kind generated by an exponential function via Dunkl generalization 1.1, $D^{\mu,\nu}_{n,m}(f;x,y,p_{1,2},q_{1,2})=D^{\mu,\nu}_{n,m}(f;x,y,p_1,p_2,q_1,q_2)$ are defined as follows:

$$\begin{aligned}&\mathcal{D}^{\mu,\nu}_{n,m}(f;x,y,p_{1,2},q_{1,2})\\ &=\sum_{\ell_1=0}^{\infty}\sum_{\ell_2=0}^{\infty}\mathcal{P}^{\mu,l_1}_{n,p_1,q_1}(x)\mathcal{Q}^{\nu,l_2}_{m,p_2,q_2}(y)\int_0^\infty\int_0^\infty\frac{t_1^{\ell_1+2\mu\theta_{\ell_1}}}{(1\oplus p_1t_1)_{p_1,q_1}^{\ell_1+2\mu\theta_{\ell_1}+n+1}}\\ &\times\frac{t_2^{\ell_2+2\nu\theta_{\ell_2}}}{(1\oplus p_2t_2)_{p_2,q_2}^{\ell_2+2\nu\theta_{\ell_2}+m+1}}f(t_1,t_2)\mathrm{d}_{p_1,q_1}t_1,\mathrm{d}_{p_2,q_2}t_2,\end{aligned} \tag{8}$$

where

$$\mathcal{P}^{\mu,l_1}_{n,p_1,q_1}(x)=\frac{1}{e_{\mu,p_1,q_1}([n]_{p_1,q_1}x)}\frac{([n]_{p_1,q_1}x)^{\ell_1}}{\gamma_{\mu,p_1,q_1}(\ell_1)}p_1^{\frac{\ell_1(\ell_1-1)}{2}}\frac{1}{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+1,n)},$$

$$\mathcal{Q}^{\nu,l_2}_{m,p_2,q_2}(y)=\frac{1}{e_{\nu,p_2,q_2}([m]_{p_2,q_2}x)}\frac{([m]_{p_2,q_2}y)^{\ell_2}}{\gamma_{\nu,p_2,q_2}(\ell_2)}p_2^{\frac{\ell_2(\ell_2-1)}{2}}\frac{1}{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+1,m)},$$

and $\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+1,n)$, $\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+1,m)$ *are the Beta functions of second kind in post quantum calculus and is defined by*

$$\mathcal{B}_{p,q}(\alpha,\beta)=\int_0^\infty\frac{t^{\alpha-1}}{(1\oplus pt)^{\alpha+\beta}_{p,q}}\mathrm{d}_{p,q}t,\quad \alpha,\ \beta\in\ \mathbb{N},\tag{9}$$

$$\mathcal{B}_{p,q}(\alpha,\beta)=\frac{[\alpha-1]_{p,q}}{p^{\alpha-1}[\beta]_{p,q}}\mathcal{B}_{p,q}(\alpha-1,\beta+1),\quad \alpha,\ \beta\in\ \mathbb{N}.\tag{10}$$

Moreover, to obtain the basic estimates here we use the following relations:

$$[\ell+1+2\tau\theta_\ell]_{p,q}=q[\ell+2\tau\theta_\ell]_{p,q}+p^{\ell+2\tau\theta_\ell},\tag{11}$$

$$[\ell+2+2\tau\theta_\ell]_{p,q}=q^2[\ell+2\tau\theta_\ell]_{p,q}+(p+q)p^{\ell+2\tau\theta_\ell}.\tag{12}$$

For more related results on (p,q)-analogues, we prefer [1, 2, 3, 16, 18, 5, 10, 7, 6]. We have the following inequalities.

Lemma 1.1. *Let* $f(t)=1,t,t^2$. *Then, the operators* $\mathcal{D}^\mu_n(\,\cdot\,;\,\cdot\,)$ *refer to* (2) *satisfy* $\mathcal{D}^\mu_n(1;x,p,q)=1$ *and the following inequalities hold:*

$$\mathcal{D}^\mu_{n,p,q}(f;x)\leqq\begin{cases}\dfrac{[n]_{p,q}}{[n-1]_{p,q}}x+\dfrac{1}{[n-1]_{p,q}}, & \text{for } f(t)=t\\[2ex] \dfrac{[n]^2_{p,q}}{[n-1]_{p,q}[n-2]_{p,q}}x^2 & \\ \quad+\dfrac{[n]_{p,q}}{[n-1]_{p,q}[n-2]_{p,q}}\left(1+[2]_{p,q}+[1+2\tau]_{p,q}\right)x & \\ \quad+\dfrac{[2]_{p,q}}{[n-1]_{p,q}[n-2]_{p,q}}, & \text{for } f(t)=t^2\end{cases}\tag{13}$$

and

$$\mathcal{D}^\mu_{n,p,q}(f;x)\geqq\begin{cases}\dfrac{q[n]_{p,q}}{[n-1]_{p,q}}x+\dfrac{1}{[n-1]_{p,q}}, & \text{for } f(t)=t\\[2ex] \dfrac{q^3[n]^2_{p,q}}{[n-1]_{p,q}[n-2]_{p,q}}x^2 & \\ \quad+\dfrac{q[n]_{p,q}}{[n-1]_{p,q}[n-2]_{p,q}}\Bigg(q+[2]_{p,q} & \\ \quad+q^{2+2\tau}[1-2\tau]_{p,q}\dfrac{e_{\tau,p,q}\left(\frac{q}{p}[n]_{p,q}x\right)}{e_{\tau,p,q}([n]_{p,q}x)}\Bigg)x & \\ \quad+\dfrac{[2]_{p,q}}{[n-1]_{p,q}[n-2]_{p,q}}, & \text{for } f(t)=t^2.\end{cases}$$

Lemma 1.2. *Let $e_{i,j} = f(t_1, t_2) = t_1^i t_2^j, 0 \leq i, j \leq 2$. Then, the operators $\mathcal{D}_{n,m}^{\mu,\nu}(\,\cdot\,;\,\cdot\,)$ refer to (??) satisfy $\mathcal{D}_{n,m}^{\mu,\nu}(e_{0,0}; x, y, p_{1,2}, q_{1,2}) = 1$ and the following inequalities hold:*

$$\mathcal{D}_{n,m}^{\mu,\nu}(e_{1,0}; x, y, p_{1,2}, q_{1,2}) \leqq \frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}} x + \frac{1}{[n-1]_{p_1,q_1}},$$

$$\mathcal{D}_{n,m}^{\mu,\nu}(e_{0,1}; x, y, p_{1,2}, q_{1,2}) \leqq \frac{[m]_{p_1,q_1}}{[n-1]_{p_1,q_1}} y + \frac{1}{[n-1]_{p_1,q_1}},$$

$$\mathcal{D}_{n,m}^{\mu,\nu}(e_{2,0}; x, y, p_{1,2}, q_{1,2}) \leqq \frac{[n]_{p_1,q_1}^2}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} x^2 + \frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} (1 + [2]_{p_1,q_1} + [1+2\mu]_{p_1,q_1})\, x + \frac{[2]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}},$$

$$\mathcal{D}_{n,m}^{\mu,\nu}(e_{0,2}; x, y, p_{1,2}, q_{1,2}) \leqq \frac{[m]_{p_2,q_2}^2}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} y^2 + \frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} (1 + [2]_{p_2,q_2} + [1+2\nu]_{p_2,q_2})\, y + \frac{[2]_{p_2,q_2}}{[m-1]_{p_2,q_2}[n-2]_{p_2,q_2}}$$

and

$$\mathcal{D}_{n,m}^{\mu,\nu}(e_{1,0}; x, y, p_{1,2}, q_{1,2}) \geqq \frac{q_1[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}} x + \frac{1}{[n-1]_{p_1,q_1}},$$

$$\mathcal{D}_{n,m}^{\mu,\nu}(e_{0,1}; x, y, p_{1,2}, q_{1,2}) \geqq \frac{q_2[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}} y + \frac{1}{[m-1]_{p_2,q_2}},$$

$$\mathcal{D}_{n,m}^{\mu,\nu}(e_{2,0}; x, y, p_{1,2}, q_{1,2}) \geqq \frac{q_1^3[n]_{p_1,q_1}^2}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} x^2 + \frac{[2]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} + \frac{q_1[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} \left(q_1 + [2]_{p_1,q_1} + q_1^{2+2\mu}[1-2\mu]_{p_1,q_1} \frac{e_{\mu,p_1,q_1}\left(\frac{q_1}{p_1}[n]_{p_1,q_1} x\right)}{e_{\mu,p_1,q_1}([n]_{p_1,q_1} x)}\right) x,$$

$$\mathcal{D}_{n,m}^{\mu,\nu}(e_{0,2}; x, y, p_{1,2}, q_{1,2}) \geqq \frac{q_2^3[m]_{p_2,q_2}^2}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} y^2 + \frac{[2]_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} + \frac{q_2[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} \left(q_2 + [2]_{p_2,q_2} + q_2^{2+2\nu}[1-2\nu]_{p_2,q_2} \frac{e_{\nu,p_2,q_2}\left(\frac{q_2}{p_2}[m]_{p_2,q_2} y\right)}{e_{\nu,p_2,q_2}([m]_{p_2,q_2} y)}\right) y.$$

Proof. To prove the results of this Lemma, we use (9)–(12). Take $f(t_1,t_2)=1$. Then,

$$\begin{aligned}
&\mathcal{D}^{\mu,\nu}_{n,m}(e_{0,0};x,y,p_{1,2},q_{1,2})\\
&=\sum_{\ell_1=0}^{\infty}\sum_{\ell_2=0}^{\infty}\mathcal{P}^{\mu,l_1}_{n,p_1,q_1}(x)\mathcal{Q}^{\nu,l_2}_{m,p_2,q_2}(y)\int_0^{\infty}\int_0^{\infty}\frac{t_1^{\ell_1+2\mu\theta_{\ell_1}}}{(1\oplus p_1t_1)_{p_1,q_1}^{\ell_1+2\mu\theta_{\ell_1}+n+1}}\\
&\times\frac{t_2^{\ell_2+2\nu\theta_{\ell_2}}}{(1\oplus p_2t_2)_{p_2,q_2}^{\ell_2+2\nu\theta_{\ell_2}+m+1}}\mathrm{d}_{p_1,q_1}t_1\mathrm{d}_{p_2,q_2}t_2\\
&=\sum_{\ell_1=0}^{\infty}\frac{1}{e_{\mu,p_1,q_1}([n]_{p_1,q_1}x)}\frac{([n]_{p_1,q_1}x)^{\ell_1}}{\gamma_{\mu,p_1,q_1}(\ell_1)}p_1^{\frac{\ell_1(\ell_1-1)}{2}}\frac{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+1,n)}{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+1,n)}\\
&\times\sum_{\ell_2=0}^{\infty}\frac{1}{e_{\nu,p_2,q_2}([m]_{p_2,q_2}x)}\frac{([m]_{p_2,q_2}y)^{\ell_2}}{\gamma_{\nu,p_2,q_2}(\ell_2)}p_2^{\frac{\ell_2(\ell_2-1)}{2}}\frac{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+1,m)}{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+1,m)}=1.
\end{aligned}$$

$$\begin{aligned}
&\mathcal{D}^{\mu,\nu}_{n,m}(e_{1,0};x,y,p_{1,2},q_{1,2})\\
&=\sum_{\ell_1=0}^{\infty}\sum_{\ell_2=0}^{\infty}\mathcal{P}^{\mu,l_1}_{n,p_1,q_1}(x)\mathcal{Q}^{\nu,l_2}_{m,p_2,q_2}(y)\int_0^{\infty}\int_0^{\infty}\frac{t_1^{\ell_1+2\mu\theta_{\ell_1}+1}}{(1\oplus p_1t_1)_{p_1,q_1}^{\ell_1+2\mu\theta_{\ell_1}+n+1}}\\
&\times\frac{t_2^{\ell_2+2\nu\theta_{\ell_2}}}{(1\oplus p_2t_2)_{p_2,q_2}^{\ell_2+2\nu\theta_{\ell_2}+m+1}}\mathrm{d}_{p_1,q_1}t_1\mathrm{d}_{p_2,q_2}t_2\\
&=\sum_{\ell_1=0}^{\infty}\frac{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+2,n-1)}{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+1,n)}\sum_{\ell_2=0}^{\infty}\frac{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+1,m)}{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+1,m)}\\
&=\frac{q_1}{[n-1]_{p_1,q_1}}\sum_{\ell=0}^{\infty}\frac{1}{p_1^{\ell_1+2\mu\theta_{\ell_1}+1}}[\ell_1+2\mu\theta_{\ell_1}]_{p_1,q_1}+\frac{1}{p_1[n-1]_{p_1,q_1}}\\
&\frac{1}{p_1[n-1]_{p_1,q_1}}+\frac{q_1[n]_{p_1,q_1}}{p_1^2[n-1]_{p_1,q_1}}\sum_{\ell_1=0}^{\infty}\left(\frac{p_1^{2\ell_1+2\mu\theta_{2\ell_1}}-q_1^{2\ell_1+2\mu\theta_{2\ell_1}}}{p_1^{2\ell_1-1}(p_1^n-q_1^n)}\right)\\
&+\frac{q_1[n]_{p_1,q_1}}{p_1^{2+2\mu}[n-1]_{p_1,q_1}}\sum_{\ell_1=0}^{\infty}\left(\frac{p_1^{2\ell_1+1+2\mu\theta_{2\ell_1+1}}-q_1^{2\ell_1+1+2\mu\theta_{2\ell_1+1}}}{p_1^{2\ell_1}(p_1^n-q_1^n)}\right).
\end{aligned}$$

Clearly, we have

$$\begin{aligned}
&\mathcal{D}^{\mu,\nu}_{n,p_1,q_1}(e_{1,0};x,y,p_{1,2},q_{1,2})\\
&\geqq\frac{1}{[n-1]_{p_1,q_1}}+\frac{q_1[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}\sum_{\ell_1=0}^{\infty}\left(\frac{p_1^{\ell_1+2\mu\theta_{\ell_1}}-q_1^{\ell_1+2\mu\theta_{\ell_1}}}{p_1^{\ell_1-1}(p_1^n-q_1^n)}\right)\\
&=\frac{1}{[n-1]_{p_1,q_1}}+\frac{q_1[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}C_{n,p_1,q_1}(t_1;x)\\
&=\frac{1}{[n-1]_{p_1,q_1}}+\frac{q_1[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}x
\end{aligned}$$

and

$$\mathcal{D}^{\mu,\nu}_{n,m}(t_1; x, y, p_{1,2}, q_{1,2}) \leqq \frac{1}{[n-1]_{p_1,q_1}} + \frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}} x,$$

$$\begin{aligned}
&\mathcal{D}^{\mu,\nu}_{n,m}(e_{0,1}; x, y, p_{1,2}, q_{1,2}) \\
&= \sum_{\ell_1=0}^{\infty}\sum_{\ell_2=0}^{\infty} \mathcal{P}^{\mu,l_1}_{n,p_1,q_1}(x)\mathcal{Q}^{\nu,l_2}_{m,p_2,q_2}(y)\int_0^{\infty}\int_0^{\infty} \frac{t_1^{\ell_1+2\mu\theta_{\ell_1}}}{(1\oplus p_1t_1)_{p_1,q_1}^{\ell_1+2\mu\theta_{\ell_1}+n+1}} \\
&\times \frac{t_2^{\ell_2+2\nu\theta_{\ell_2}+1}}{(1\oplus p_2t_2)_{p_2,q_2}^{\ell_2+2\nu\theta_{\ell_2}+m+1}} \mathrm{d}_{p_1,q_1}t_1 \mathrm{d}_{p_2,q_2}t_2 \\
&= \sum_{\ell_2=0}^{\infty} \frac{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+2, m-1)}{\mathcal{B}_{p_2,q_1}(\ell_2+2\nu\theta_{\ell_2}+1, m)} \sum_{\ell_1=0}^{\infty} \frac{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+1, n)}{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+1, n)} \\
&= \frac{q_2}{[m-1]_{p_2,q_2}} \sum_{\ell_2=0}^{\infty} \frac{1}{p_2^{\ell_2+2\nu\theta_{\ell_2}+1}} [\ell_2+2\nu\theta_{\ell_2}]_{p_2,q_2} + \frac{1}{p_2[m-1]_{p_2,q_2}} \\
&= \frac{1}{p_2[m-1]_{p_2,q_2}} + \frac{q_2[m]_{p_2,q_2}}{p_2^2[m-1]_{p_2,q_2}} \sum_{\ell_2=0}^{\infty} \left(\frac{p_2^{2\ell_2+2\nu\theta_{2\ell_2}} - q_2^{2\ell_2+2\nu\theta_{2\ell_2}}}{p_2^{2\ell_2-1}(p_2^m - q_2^m)} \right) \\
&+ \frac{q_2[m]_{p_2,q_2}}{p_2^{2+2\nu}[m-1]_{p_2,q_2}} \sum_{\ell_2=0}^{\infty} \left(\frac{p_2^{2\ell_2+1+2\nu\theta_{2\ell_2+1}} - q_2^{2\ell_2+1+2\nu\theta_{2\ell_2+1}}}{p_2^{2\ell_2}(p_2^m - q_2^m)} \right).
\end{aligned}$$

Clearly, we have

$$\begin{aligned}
&\mathcal{D}^{\mu,\nu}_{n,p_1,q_1}(e_{0,1}; x, y, p_{1,2}, q_{1,2}) \\
&\geqq \frac{1}{[m-1]_{p_2,q_2}} + \frac{q_2[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}} \sum_{\ell_2=0}^{\infty} \left(\frac{p_2^{\ell_2+2\nu\theta_{\ell_2}} - q_2^{\ell_2+2\nu\theta_{\ell_2}}}{p_2^{\ell_2-1}(p_2^m - q_2^m)} \right) \\
&= \frac{1}{[m-1]_{p_2,q_2}} + \frac{q_2[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}} C_{m,p_2,q_2}(t_2; y) \\
&= \frac{1}{[m-1]_{p_2,q_2}} + \frac{q_2[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}} y
\end{aligned}$$

and

$$\mathcal{D}^{\mu,\nu}_{n,m}(t_2; x, y, p_{1,2}, q_{1,2}) \leqq \frac{1}{[m-1]_{p_2,q_2}} + \frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}} y.$$

Similarly for $e_{2,0} = f(t) = t_1^2$, we have

$$\begin{aligned}
&\mathcal{D}^{\mu,\nu}_{n,m}(e_{2,0}; x, y, p_{1,2}, q_{1,2}) \\
&= \sum_{\ell_1=0}^{\infty} \mathcal{P}^{\mu,l_1}_{n,p_1,q_1}(x) \int_0^{\infty} \frac{t_1^{\ell_1+2\mu\theta_{\ell_1}+2}}{(1\oplus p_1t_1)_{p_1,q_1}^{\ell_1+2\mu\theta_{\ell_1}+n+1}} \mathrm{d}_{p_1,q_1}t_1 \\
&= \sum_{\ell_1=0}^{\infty} \frac{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+3, n-2)}{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+1, n)} \sum_{\ell_2=0}^{\infty} \frac{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+1, m)}{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+1, m)}
\end{aligned}$$

$$= \sum_{\ell_1=0}^{\infty} \frac{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+3,n-2)}{\mathcal{B}_{p_1,q_1}(\ell_1+2\mu\theta_{\ell_1}+1,n)}$$

$$= \frac{1}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} \sum_{\ell_1=0}^{\infty} \frac{1}{p_1^{3+2\ell_1+4\mu\theta_{\ell_1}+1}} [\ell_1+2\mu\theta_{\ell_1}+1]_{p_1,q_1}[\ell_1+2\mu\theta_{\ell_1}+2]_{p_1,q_1}$$

$$= \frac{q_1^3[n]_{p_1,q_1}^2}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} \sum_{\ell_1=0}^{\infty} \frac{1}{p_1^{5+4\mu\theta_{\ell_1}}} \left(\frac{p_1^{\ell_1+2\mu\theta_{\ell_1}}-q_1^{\ell_1+2\mu\theta_{\ell_1}}}{p_1^{\ell_1-1}(p_1^n-q_1^n)}\right)^2$$

$$+ \frac{q_1(p_1+2q_1)[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} \sum_{\ell_1=0}^{\infty} \frac{1}{p_1^{4+2\mu\theta_{\ell_1}}} \left(\frac{p_1^{\ell_1+2\mu\theta_{\ell_1}}-q_1^{\ell_1+2\mu\theta_{\ell_1}}}{p_1^{\ell_1-1}(p_1^n-q_1^n)}\right)$$

$$+ \frac{(p_1+q_1)}{p_1^3[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} \sum_{\ell_1=0}^{\infty} \mathcal{P}_{n,p-1,q_1}(x).$$

Now, by separating it into even and odd terms and applying θ_{ℓ_1} from (7), i.e., taking $\ell_1 = 2r$ and $\ell_1 = 2r+1$ for all $r = 0,1,2,\ldots$, we have

$$\mathcal{D}_{n,m}^{\mu,\nu}(te_{2,0};x,y,p_{1,2},q_{1,2}) \geqq \frac{q_1^3[n]_{p_1,q_1}^2}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} \sum_{\ell_1=0}^{\infty} \left(\frac{p_1^{\ell_1+2\mu\theta_{\ell_1}}-q_1^{\ell_1+2\mu\theta_{\ell_1}}}{p_1^{\ell_1-1}(p_1^n-q_1^n)}\right)^2$$

$$+ \frac{q_1(q_1+[2]_{p_1,q_1})[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} \sum_{\ell_1=0}^{\infty} \left(\frac{p_1^{\ell_1+2\mu\theta_{\ell_1}}-q_1^{\ell_1+2\mu\theta_{\ell_1}}}{p_1^{\ell_1-1}(p_1^n-q_1^n)}\right)$$

$$+ \frac{[2]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_2}}$$

$$= \frac{q_1^3[n]_{p_1,q_1}^2}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} C_{n,p_1,q_1}(t_1^2;x) + \frac{q_1(q_1+[2]_{p_1,q_1})[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} C_{n,p_1,q_1}(t_1;x)$$

$$+ \frac{[2]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}}.$$

Similarly,

$$\mathcal{D}_{n,m}^{\mu,\nu}(e_{2,0};x,y,p_{1,2},q_{1,2}) \leqq \frac{[n]_{p_1,q_1}^2}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} C_{n,p_1,q_1}(t_1^2;x)$$
$$+ \frac{(1+[2]_{p_1,q_1})[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} C_{n,p_1,q_1}(t_1;x) + \frac{[2]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}}.$$

Similarly, for $e_{0,2} = f(t) = t_2^2$, we have

$$\mathcal{D}_{n,m}^{\mu,\nu}(e_{0,2};x,y,p_{1,2},q_{1,2}) = \sum_{\ell_2=0}^{\infty} \mathcal{Q}_{m,p_2,q_2}^{\nu,l_2}(y) \int_0^{\infty} \frac{t_2^{\ell_2+2\nu\theta_{\ell_2}+2}}{(1\oplus p_2t_2)_{p_2,q_2}^{\ell_2+2\nu\theta_{\ell_2}+m+1}} \mathrm{d}_{p_2,q_2}t_2$$

$$= \sum_{\ell_2=0}^{\infty} \frac{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+3,m-2)}{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+1,m)}$$

$$= \sum_{\ell_2=0}^{\infty} \frac{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+3, m-2)}{\mathcal{B}_{p_2,q_2}(\ell_2+2\nu\theta_{\ell_2}+1, m)}$$

$$= \frac{1}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} \sum_{\ell_2=0}^{\infty} \frac{1}{p_2^{3+2\ell_2+4\nu\theta_{\ell_2}+1}} [\ell_2+2\nu\theta_{\ell_2}+1]_{p_2,q_2}$$

$$[\ell_2+2\nu\theta_{\ell_2}+2]_{p_2,q_2}$$

$$= \frac{q_1^3[m]^2_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} \sum_{\ell_2=0}^{\infty} \frac{1}{p_2^{5+4\nu\theta_{\ell_2}}} \left(\frac{p_2^{\ell_2+2\nu\theta_{\ell_2}} - q_2^{\ell_2+2\nu\theta_{\ell_2}}}{p_2^{\ell_2-1}(p_2^m - q_2^m)} \right)^2$$

$$+ \frac{q_2(p_2+2q_2)[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} \sum_{\ell_2=0}^{\infty} \frac{1}{p_2^{4+2\nu\theta_{\ell_2}}} \left(\frac{p_2^{\ell_2+2\nu\theta_{\ell_2}} - q_2^{\ell_2+2\nu\theta_{\ell_2}}}{p_2^{\ell_2-1}(p_2^m - q_2^m)} \right)$$

$$+ \frac{(p_2+q_2)}{p_2^3[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} \sum_{\ell_2=0}^{\infty} \mathcal{P}_{m,p-2,q_2}(y).$$

Now, by separating it into even and odd terms and applying θ_{ℓ_2} from (7), i.e., taking $\ell_2 = 2r$ and $\ell_2 = 2r+1$ for all $r = 0, 1, 2, \ldots$, we have

$$\begin{aligned}
\mathcal{D}^{\mu,\nu}_{n,m}(e_{0,2}; x, y, p_{1,2}, q_{1,2}) \geqq{} & \frac{q_2^3[m]^2_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} \sum_{\ell_2=0}^{\infty} \left(\frac{p_2^{\ell_2+2\nu\theta_{\ell_2}} - q_2^{\ell_2+2\nu\theta_{\ell_2}}}{p_2^{\ell_2-1}(p_2^m - q_2^m)} \right)^2 \\
& + \frac{q_2(q_2+[2]_{p_2,q_2})[m]_{p_2,q_2}}{[m-1]_{2_1,q_2}[m-2]_{p_2,q_2}} \sum_{\ell_2=0}^{\infty} \left(\frac{p_2^{\ell_2+2\nu\theta_{\ell_2}} - q_2^{\ell_2+2\nu\theta_{\ell_2}}}{p_2^{\ell_2-1}(p_2^m - q_2^m)} \right) \\
& + \frac{[2]_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} \\
={} & \frac{q_2^3[m]^2_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} C_{m,p_2,q_2}(t_2^2; x, y) \\
& + \frac{q_2(q_2+[2]_{p_2,q_2})[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} C_{m,p_2,q_2}(t_2; y) \\
& + \frac{[2]_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}}.
\end{aligned}$$

Similarly,

$$\mathcal{D}^{\mu,\nu}_{n,m}(e_{0,2}; x, y, p_{1,2}, q_{1,2}) \leqq \frac{[m]^2_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} C_{m,p_2,q_2}(t_2^2; y)$$

$$+ \frac{(1+[2]_{p_2,q_2})[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} C_{m,p_2,q_2}(t_2; y) + \frac{[2]_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}}.$$

This completes the proof of Lemma 1.2. □

Lemma 1.3. *Let $\Psi_{i,j} = (t_1 - x)^i (t_2 - x)^j$ for $i, j = 1, 2$, then we have following inequalities:*

1. $\mathcal{D}^{\mu,\nu}_{n,m}(\Psi_{1,0}; x, y, p_{1,2}, q_{1,2}) \leqq \left(\frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}} - 1 \right) x + \frac{1}{[n-1]_{p_1,q_1}},$
   for $n > 1,\ n \in \mathbb{N}$
2. $\mathcal{D}^{\mu,\nu}_{n,m}(\Psi_{0,1}; x, y, p_{1,2}, q_{1,2}) \leqq \left(\frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}} - 1 \right) y + \frac{1}{[m-1]_{p_2,q_2}},$
   for $n > 1,\ m \in \mathbb{N}$
3. $$\mathcal{D}^{\mu,\nu}_{n,m}(\Psi_{2,0}; x, y, p_{1,2}, q_{1,2}) \leqq \left(\frac{[n]^2_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}} - \frac{2[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}} + 1 \right) x^2 + \frac{1}{[n-1]_{p_1,q_1}} \left(\frac{[n]_{p_1,q_1}}{[n-2]_{p_1,q_1}} \big(1 + [2]_{p_1,q_1} + [1+2\mu]_{p_1,q_1}\big) - 2 \right) x + \frac{[2]_{p_1,q_1}}{[n-1]_{p_1,q_1}[n-2]_{p_1,q_1}}, \qquad \text{for} \quad n > 2,\ n \in \mathbb{N}.$$
4. $$\mathcal{D}^{\mu,\nu}_{n,m}(\Psi_{0,2}; x, y, p_{1,2}, q_{1,2}) \leqq \left(\frac{[m]^2_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}} - \frac{2[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}} + 1 \right) y^2 + \frac{1}{[m-1]_{p_2,q_2}} \left(\frac{[m]_{p_2,q_2}}{[m-2]_{p_2,q_2}} \big(1 + [2]_{p_2,q_2} + [1+2\nu]_{p_2,q_2}\big) - 2 \right) y + \frac{[2]_{p_2,q_2}}{[m-1]_{p_2,q_2}[m-2]_{p_2,q_2}}, \qquad \text{for} \quad m > 2,\ m \in \mathbb{N}.$$

Definition 1.2. *Let $X, Y \subset \mathbb{R}$ be any two given intervals and the set $B(X \times Y) = \{f : X \times Y \to \mathbb{R} | f$ is bounded on $X \times Y\}$. For $f \in B(X \times Y)$, let the function $\omega_{total}(f; \cdot, *) : [0,\infty) \times [0,\infty) \to \mathbb{R}$, defined for any $(\delta_1, \delta_2) \in [0,\infty) \times [0,\infty)$ by $\omega_{total}(f; \delta_1, \delta_2) = \sup_{|x-x'| \leq \delta_1, |y-y'| \leq \delta_2} \{|f(x,y) - f(x',y')| : (x,y), (x',y') \in [0,\infty) \times [0,\infty)\}$, is called the first order modulus of smoothness of the function f or the total modulus of continuity of the function f.*

In order to get the rate of convergence and degree of approximation for the operators $\mathcal{D}^{\mu,\nu}_{n,m}$, we consider $p_1 = p_n$, $p_2 = p_m$ and $q_1 = q_n$, $q_2 = q_m$ such that $0 < q_n < p_n \leq 1$ and $0 < q_m < p_m \leq 1$ satisfying

$$\lim_{n\to\infty} q_n^n \to a, \lim_{m\to\infty} q_m^m \to b, \lim_{n\to\infty} p_n^n \to c, \lim_{m\to\infty} p_m^m \to d \tag{14}$$

and

$$\lim_{n\to\infty} p_n \to 1, \lim_{m\to\infty} p_m \to 1, \lim_{n\to\infty} q_n \to 1, \lim_{m\to\infty} q_m \to 1, \tag{15}$$

where $0 \leq a, b < c, d < 1$. Here, we recall the following result due to Volkov [14]:

Theorem 1.1. *Let I and J be compact intervals of the real line. Let $L_{n,m} : C(I \times J) \to C(I \times J), (n,m) \in \mathbb{N} \times \mathbb{N}$ be linear positive operators. If*

$$\lim_{n,m\to\infty} L_{n,m}(e_{ij}) \quad = \quad e_{x,y}, (i,j) \in \{(0,0), (1,0), (0,1)\}$$

and

$$\lim_{n,m\to\infty} L_{n,m}(e_{20}+e_{02}) = e_{20}+e_{02},$$

uniformly on $I \times J$, then the sequence $(L_{n,m}f)$ converges to f uniformly on $I \times J$ for any $f \in C(I \times J)$.

Theorem 1.2. *Let $e_{ij}(t_1,t_2) = t_1^i t_2^j$ $(0 \le i+j \le 2, i,j \in \mathbb{N})$ be the test functions defined on $J_1 \times J_2$ and $(p_n),(q_n),(p_m),(q_m)$ be the sequences defined by (14) and (15). If*

$$\lim_{n,m\to\infty} (\mathcal{D}^{\mu,\nu}_{n,m} e_{ij})(t_1,t_2) = e_{ij}(t_1,t_2), (i,j) \in \{(0,0),(1,0),(0,1)\}$$

and

$$\lim_{n,m\to\infty} (\mathcal{D}^{\mu,\nu}_{n,m}(e_{20}+e_{02})(t_1,t_2) = e_{20}(t_1,t_2)+e_{02}(t_1,t_2),$$

uniformly on $J_1 \times J_2$, then

$$\lim_{n,m\to\infty} (\mathcal{D}^{\mu,\nu}_{n,m} f)(t_1,t_2) = f(t_1,t_2),$$

uniformly for any $f \in C(J_1 \times J_2)$.

Proof. Using Lemma 1.2, it is obvious for $i=j=0$

$$\lim_{n,m\to\infty} (\mathcal{D}^{\mu,\nu}_{n,m} e_{00})(t_1,t_2) = e_{00}(t_1,t_2).$$

For $i=1$ and $j=0$, we have

$$\begin{aligned} \lim_{n,m\to\infty} (\mathcal{D}^{\mu,\nu}_{n,m} e_{10})(t_1,t_2) &= t_1, \\ \lim_{n,m\to\infty} (\mathcal{D}^{\mu,\nu}_{n,m} e_{10})(t_1,t_2) &= e_{10}(t_1,t_2), \end{aligned}$$

For $i=0$ and $j=1$, we have

$$\begin{aligned} \lim_{n,m\to\infty} (\mathcal{D}^{\mu,\nu}_{n,m} e_{01})(t_1,t_2) &= t_2, \\ \lim_{n,m\to\infty} (\mathcal{D}^{\mu,\nu}_{n,m} e_{01})(t_1,t_2) &= e_{01}(t_1,t_2), \end{aligned}$$

and

$$\begin{aligned} \lim_{n,m\to\infty} (\mathcal{D}^{\mu,\nu}_{n,m}(e_{20}+e_{02})(t_1,t_2) &= \lim_{n,m\to\infty} \left\{ \frac{p_1^{n-1} b_n}{[n]_{p_1,q_1}} x + \frac{q_1[n-1]_{p_1,q_1}}{[n]_{p_1,q_1}} x^2 \right. \\ &+ \left. \frac{p_2^{m-1} b_m}{[m]_{p_2,q_2}} y + \frac{q_2[m-1]_{p_2,q_2}}{[n]_{p_2,q_2}} y^2 \right\}, \\ \lim_{n,m\to\infty} (\mathcal{D}^{\mu,\nu}_{n,m}(e_{20}+e_{02})(x,y) &= e_{20}(x,y)+e_{02}(x,y). \end{aligned}$$

From Theorem 1.1, the proof of theorem 1.2 is completed. □

Theorem 1.3 ([13]). *Let $L : C([0,\infty) \times [0,\infty)) \to B([0,\infty) \times [0,\infty))$ be a linear positive operator. For any $f \in C(X \times Y)$, any $(x,y) \in X \times Y$ and any $\delta_1, \delta_2 > 0$, the following inequality*

$$\begin{aligned}|(Lf)(x,y) - f(x,y)| \leq |Le_{0,0}(x,y) - 1||f(x,y)| + \Big[Le_{0,0}(x,y) \\ + \delta_1^{-1}\sqrt{Le_{0,0}(x,y)(L(\cdot - x))^2(x,y)} \\ + \delta_2^{-1}\sqrt{Le_{0,0}(x,y)(L(* - y))^2(x,y)} \\ + \delta_1^{-1}\delta_2^{-1}\sqrt{(Le_{0,0})^2(x,y)(L(\cdot - x))^2(x,y)(L(* - y))^2(x,y)}\Big] \\ \times \omega_{total}(f;\delta_1,\delta_2),\end{aligned}$$

holds.

Theorem 1.4. *Let $f \in C(J_1 \times J_2)$ and $(x,y) \in J_1 \times J_2$. Then, for $(n,m) \in \mathbb{N}$ and for any $\delta_1, \delta_2 > 0$, we have*

$$|(C_{n,m}f)(x,y) - f(x,y)| \leq 4\omega_{total}(f;\delta_1,\delta_2),$$

where $\delta_1 = \sqrt{\mathcal{D}_{n,m}^{\mu,\nu}((t_1-x)^2, x, y, p_{12}, q_{12})}$ and $\delta_2 = \sqrt{\mathcal{D}_{n,m}^{\mu,\nu}((t_2-y)^2, x, y, p_{12}, q_{12})}$.

Proof. From Theorem 1.3, we have

$$\begin{aligned}|(\mathcal{D}_{n,m}^{\mu,\nu}f)(x,y) - f(x,y)| \leq \Big[1 + +\delta_1^{-1}\sqrt{\mathcal{D}_{n,m}^{\mu,\nu}((t_1-x)^2)(x,y))} \\ + \delta_2^{-1}\sqrt{\mathcal{D}_{n,m}^{\mu,\nu}((t_2-y)^2)(x,y)} \\ + \delta_1^{-1}\delta_2^{-1}\sqrt{\mathcal{D}_{n,m}^{\mu,\nu}((t_1-x)^2)(x,y))\mathcal{D}_{n,m}^{\mu,\nu}((t_2-y)^2)(x,y)}\Big] \times \omega_{total}(f;\delta_1,\delta_2).\end{aligned}$$

On choosing $\delta = \sqrt{\mathcal{D}_{n,m}^{\mu,\nu}((t_1-x)^2)(x,y)}$ and $\delta_2 = \sqrt{\mathcal{D}_{n,m}^{\mu,\nu}((t_2-y)^2)(x,y)}$, we get the required result. □

Now, we shall investigate degree of approximation for the operators $C_{n,m}$ in Lipschitz class. We consider the Lipschitz class $Lip_M(\gamma_1,\gamma_2)$ in terms of two variables as follows:

$$|f(t_1,t_2) - f(x,y)| \leq M|t_1 - x|^{\gamma_1}|t_2 - y|^{\gamma_2},$$

where $M > 0$, $0 < \gamma_1, \gamma_2 \leq 1$ and for any $(t_1,t_2), (x,y) \in J_1 \times J_2$.

Theorem 1.5. *For $f \in Lip_M(\gamma_1,\gamma_2)$, we have*

$$|\mathcal{D}_{n,m}^{\mu,\nu}(f;q_n,q_m,p_n,p_m;x,y) - f(x,y)| \leq M\delta_n^{\gamma_1/2}(x)\delta_m^{\gamma_2/2}(y),$$

where $\delta_n(x) = \mathcal{D}_{n,m}^{\mu,\nu}((t_1-x)^2;q_n,q_m,p_n,p_m;x,y)$ and

$$\delta_m(y) = \mathcal{D}_{n,m}^{\mu,\nu}((t_2-y)^2;q_n,q_m,p_n,p_m;x,y).$$

Proof. Since $f \in Lip_M(\gamma_1, \gamma_2)$, we can write

$$\begin{aligned}
&|\mathcal{D}_{n,m}^{\mu,\nu}(f; q_n, q_m, p_n, p_m; x, y) - f(x,y)| \\
&\leq \mathcal{D}_{n,m}^{\mu,\nu}(|f(t_1,t_2) - f(x,y)|; q_n, q_m, p_n, p_m; x, y) \\
&\leq M\mathcal{D}_{n,m}^{\mu,\nu}(|t_1 - x|^{\gamma_1}|t_2 - y|^{\gamma_2}; q_n, q_m, p_n, p_m; x, y) \\
&= M\mathcal{D}_{n,m}^{\mu,\nu}(|t_1 - x|^{\gamma_1}; q_n, q_m, p_n, p_m; x, y) \times (|t_2 - y|^{\gamma_2}; q_n, q_m, p_n, p_m; x, y).
\end{aligned}$$

Using Hölder inequality with $\alpha_1 = \frac{2}{\gamma_1}$, $\beta_1 = \frac{2}{2-\gamma_1}$ and $\alpha_2 = \frac{2}{\gamma_2}$, $\beta_2 = \frac{2}{2-\gamma_2}$, respectively, we get

$$\begin{aligned}
|\mathcal{D}_{n,m}^{\mu,\nu}(f; q_n, q_m, p_n, p_m; x, y) - f(x,y)| &\leq \left\{\mathcal{D}_{n,m}^{\mu,\nu}((t_1 - x)^2; q_n, q_m, p_n, p_m; x, y)\right\}^{\frac{\gamma_1}{2}} \\
&\times \left\{\mathcal{D}_{n,m}^{\mu,\nu}(1; q_n, q_m, p_n, p_m; x, y)\right\}^{\frac{2}{2-\gamma_1}} \\
&\times \left\{\mathcal{D}_{n,m}^{\mu,\nu}((t_2 - x_2)^2; q_n, q_m, p_n, p_m; x, y)\right\}^{\frac{\gamma_2}{2}} \\
&\times \left\{\mathcal{D}_{n,m}^{\mu,\nu}(1; q_n, q_n, p_n, p_m; x, y)\right\}^{\frac{2}{2-\gamma_2}} \\
&= M\delta_n^{\gamma_1/2}(x)\delta_m^{\gamma_2/2}(y),
\end{aligned}$$

which completes the proof of Theorem 1.5. □

Here, we discuss degree of approximation in weighted space for the operators defined by (**??**). We recall some basic notions from [**?**] as follows

$B_\rho([0,\infty) \times [0,\infty))$ is the space of all functions defined on $\mathbb{R}_+^2 = [0,\infty) \times [0,\infty)$ with the condition $|f(x,y)| \leq M_f \rho(x,y)$, where M_f is a positive constant depending on f and $\rho(x,y) = 1 + x^2 + y^2$ is a weight function. $C_\rho([0,\infty) \times [0,\infty)) = \{f : f$ is a continuous function in to $B_\rho([0,\infty) \times [0,\infty))\}$ equipped with the norm $\|f\|_\rho = \sup_{(x,y)\in\mathbb{R}_+^2} \frac{|f(x,y)|}{\rho(x,y)}$ and $C_\rho^k([0,\infty) \times [0,\infty)) = \{f : f \in C_\rho$ and $\lim_{x,y\to\infty} \frac{|f(x,y)|}{\rho(x,y)} < k\}$. For all $f \in C_\rho^k$, the weighted modulus of continuity is defined as

$$\omega_\rho(f; \delta_1, \delta_2) = \sup_{(x,y)\in\mathbb{R}_+^2} \sup_{|h_1|\leq\delta_1, |h_2|\leq\delta_2} \frac{|f(x+h_1, y+h_2) - f(x,y)|}{\rho(x,y)\rho(h_1,h_2)}$$

and

$$\begin{aligned}
&|f(t_1,t_2) - f(x,y)| \leq 8(1 + x^2 + y^2)\omega_\rho(f; \delta_n, \delta_m) \\
&\times \left(1 + \frac{|t_1 - x|}{\delta_n}\right)\left(1 + \frac{|t_2 - y|}{\delta_m}\right)(1 + (t_1 - x)^2)(1 + (t_2 - y)^2). \qquad (16)
\end{aligned}$$

Theorem 1.6. *If the operators $\mathcal{D}_{n,m}^{\mu,\nu}(.;\ .)$ defined by* (**??**) *satisfying the conditions*

$$\begin{aligned}
\lim_{n,m\to\infty} \|\mathcal{D}_{n,m}^{\mu,\nu}(e_{0,0}; .) - e_{0,0}|\| = 0, \\
\lim_{n,m\to\infty} \|\mathcal{D}_{n,m}^{\mu,\nu}(e_{1,0}; .) - e_{1,0}|\| = 0, \\
\lim_{n,m\to\infty} \|\mathcal{D}_{n,m}^{\mu,\nu}(e_{0,1}; .) - e_{0,1}|\| = 0,
\end{aligned}$$

and

$$\lim_{n,m\to\infty} \|\mathcal{D}^{\mu,\nu}_{n,m}(e_{2,0}+e_{0,2};.)-(e_{2,0}+e_{0,2})|\| = 0.$$

Then

$$\lim_{n,m\to\infty} \|\mathcal{D}^{\mu,\nu}_{n,m}(f;.)-f|\| = 0,$$

for each $f \in C^k_\rho([0,\infty)\times[0,\infty))$.

Proof. In view of Lemma 1.2, we completes the proof of Theorem 1.6. □

Theorem 1.7. *Let* $f \in C^k_\rho([0,\infty)\times[0,\infty))$. *Then,*

$$\sup_{(x,y)\in\mathbb{R}^2_+} \frac{|\mathcal{D}^{\mu,\nu}_{n,m}(f;x,y,p_n,p_m,q_n,q_m)-f(x,y)|}{(1+x^2+y^2)^3} \leq K\omega_\rho(f;\delta_n,\delta_m)$$

holds for large values of n,m, *where*

$$\delta_n = o\left(\frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}\right) \quad and \quad \delta_m = o\left(\frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}}\right).$$

Proof. From (16) and the operators (**??**), we have

$$\begin{aligned}
&|\mathcal{D}^{\mu,\nu}_{n,m}(f;x,y,p_n,p_m,q_n,q_m)-f(x,y)|\\
&\leq 8(1+x^2+y^2)\omega_\rho(f;\delta_n,\delta_m)\\
&\times\left(1+\frac{\mathcal{D}^{\mu,\nu}_{n,m}(|t_1-x|;x,y,p_n,p_m,q_n,q_m)}{\delta_n}\right)\\
&\times\left(1+\frac{\mathcal{D}^{\mu,\nu}_{n,m}(|t_2-y|;x,y,p_n,p_m,q_n,q_m)}{\delta_m}\right)\\
&\times(1+\mathcal{D}^{\mu,\nu}_{n,m}((t_1-x)^2;x,y,p_n,p_m,q_n,q_m))\\
&\times(1+\mathcal{D}^{\mu,\nu}_{n,m}((t_2-y)^2;x,y,p_n,p_m,q_n,q_m)).
\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&|\mathcal{D}^{\mu,\nu}_{n,m}(f;x,y,p_n,p_m,q_n,q_m)-f(x,y)|\\
&\leq 8(1+x^2+y^2)\omega_\rho(f;\delta_n,\delta_m)\Bigg[1+\mathcal{D}^{\mu,\nu}_{n,m}((t_1-x)^2;x,y,p_n,p_m,q_n,q_m))\\
&\frac{\sqrt{\mathcal{D}^{\mu,\nu}_{n,m}((t_1-x)^2;x,y,p_n,p_m,q_n,q_m))}}{\delta_n}\\
&\times\frac{\sqrt{\mathcal{D}^{\mu,\nu}_{n,m}((t_1-x)^2;x,y,p_n,p_m,q_n,q_m))\mathcal{D}^{\mu,\nu}_{n,m}((t_1-x)^4;x,y,p_n,p_m,q_n,q_m))}}{\delta_n}\Bigg]
\end{aligned}$$

$$\times\Bigg[1+\mathcal{D}^{\mu,\nu}_{n,m}((t_2-y)^2;x,y,p_n,p_m,q_n,q_m)) + \frac{\sqrt{\mathcal{D}^{\mu,\nu}_{n,m}((t_2-y)^2;x,y,p_n,p_m,q_n,q_m))}}{\delta_m} \times \frac{\sqrt{\mathcal{D}^{\mu,\nu}_{n,m}((t_2-y)^2;x,y,p_n,p_m,q_n,q_m))\mathcal{D}^{\mu,\nu}_{n,m}((t_2-y)^4; x,y,p_n,p_m,q_n,q_m))}}{\delta_m}\Bigg]. \tag{17}$$

From Lemma 1.2, we have

$$\begin{aligned}
&\mathcal{D}^{\mu,\nu}_{n,m}((t_1-x);q_n,q_m,p_n,p_m,x,y) &&\leq o\left(\frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}\right)x,\\
&\mathcal{D}^{\mu,\nu}_{n,m}((t_2-y)^2;q_n,q_m,p_n,p_m,x,y) &&\leq o\left(\frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}}\right)y,\\
&\mathcal{D}^{\mu,\nu}_{n,m}((t_1-x)^2;q_n,q_m,p_n,p_m,x,y) &&\leq o\left(\frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}\right)(x^2+x),\\
&\mathcal{D}^{\mu,\nu}_{n,m}((t_2-x)^2;q_n,q_m,p_n,p_m,x,y) &&\leq o\left(\frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}}\right)(y^2+y).
\end{aligned} \tag{18}$$

Combining (17) and all identities in (18), we obtain

$$\begin{aligned}
&|\mathcal{D}^{\mu,\nu}_{n,m}(f;x,y,p_n,p_m,q_n,q_m)-f(x,y)|\\
&\leq 8(1+x^2+y^2)\omega_\rho(f;\delta_n,\delta_m)\left[1+o\left(\frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}\right)x\right.\\
&\left.+\frac{\sqrt{o\left(\frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}\right)x}}{\delta_n}\times\frac{\sqrt{o\left(\frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}}\right)xo\left(\right)(x^2+x)}}{\delta_n}\right]\\
&\times\left[1+o\left(\frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}}\right)y+\frac{\sqrt{o\left(\frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}}\right)y}}{\delta_m}\right.\\
&\left.\times\frac{\sqrt{o\left(\frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}}\right)yo\left(\frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}}\right)(y^2+y)}}{\delta_n}\right].
\end{aligned}$$

□

Choosing $\delta_n=o(\frac{[n]_{p_1,q_1}}{[n-1]_{p_1,q_1}})$ and $\delta_m=o(\frac{[m]_{p_2,q_2}}{[m-1]_{p_2,q_2}})$, we find

$$|\mathcal{D}^{\mu,\nu}_{n,m}(f;x,y,p_n,p_m,q_n,q_m)-f(x,y)|$$

$$\leq 8(1+x^2+y^2)\omega_\rho(f;\delta_n,\delta_m)\left[1+\delta_n x+\sqrt{x}\frac{\sqrt{x(x^2+x)}}{\delta_n}\right]$$
$$\times\left[1+\delta_m y+\sqrt{y}\frac{\sqrt{y(y^2+y)}}{\delta_n}\right].$$

For sufficiently large values of n and m, we have

$$\sup_{(x,y)\in\mathbb{R}_+^2}\frac{|\mathcal{D}_{n,m}^{\mu,\nu}(f;x,y,p_n,p_m,q_n,q_m)-f(x,y)|}{(1+x^2+y^2)^3}\leq K\omega_\rho(f;\delta_n,\delta_m),$$

where K is a positive constant independent of n, m and $\delta_n<1$, $\delta_m<1$.

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