Error estimates of two-grid method for second-order nonlinear hyperbolic equation

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Abstract. In this paper, the full discrete scheme of mixed finite element approximation is introduced for second-order nonlinear hyperbolic equation. In order to deal with the nonlinear mixed-method equations efficiently, a two-grid algorithm is considered. Numerical stability and error estimate are proved on both the coarse grid and fine grid. It is shown that the two-grid method can achieve asymptotically optimal approximation as long as the mesh sizes satisfy $h = \mathcal{O}(H^{(2k+1)/(k+1)})$. Some numerical results are provided to confirm the theoretical analysis.
Keywords: nonlinear hyperbolic equation, mixed finite element method, two-grid method, error estimate.

1. Introduction

In this paper, we consider the following nonlinear hyperbolic equation

$$u_{tt} - \nabla \cdot (K(u)\nabla u) = f, \quad (\boldsymbol{x}, t) \in \Omega \times J, \tag{1.1}$$

$$u(\boldsymbol{x}, t) = 0, \quad (\boldsymbol{x}, t) \in \partial\Omega \times J, \tag{1.2}$$

$$u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}),\ u_t(\boldsymbol{x}, 0) = u_1(\boldsymbol{x}), \quad \boldsymbol{x} \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain, $J = (0, T]$, $K(u) : \Omega \times \mathbb{R} \to \mathbb{R}^{2\times 2}$ is a symmetric and uniformly positive definite bounded tensor.

Hyperbolic equations can demonstrate many physical processes and phenomena such as vibrations of a membrane, acoustic vibrations of a gas, hydrodynamics, displacement problems in porous media, etc. Lots of numerical methods have been developed for solving these model problems. Such as finite difference

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methods [1, 2], finite element methods [3, 4, 15, 18], mixed finite element methods [5-7] and so on. In this paper, we consider a mixed element method for nonlinear hyperbolic equation in which the coefficient K is nonlinear.

The mixed finite element method (MFEM), as a type of powerful numerical tool for solving differential problems, was extensively used in the analysis of engineering and scientific computation. In the past decades, the theoretical framework and the basic tools for the analysis of the MFEM have been developed. Perhaps the most important property of the MFEM is that it can simultaneously approximate both the scalar (pressure) and vector (flux) functions. The advantage of this approach has attracted many researchers to do research in this field. For example, there are some papers such as [8, 11, 19] on elliptic equations and parabolic equations. There are also some papers such as [5-7] on the MFEM for the linear and semilinear hyperbolic problems.

For the mixed method, the problem (1.1) is often rewritten by introducing a new variable

$$\boldsymbol{z} = -K(u)\nabla u,$$

or equivalently

$$\kappa(u)\boldsymbol{z} = -\nabla u, \tag{1.4}$$

as

$$u_{tt} + \nabla \cdot \boldsymbol{z} = f, \tag{1.5}$$

where $\kappa(u) = K^{-1}(u)$ is a square-integrable, symmetric, uniformly positive-definite tensor defined on Ω, and there exist constants K_*, $K^* > 0$, such that

$$K_*|\boldsymbol{y}|^2 \leq \boldsymbol{y}^T\kappa(u)\boldsymbol{y} \leq K^*|\boldsymbol{y}|^2,\ \boldsymbol{y} \in \mathbb{R}^2. \tag{1.6}$$

As we know, the resulting algebraic system of equations is a large systems of nonlinear equations. Therefore, it is necessary for us to study an effective algorithm for this essential system. We will consider a two-grid method inspired by Xu [9, 10]. The key feature of this method is that it can reduce the complexity of the original problem and save the computational time. Thus, many articles utilize this method to numerically solve differential equations and developed some new numerical techniques based on the idea of two-grid algorithm [11-18]. Now, the two-grid methods have been proved to be efficient discretization techniques for the complicated problems (nonsymmetric indefinite or nonlinear, etc.) of various type.

For the hyperbolic equations, Chen et al. [16] discussed a two-grid method for semilinear problem by using finite volume element method. Later on, they also investigate this method for the nonlinear case [17]. Recently, in [18], the two-grid method was presented to solve the two-dimensional nonlinear hyperbolic equation by the bilinear finite element. In this work, we use a two-grid method based on MFEM to approximate the solution of (1.1). We first solve a nonlinear MFE system on a coarse grid, then we use the known coarse grid solution and a Taylor expansion to get the solution of a linear system on the fine grid. As shown in [9, 10], the coarse mesh can be quite coarse and still maintain a good accuracy approximation. The novelty and major achievement of this paper is that we successfully extend the two-grid method to solve the nonlinear hyperbolic problems by the MFEM. Convergence rate in both time and space is proved.

This paper is organized as follows. In Section 2, we present a two-grid algorithm combined with the fully discrete MFEM for (1.1). In Section 3, we carry out the stability analysis for two-grid method. In Section 4, we deduce the error estimates for both the coarse grid and fine grid. In Section 5, we give some numerical experiments to verify the theoretical results.

Throughout this paper, let C denote a generic positive constant independent of mesh parameters with possibly different values in different contexts. Let $L^p(\Omega)$ for $1 \leq p < \infty$ denote the standard Banach space defined on Ω, with norm $||\cdot||_p$. For any nonnegative integer m, let $W^{m,p}(\Omega) = \{\mu \in L^p(\Omega),\ D^\vartheta\mu \in L^p(\Omega),\ |\vartheta| \leq m\}$ denote the Sobolev spaces endowed with the norm $||\mu||^p_{m,p} = \sum_{|\vartheta|\leq m} ||D^\vartheta\mu||^p_{L^p(\Omega)}$. When $p = 2$, we omit the subscript.

2. The two-grid algorithm based on MFEM

Let $W = L^2(\Omega)$ and $\boldsymbol{V} = H(div;\Omega)$. The weak form for the mixed problem (1.4)-(1.5) is to seek a pair of functions: $(u, \boldsymbol{z}) : (0, T) \to W \times \boldsymbol{V}$ satisfying

$$(u_{tt}, w) + (\nabla\cdot\boldsymbol{z}, w) = (f, w), \quad \forall w \in W, \tag{2.7}$$

$$(\kappa(u)\boldsymbol{z}, \boldsymbol{v}) - (\nabla\cdot\boldsymbol{v}, u) = 0, \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{2.8}$$

with $u(0) = u_0$ and $u_t(0) = u_1$.

Let $\mathcal{T}_h$ be a quasi-uniform family of finite element partition of Ω into triangles or rectangles with the mesh size h. We take finite-dimensional subspaces $W_h \times \boldsymbol{V}_h \subset W \times \boldsymbol{V}$, using Raviart-Thomas ($RT$) mixed finite element space [19] of index k, where k is fixed nonnegative integer, associated with $\mathcal{T}_h$. The following inclusion holds for the RT_k spaces

$$\nabla\cdot\boldsymbol{v}_h \in W_h, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{2.9}$$

Let Q_h be the L^2 projection of W onto W_h such that

$$(\alpha, w_h) = (Q_h\alpha, w_h), \quad \forall w_h \in W_h,\ \alpha \in L^2(\Omega). \tag{2.10}$$

Associated with the standard mixed finite element spaces is Fortin projection $\Pi_h : (H^1(\Omega))^2 \to \boldsymbol{V}_h$, such that for $\boldsymbol{q} \in H(div,\Omega)$

$$(\nabla\cdot\Pi_h\boldsymbol{q}, w_h) = (\nabla\cdot\boldsymbol{q}, w_h), \quad \forall w_h \in W_h. \tag{2.11}$$

The following approximation properties hold for the projections Q_h and Π_h (see [19])

$$\|Q_h\alpha\|_{0,q} \le C\|\alpha\|_{0,q}, \quad 2 \le q < \infty, \tag{2.12}$$
$$\|\alpha - Q_h\alpha\|_{0,q} \le C\|\alpha\|_{r,q}h^r, \quad 0 \le r \le k+1, \tag{2.13}$$
$$\|\boldsymbol{q} - \Pi_h\boldsymbol{q}\|_{0,q} \le C\|\boldsymbol{q}\|_{r,q}h^r, \quad 1/q < r \le k+1, \tag{2.14}$$
$$\|\nabla\cdot(\boldsymbol{q} - \Pi_h\boldsymbol{q})\|_{0,q} \le C\|\nabla\cdot\boldsymbol{q}\|_{r,q}h^r, \quad 0 \le r \le k+1. \tag{2.15}$$

For discretization of time variable, let

$$t^n = n\Delta t,\ n = 0, 1, \cdots, N,$$

where $\Delta t = T/N$ is the step size of time variable.

For any function φ of time, let φ^n denote $\varphi(\cdot, t^n)$. Moreover, we describe some of the notations which will be frequently used in our analysis:

$$\begin{aligned} \varphi^{n+\frac{1}{2}} &= \frac{1}{2}(\varphi^{n+1} + \varphi^n), \quad \partial_t\varphi^{n+\frac{1}{2}} = \frac{1}{\Delta t}(\varphi^{n+1} - \varphi^n), \\ \partial_t\varphi^n &= \frac{1}{2\Delta t}(\varphi^{n+1} - \varphi^{n-1}),\ \partial_{tt}\varphi^n = \frac{1}{(\Delta t)^2}(\varphi^{n+1} - 2\varphi^n + \varphi^{n-1}), \end{aligned} \tag{2.16}$$

obviously, we have

$$\partial_t\varphi^n = \frac{1}{2}(\partial_t\varphi^{n+\frac{1}{2}} + \partial_t\varphi^{n-\frac{1}{2}}),\ \partial_{tt}\varphi^n = \frac{1}{\Delta t}(\partial_t\varphi^{n+\frac{1}{2}} - \partial_t\varphi^{n-\frac{1}{2}}).$$

The fully discrete scheme of (2.7)-(2.8) is as follows: find $(u_h^{n+1}, \boldsymbol{z}_h^{n+1}) \in W_h \times \boldsymbol{V}_h$ such that

$$(u_h^0, w_h) = (Q_h u_0, w_h), \quad \forall w_h \in W_h, \tag{2.17}$$
$$(\boldsymbol{z}_h^0, \boldsymbol{v}_h) = (\boldsymbol{z}^0, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.18}$$
$$\left(\frac{2}{\Delta t}\partial_t u_h^{\frac{1}{2}}, w_h\right) + (\nabla\cdot\boldsymbol{z}_h^0, w_h) = \left(f^0 + \frac{2}{\Delta t}Q_h u_1, w_h\right), \quad \forall w_h \in W_h, \tag{2.19}$$
$$(\partial_{tt}u_h^n, w_h) + (\nabla\cdot\boldsymbol{z}_h^n, w_h) = (f^n, w_h), \quad \forall w_h \in W_h, \tag{2.20}$$
$$(\kappa(u_h^{n+1})\boldsymbol{z}_h^{n+1}, \boldsymbol{v}_h) - (\nabla\cdot\boldsymbol{v}_h, u_h^{n+1}) = 0, \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{2.21}$$

In order to prove the existence and uniqueness of the discrete problem (2.17)-(2.21), we rewrite (2.20) as

$$\left(\frac{1}{(\Delta t)^2}u_h^{n+1}, w_h\right) = -(\nabla\cdot\boldsymbol{z}_h^n, w_h) + \left(\frac{u_h^n - u_h^{n-1}}{(\Delta t)^2}, w_h\right) + (f^n, w_h), \quad \forall w_h \in W_h. \tag{2.22}$$

Let B_u and B_z be bases of W_h and $\boldsymbol{V}_h$, respectively. So, $u_h = Y\cdot B_u$ and $\boldsymbol{z}_h = X\cdot B_z$, where X and Y are nodal variables. Let $(u_h, w_h) = (Y\cdot B_u, \alpha\cdot B_u) =$

$\alpha \cdot LY$, where L is the matrix associated with the operator whose quadratic form is the L^2 inner products. Similarly, to L, introduce matrices A, B and D,

$$\begin{aligned}
&(\kappa(u_h^{n+1})\boldsymbol{z}_h^{n+1}, \boldsymbol{v}_h) = \chi \cdot AX,\\
&-(\nabla \cdot \boldsymbol{v}_h, u_h^{n+1}) = B\chi \cdot Y = B^T Y \cdot \chi,\\
&\left(\frac{1}{(\Delta t)^2} u_h^{n+1}, w_h\right) = DY \cdot \alpha,
\end{aligned}$$

where $\boldsymbol{v}_h = \chi \cdot B_z$ and $w_h = \alpha \cdot B_u$. Then, the matrix form of (2.17)-(2.21), relative to the bases B_u and B_z, is

$$\begin{bmatrix} A & B^T \\ \mathbf{0} & D \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ G \end{bmatrix}. \tag{2.23}$$

Recalling the assumptions on $\kappa(u)$, and noting that A and D are positive definite, as required by [20], there exists a unique solution (X, Y) to the system (2.23). Therefore, we can deduce that there exists a unique solution $(u_h^{n+1}, \boldsymbol{z}_h^{n+1})$ to (2.17)-(2.21).

To speed up the scheme (2.17)-(2.21), we present two-grid algorithm for problem (2.17)-(2.21) based on another mixed finite element space $W_H \times \boldsymbol{V}_H$ ($\subset W_h \times \boldsymbol{V}_h$), having mesh size $h \ll H < 1$. The basic idea in our approach is to solve the original nonlinear problem on a coarse grid $\mathcal{T}_H(\Omega)$, and then solve a corresponding linear problem on the fine grid $\mathcal{T}_h(\Omega)$.

Now, we give the two-grid algorithm which has two steps:

Algorithm 2.1.

Step 1. On the coarse grid $\mathcal{T}_H$, find $(u_H^{n+1}, \boldsymbol{z}_H^{n+1}) \in W_H \times \boldsymbol{V}_H$, solve the following nonlinear system:

$$(u_H^0, w_H) = (Q_H u_0, w_H), \quad \forall w_H \in W_H, \tag{2.24}$$

$$(\boldsymbol{z}_H^0, \boldsymbol{v}_H) = (\boldsymbol{z}^0, \boldsymbol{v}_H), \qquad \forall \boldsymbol{v}_H \in \boldsymbol{V}_H, \tag{2.25}$$

$$\left(\frac{2}{\Delta t}\partial_t u_H^{\frac{1}{2}}, w_H\right) + (\nabla \cdot \boldsymbol{z}_H^0, w_H) = \left(f^0 + \frac{2}{\Delta t} Q_H u_1, w_H\right), \forall w_H \in W_H, \tag{2.26}$$

$$(\partial_{tt} u_H^n, w_H) + (\nabla \cdot \boldsymbol{z}_H^n, w_H) = (f^n, w_H), \quad \forall w_H \in W_H, \tag{2.27}$$

$$(\kappa(u_H^{n+1})\boldsymbol{z}_H^{n+1}, \boldsymbol{v}_H) - (\nabla \cdot \boldsymbol{v}_H, u_H^{n+1}) = 0, \qquad \forall \boldsymbol{v}_H \in \boldsymbol{V}_H. \tag{2.28}$$

Step 2. On the fine grid $\mathcal{T}_h$, find $(U_h^{n+1}, \boldsymbol{Z}_h^{n+1}) \in W_h \times \boldsymbol{V}_h$, solve the following linear system:

$$(U_h^0, w_h) = (Q_h u_0, w_h), \quad \forall w_h \in W_h, \tag{2.29}$$

$$(\boldsymbol{Z}_h^0, \boldsymbol{v}_h) = (\boldsymbol{z}^0, \boldsymbol{v}_h), \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{2.30}$$

$$\left(\frac{2}{\Delta t}\partial_t U_h^{\frac{1}{2}}, w_h\right) + (\nabla \cdot \boldsymbol{Z}_h^0, w_h) = \left(f^0 + \frac{2}{\Delta t} Q_h u_1, w_h\right), \quad \forall w_h \in W_h, \tag{2.31}$$

$$(\partial_{tt} U_h^n, w_h) + (\nabla \cdot \boldsymbol{Z}_h^n, w_h) = (f^n, w_h), \forall w_h \in W_h, \tag{2.32}$$

$$\begin{aligned}
&(\kappa'(u_H^{n+1})\boldsymbol{z}_H^{n+1}(U_h^{n+1} - u_H^{n+1}) + \kappa(u_H^{n+1})\boldsymbol{Z}_h^{n+1}, \boldsymbol{v}_h)\\
&= (\nabla \cdot \boldsymbol{v}_h, U_h^{n+1}), \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.
\end{aligned} \tag{2.33}$$

3. Stability analysis

In this section, we will carry out the stability analysis for two-grid scheme (2.24)-(2.33). We suppose that $\kappa(u)$ is triple continuously differentiable with bounded derivatives up to the second order on Ω, i.e., there exists M_1, $M_2 > 0$, such that $\|\kappa_u\|_{0,\infty} \le M_1$, $\|\kappa_{uu}\|_{0,\infty} \le M_2$. Moreover, we also assume $\|\boldsymbol{z}\|_{0,\infty} \le M_3$, where $M_3 > 0$. As in [6], we use the "inverse assumption", which states that there exists a constant C_0 independent of $\hbar$, such that

$$\|\nabla \cdot \varphi\| \le C_0 \hbar^{-1} \|\varphi\|, \tag{3.34}$$

for $\varphi \in W_\hbar$, where $\hbar$ is either h or H depending on whether we work on the fine grid space or coarse grid space.

In order to derive the stability for our two-grid method, we need to obtain a stability result first for the coarse grid system (2.24)-(2.28).

Theorem 3.1. *The scheme defined by* (2.24)-(2.28) *is stable for* $\Delta t < \frac{2H}{C_0}$, *and*

$$\begin{aligned} \|u_H^{N+1}\|^2 + \|\boldsymbol{z}_H^{N+1}\|^2 \le C(\|u_H^1\|^2 + \|\boldsymbol{z}_H^1\|^2 + \|\partial_t u_H^{\frac{1}{2}}\|^2 \\ + \|\nabla \cdot \boldsymbol{z}_H^0\|^2) + C\Delta t \sum_{n=1}^{N} \max_{1\le i\le n} \|f^i\|^2 \end{aligned} \tag{3.35}$$

holds.

Proof. Let

$$\overline{\boldsymbol{z}}_H^0 = \frac{\Delta t}{2}\boldsymbol{z}_H^0, \quad \overline{\boldsymbol{z}}_H^n = \frac{\Delta t}{2}\boldsymbol{z}_H^0 + \Delta t \sum_{i=1}^{n} \boldsymbol{z}_H^i.$$

Summing over time levels and multiplying (2.27) by Δt, we have

$$\begin{aligned} &(\partial_t u_H^{n+\frac{1}{2}} - \partial_t u_H^{\frac{1}{2}}, w_H) + (\nabla \cdot (\overline{\boldsymbol{z}}_H^n - \overline{\boldsymbol{z}}_H^0), w_H) \\ &= \left(\Delta t \sum_{i=1}^{n} f^i, w_H\right), \quad \forall w_H \in W_H. \end{aligned} \tag{3.36}$$

We rewrite (2.28) by noting that $\boldsymbol{z}_H^{n+1} = \partial_t \overline{\boldsymbol{z}}_H^{n+\frac{1}{2}}$, so that

$$(\kappa(u_H^{n+1})\partial_t \overline{\boldsymbol{z}}_H^{n+\frac{1}{2}}, \boldsymbol{v}_H) - (\nabla \cdot \boldsymbol{v}_H, u_H^{n+1}) = 0, \qquad \forall \boldsymbol{v}_H \in \boldsymbol{V}_H. \tag{3.37}$$

Let $w_h = u_H^{n+\frac{1}{2}}$ and $\boldsymbol{v}_h = \overline{\boldsymbol{z}}_H^{n+\frac{1}{2}}$ are the test functions in (3.36) and (3.37), then add those equations to get

$$\begin{aligned} &(u_H^{n+1} - u_H^n, u_H^{n+1} + u_H^n) + (\kappa(u_H^{n+1})(\overline{\boldsymbol{z}}_H^{n+1} - \overline{\boldsymbol{z}}_H^n), \overline{\boldsymbol{z}}_H^{n+1} + \overline{\boldsymbol{z}}_H^n) \\ &+ \Delta t(\nabla \cdot \overline{\boldsymbol{z}}_H^n, u_H^n) - \Delta t(\nabla \cdot \overline{\boldsymbol{z}}_H^{n+1}, u_H^{n+1}) \\ &= 2\Delta t \left\{ \left(\partial_t u_H^{\frac{1}{2}}, u_H^{n+\frac{1}{2}}\right) + \left(\nabla \cdot \overline{\boldsymbol{z}}_H^0, u_H^{n+\frac{1}{2}}\right) + \left(\Delta t \sum_{i=1}^{n} f^i, u_H^{n+\frac{1}{2}}\right) \right\}. \end{aligned} \tag{3.38}$$

Using the Cauchy-Schwarz inequality, the terms on the right-hand side of the previous inequality are bounded as

$$\begin{aligned}&\left(\partial_t u_H^{\frac{1}{2}}, u_H^{n+\frac{1}{2}}\right) + \left(\nabla\cdot\overline{\boldsymbol{z}}_H^0, u_H^{n+\frac{1}{2}}\right) + \left(\Delta t\sum_{i=1}^n f^i, u_H^{n+\frac{1}{2}}\right)\\ &\leq C(\|\partial_t u_H^{\frac{1}{2}}\| + \|\nabla\cdot\overline{\boldsymbol{z}}_H^0\| + \|\sum_{i=1}^n f^i\|)\|u_H^{n+\frac{1}{2}}\|.\end{aligned}\tag{3.39}$$

In addition, the first two terms in the left-hand side of (3.38) are evaluated as

$$\begin{aligned}&(u_H^{n+1} - u_H^n, u_H^{n+1} + u_H^n) + (\kappa(u_H^{n+1})(\overline{\boldsymbol{z}}_H^{n+1} - \overline{\boldsymbol{z}}_H^n), \overline{\boldsymbol{z}}_H^{n+1} + \overline{\boldsymbol{z}}_H^n)\\ &\geq \|u_H^{n+1}\|^2 - \|u_H^n\|^2 + K_*(\|\overline{\boldsymbol{z}}_H^{n+1}\|^2 - \|\overline{\boldsymbol{z}}_H^n\|^2).\end{aligned}\tag{3.40}$$

Summing (3.38) from $n = 1, \cdots, N$, and using (3.39) and (3.40), we get

$$\begin{aligned}&\|u_H^{N+1}\|^2 - \|u_H^1\|^2 + \|\overline{\boldsymbol{z}}_H^{N+1}\|^2 - \|\overline{\boldsymbol{z}}_H^1\|^2 - \Delta t\left[\left(\nabla\cdot\overline{\boldsymbol{z}}_H^{N+1}, u_H^{N+1}) - (\nabla\cdot\overline{\boldsymbol{z}}_H^1, u_H^1\right)\right]\\ &\leq C\Delta t\sum_{n=1}^N(\|\partial_t u_H^{\frac{1}{2}}\| + \|\nabla\cdot\overline{\boldsymbol{z}}_H^0\| + \|\sum_{i=1}^n f^i\|)\|u_H^{n+\frac{1}{2}}\|.\end{aligned}$$

Employing the Cauchy-Schwarz inequality, the inverse assumption (3.34), and choosing H and Δt such that $\Delta t < \frac{2H}{C_0}$, we obtain

$$\begin{aligned}\Delta t(\nabla\cdot\overline{\boldsymbol{z}}_H^{N+1}, u_H^{N+1}) &\leq \Delta t\|\nabla\cdot\overline{\boldsymbol{z}}_H^{N+1}\|\cdot\|u_H^{N+1}\| \leq \Delta t C_0 H^{-1}\|\overline{\boldsymbol{z}}_H^{N+1}\|\cdot\|u_H^{N+1}\|\\ &\leq \frac{\Delta t C_0}{2H}\left(\|\overline{\boldsymbol{z}}_H^{N+1}\|^2 + \|u_H^{N+1}\|^2\right)\\ &< \|\overline{\boldsymbol{z}}_H^{N+1}\|^2 + \|u_H^{N+1}\|^2.\end{aligned}\tag{3.41}$$

Thus, we have

$$\begin{aligned}&\|u_H^{N+1}\|^2 + \|\overline{\boldsymbol{z}}_H^{N+1}\|^2 \leq \|u_H^1\|^2 + \|\overline{\boldsymbol{z}}_H^1\|^2\\ &+ C\Delta t\sum_{n=1}^N(\|\partial_t u_H^{\frac{1}{2}}\| + \|\nabla\cdot\overline{\boldsymbol{z}}_H^0\| + \|\sum_{i=1}^n f^i\|)\|u_H^{n+\frac{1}{2}}\|\\ &\leq \|u_H^1\|^2 + \|\overline{\boldsymbol{z}}_H^1\|^2 + C\Delta t\sum_{n=1}^N(\|u_H^{n+1}\|^2 + \|\partial_t u_H^{\frac{1}{2}}\|^2 + \|\nabla\cdot\overline{\boldsymbol{z}}_H^0\|^2 + \max_{1\leq i\leq n}\|f^i\|^2).\end{aligned}\tag{3.42}$$

Note that $\Delta t\sum_{n=1}^N \leq T$, use Gronwall's lemma to get

$$\begin{aligned}\|u_H^{N+1}\|^2 + \|\overline{\boldsymbol{z}}_H^{N+1}\|^2 \leq \|u_H^1\|^2 + \|\overline{\boldsymbol{z}}_H^1\|^2 + C(\|\partial_t u_H^{\frac{1}{2}}\|^2 + \|\nabla\cdot\overline{\boldsymbol{z}}_H^0\|^2)\\ + C\Delta t\sum_{n=1}^N\max_{1\leq i\leq n}\|f^i\|^2.\end{aligned}$$

The desired inequality (3.35) follows from the above inequality, and the proof is completed. □

Following a similar analysis as that carried above for the coarse grid, we can obtain the following stability on the fine grid $\mathcal{T}_h$.

Theorem 3.2. *For the scheme (2.29)-(2.33), we have the following stable inequality*

$$\|U_h^{N+1}\|^2 + \|\boldsymbol{Z}_h^{N+1}\|^2 \leq C(\|u_H^1\|^2 + \|\boldsymbol{z}_H^1\|^2 + \|\partial_t u_H^{\frac{1}{2}}\|^2 + \|\nabla\cdot\boldsymbol{z}_H^0\|^2 + \|U_h^1\|^2 + \|\boldsymbol{Z}_h^1\|^2 + \|\partial_t U_h^{\frac{1}{2}}\|^2 + \|\nabla\cdot\boldsymbol{Z}_h^0\|^2) + C\Delta t\sum_{n=1}^{N}\max_{1\leq i\leq n}\|f^i\|^2.$$

Proof. Let

$$\overline{\boldsymbol{Z}}_h^0 = \frac{\Delta t}{2}\boldsymbol{Z}_h^0, \quad \overline{\boldsymbol{Z}}_h^n = \frac{\Delta t}{2}\boldsymbol{Z}_h^0 + \Delta t\sum_{i=1}^{n}\boldsymbol{Z}_h^i.$$

Similarly as in Theorem 3.1, we have (cf. (3.38)):

$$\begin{aligned}
&(U_h^{n+1}-U_h^n, U_h^{n+1}+U_h^n) + (\kappa(u_H^{n+1})(\overline{\boldsymbol{Z}}_h^{n+1}-\overline{\boldsymbol{Z}}_h^n), \overline{\boldsymbol{Z}}_h^{n+1}+\overline{\boldsymbol{Z}}_h^n) + \Delta t(\nabla\cdot\overline{\boldsymbol{Z}}_h^n, U_h^n)\\
&- \Delta t(\nabla\cdot\overline{\boldsymbol{Z}}_h^{n+1}, U_h^{n+1})\\
&= 2\Delta t\{(\partial_t U_h^{\frac{1}{2}}, U_h^{n+\frac{1}{2}}) + (\nabla\cdot\overline{\boldsymbol{Z}}_h^0, U_h^{n+\frac{1}{2}}) - (\kappa'(u_H^{n+1})\boldsymbol{z}_H^{n+1}(U_h^{n+1}-u_H^{n+1}), \overline{\boldsymbol{Z}}_h^{n+\frac{1}{2}}\\
&+ (\Delta t\sum_{i=1}^{n} f^i, U_h^{n+\frac{1}{2}})\}.
\end{aligned}$$

Following a similar analysis as that carried out for (3.42), using the boundedness assumption on $\|\boldsymbol{z}\|_{0,\infty} \leq M_3$, we see that

$$\begin{aligned}
&\|U_h^{N+1}\|^2 + \|\overline{\boldsymbol{Z}}_h^{N+1}\|^2\\
&\leq \|U_h^1\|^2 + \|\overline{\boldsymbol{Z}}_h^1\|^2 + 2\Delta t\sum_{n=1}^{N}(\|\partial_t U_h^{\frac{1}{2}}\| + \|\nabla\cdot\overline{\boldsymbol{Z}}_h^0\| + \|\sum_{i=1}^{n} f^i\|)\|U_h^{n+\frac{1}{2}}\|\\
&+ C\Delta t\sum_{n=1}^{N}\|\overline{\boldsymbol{z}}_H^{n+1}\|_{0,\infty}(\|U_h^{n+1}\| + \|u_H^{n+1}\|)\|\overline{\boldsymbol{Z}}_h^{n+\frac{1}{2}}\|\\
&\leq \|U_h^1\|^2 + \|\overline{\boldsymbol{Z}}_h^1\|^2 + C\Delta t\sum_{n=1}^{N}(\|U_h^{n+1}\|^2 + \|\partial_t U_h^{\frac{1}{2}}\|^2 + \|\nabla\cdot\overline{\boldsymbol{Z}}_h^0\|^2 + \|\overline{\boldsymbol{Z}}_h^{n+1}\|^2 + \|u_H^{n+1}\|^2)\\
&+ C\Delta t\sum_{n=1}^{N}\max_{1\leq i\leq n}\|f^i\|^2.
\end{aligned}$$

Noting that $\Delta t\sum_{n=1}^{N} \leq T$, and using Gronwall's lemma and (3.35), we derive that

$$\|U_h^{N+1}\|^2 + \|\overline{\boldsymbol{Z}}_h^{N+1}\|^2 \leq \|U_h^1\|^2 + \|\overline{\boldsymbol{Z}}_h^1\|^2 + C(\|\partial_t U_h^{\frac{1}{2}}\|^2 + \|\nabla\cdot\overline{\boldsymbol{Z}}_h^0\|^2 + \|u_H^1\|^2 + \|\boldsymbol{z}_H^1\|^2 + \|\partial_t u_H^{\frac{1}{2}}\|^2 + \|\nabla\cdot\boldsymbol{z}_H^0\|^2) + C\Delta t\sum_{n=1}^{N}\max_{1\leq i\leq n}\|f^i\|^2.$$

Thus, the proof of this theorem is completed. □

4. Error analysis based on two-grid algorithm

In this section, we will prove the optimal a priori error estimate for schemes on both coarse and fine grids. As in [21], we shall use the following result

$$\|\varphi\|_{0,\infty} \le C\hbar^{-1}\|\varphi\|. \tag{4.43}$$

The time-space norms $\|\cdot\|_{l^\infty(L^2)}$ and $\|\cdot\|_{L^p(L^2)}$ are defined as

$$\|\varphi\|_{l^\infty(L^2)} = \|\varphi\|_{l^\infty(0,T;L^2(\Omega))} = \max_{1\le n\le N} ||\varphi^n||_{L^2(\Omega)},$$

$$\|\varphi\|_{L^p(L^2)} = \|\varphi\|_{L^p(0,T;L^2(\Omega))} = \left(\int_0^T ||\varphi||^2_{L^2(\Omega)}\right)^{\frac{1}{p}},$$

in the case $1 \le p < \infty$, and in the case $p = \infty$, the integral is replaced by the essential supremum.

In order to derive the error estimates for our two-grid method, we need to obtain an error estimate for the coarse grid system (2.24)-(2.28).

Theorem 4.1. *Define* $(u_H^n, \boldsymbol{z}_H^n) \in W_H \times \boldsymbol{V}_H$ *by* (2.24)-(2.28). *If the time step satisfies* $\Delta t < \frac{2H}{C_0}$, *then there exists a positive constant* C *such that*

$$||u - u_H||_{l^\infty(L^2)} + ||\boldsymbol{z} - \boldsymbol{z}_H||_{l^\infty(L^2)} \le C((\Delta t)^2 + H^{k+1}), \tag{4.44}$$

where k *is associated with the degree of the finite element polynomial.*

Proof. Set $\xi^n = u_H^n - Q_H u^n$, $\eta^n = \boldsymbol{z}_H^n - \Pi_H \boldsymbol{z}^n$, $\zeta^n = u^n - Q_H u^n$ and $\delta^n = \boldsymbol{z}^n - \Pi_H \boldsymbol{z}^n$. Subtracting (2.7) from (2.27), (2.8) from (2.28), respectively, we obtain the error equations

$$(\partial_{tt}\xi^n, w_H) + (\nabla\cdot\eta^n, w_H) = (\partial_{tt}\zeta^n, w_H) + (u_{tt}^n - \partial_{tt}u^n, w_H), \forall w_H \in W_H, \tag{4.45}$$

$$(\kappa(u_H^{n+1})\eta^{n+1}, \boldsymbol{v}_H) - (\nabla\cdot\boldsymbol{v}_H, \xi^{n+1}) = (I, \boldsymbol{v}_H), \quad \forall \boldsymbol{v}_H \in \boldsymbol{V}_H, \tag{4.46}$$

where

$$\begin{aligned} I =&(\kappa(u^{n+1}) - \kappa(u_H^{n+1}))\boldsymbol{z}^{n+1} - (\kappa(u^{n+1}) - \kappa(u_H^{n+1}))(\boldsymbol{z}^{n+1} - \Pi_H \boldsymbol{z}^{n+1}) \\ &+ \kappa(u^{n+1})(\boldsymbol{z}^{n+1} - \Pi_H \boldsymbol{z}^{n+1}) = \sum_{i=1}^{3} I_i. \end{aligned}$$

Using (2.17) in (4.45) yields

$$\begin{aligned} &\left(\frac{\partial_t \xi^{n+\frac{1}{2}} - \partial_t \xi^{n-\frac{1}{2}}}{\Delta t}, w_H\right) + (\nabla\cdot\eta^n, w_H) \\ =& \left(\frac{\partial_t \zeta^{n+\frac{1}{2}} - \partial_t \zeta^{n-\frac{1}{2}}}{\Delta t}, w_H\right) + (\beta_1^n, w_H), \end{aligned} \tag{4.47}$$

for any $w_H \in W_H$, where

$$\beta_1^n = u_{tt}^n - \partial_{tt} u^n = \frac{1}{6(\Delta t)^2} \int_{-\Delta t}^{\Delta t} (|t| - \Delta t)^3 \frac{\partial^4 u}{\partial t^4}(t^n + t) dt.$$

We introduce

$$\phi^0 = \frac{\Delta t}{2} \eta^0, \quad \phi^n = \frac{\Delta t}{2} \eta^0 + \Delta t \sum_{i=1}^{n} \eta^i.$$

Summing over time levels and multiplying both sides of (4.47) by Δt, we find that

$$\begin{aligned}(4.48)\qquad & (\partial_t \xi^{n+\frac{1}{2}} - \partial_t \xi^{\frac{1}{2}}, w_H) + (\nabla \cdot (\phi^n - \phi^0), w_H) \\ & = (\partial_t \zeta^{n+\frac{1}{2}} - \partial_t \zeta^{\frac{1}{2}}, w_H) + \left(\Delta t \sum_{i=1}^{n} \beta_1^i, w_H \right), \quad \forall w_H \in W_H,\end{aligned}$$

where $\Delta t \sum\limits_{i=1}^{n} \eta^i = \phi^n - \phi^0$. For $t = 0$, by (2.7), we have

$$(4.49)\qquad (u_{tt}^0, w_H) + (\nabla \cdot \boldsymbol{z}^0, w_H) = (f^0, w_H), \quad \forall w_H \in W_H.$$

It is simple to see

$$\begin{aligned}(4.50)\qquad \frac{1}{2\Delta t} \int_0^{\Delta t} (\Delta t - t)^2 \frac{\partial^3 u}{\partial t^3}(t) dt &= -\frac{\Delta t}{2} u_{tt}^0 + \frac{1}{\Delta t} \int_0^{\Delta t} (\Delta t - t) \frac{\partial^2 u}{\partial t^2}(t) dt \\ &= -\frac{\Delta t}{2} u_{tt}^0 - u_t^0 - \frac{1}{\Delta t} \int_0^{\Delta t} \frac{\partial u}{\partial t}(t) dt \\ &= -\frac{\Delta t}{2} u_{tt}^0 - u_t^0 - \frac{1}{\Delta t}(u^1 - u^0) \\ &= -\frac{\Delta t}{2} u_{tt}^0 - u_1 - \partial_t u^{\frac{1}{2}}.\end{aligned}$$

Using the projection operators of Q_H and Π_H, (2.11), (4.49) and (4.50), (2.26) can be transformed into the following:

$$\begin{aligned}(4.51)\qquad & (\partial_t \xi^{\frac{1}{2}}, w_H) + \frac{\Delta t}{2} (\nabla \cdot \eta^0, w_H) \\ & = -(\partial_t Q_H u^{\frac{1}{2}}, w_H) - \frac{\Delta t}{2} (\nabla \cdot \Pi_H z^0, w_H) + \left(\frac{\Delta t}{2} f^0 + Q_H u_1, w_H \right) \\ & = -(\partial_t Q_H u^{\frac{1}{2}}, w_H) + \left(\frac{\Delta t}{2} u_{tt}^0, w_H \right) + (Q_H u_1, w_H) \\ & = (\partial_t \zeta^{\frac{1}{2}}, w_H) + (Q_H u_1 - u_1, w_H) + \left(\frac{\Delta t}{2} u_{tt}^0 + u_1 + \partial_t u^{\frac{1}{2}}, w_H \right) \\ & = (\partial_t \zeta^{\frac{1}{2}}, w_H) + (Q_H u_1 - u_1, w_H) - \frac{1}{2\Delta t} \int_0^{\Delta t} (\Delta t - t)^2 \left(\frac{\partial^3 u}{\partial t^3}, w_H \right) dt, \\ & \forall w_H \in W_H.\end{aligned}$$

Thus, it follows from (4.48) and (4.51) that

$$(\partial_t\xi^{n+\frac{1}{2}}, w_H) + (\nabla\cdot\phi^n, w_H) = (\partial_t\zeta^{n+\frac{1}{2}}, w_H) + (\beta_2^n, w_H), \quad \forall w_H \in W_H, \tag{4.52}$$

where

$$\beta_2^n = Q_h u_1 - u_1 + \Delta t\sum_{i=1}^{n}\beta_1^i - \frac{1}{2\Delta t}\int_0^{\Delta t}(\Delta t - t)^2\frac{\partial^3 u}{\partial t^3}(t)dt.$$

Noting that $\eta^{n+1} = \partial_t\phi^{n+\frac{1}{2}}$, we rewrite (4.46) as follows:

$$(\kappa(u_H^{n+1})\partial_t\phi^{n+\frac{1}{2}}, \boldsymbol{v}_H) - (\nabla\cdot\boldsymbol{v}_H, \xi^{n+1}) = (I, \boldsymbol{v}_H), \quad \forall\boldsymbol{v}_H \in \boldsymbol{V}_H. \tag{4.53}$$

Choosing the test functions$w_H = \xi^{n+\frac{1}{2}}$ and $\boldsymbol{v}_H = \phi^{n+\frac{1}{2}}$ in (4.52) and (4.53), respectively. Then, multiplying the two resulting equations by $2\Delta t$, we have

$$\begin{aligned}&(\xi^{n+1} - \xi^n, \xi^{n+1} + \xi^n) + \Delta t(\nabla\cdot\phi^n, \xi^{n+1} + \xi^n)\\ &= 2\Delta t(\partial_t\zeta^{n+\frac{1}{2}} + \beta_2^n, \xi^{n+\frac{1}{2}}),\end{aligned} \tag{4.54}$$

$$\begin{aligned}&(\kappa(u_H^{n+1})(\phi^{n+1} - \phi^n), \phi^{n+1} + \phi^n) - \Delta t(\nabla\cdot(\phi^{n+1} + \phi^n), \xi^{n+1})\\ &= 2\Delta t(I, \phi^{n+\frac{1}{2}}).\end{aligned} \tag{4.55}$$

Combine (4.54) and (4.55) to obtain

$$\begin{aligned}&\|\xi^{n+1}\|^2 - \|\xi^n\|^2 + (\kappa(u_H^{n+1})(\phi^{n+1} - \phi^n), \phi^{n+1} + \phi^n) + \Delta t(\nabla\cdot\phi^n, \xi^n)\\ &- \Delta t(\nabla\cdot\phi^{n+1}, \xi^{n+1})\\ &= 2\Delta t(\partial_t\zeta^{n+\frac{1}{2}} + \beta_2^n, \xi^{n+\frac{1}{2}}) + 2\Delta t(I, \phi^{n+\frac{1}{2}}).\end{aligned} \tag{4.56}$$

Using (1.6), the third term on the left-hand side of (4.56) can be bounded as

$$(\kappa(u_H^{n+1})(\phi^{n+1} - \phi^n), \phi^{n+1} + \phi^n) \geq K_*(\|\phi^{n+1}\|^2 - \|\phi^n\|^2). \tag{4.57}$$

Next, we estimate the right-hand terms of (4.56). For the first term, using the Cauchy-Schwarz inequality, we have the following estimation

$$(\partial_t\zeta^{n+\frac{1}{2}} + \beta_2^n, \xi^{n+\frac{1}{2}}) \leq (\|\partial_t\zeta^{n+\frac{1}{2}}\| + \|\beta_2^n\|)\|\xi^{n+\frac{1}{2}}\|. \tag{4.58}$$

For the second term, by the assumptions on $\kappa(u)$ and $\boldsymbol{z}$, the inverse inequality and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}|(I_1, \phi^{n+\frac{1}{2}})| &= |((\kappa(u^{n+1}) - \kappa(Q_H u^{n+1}) + \kappa(Q_H u^{n+1}) - \kappa(u_H^{n+1}))\boldsymbol{z}^{n+1}, \phi^{n+\frac{1}{2}})|\\ &\leq C(\|\xi^{n+1}\| + \|\zeta^{n+1}\|)\|\phi^{n+\frac{1}{2}}\|,\\ |(I_2, \phi^{n+\frac{1}{2}})| &= |((\kappa(u^{n+1}) - \kappa(Q_H u^{n+1}) + \kappa(Q_H u^{n+1})\\ &\quad - \kappa(u_H^{n+1}))(\boldsymbol{z}^{n+1} - \Pi_H\boldsymbol{z}^{n+1}), \phi^{n+\frac{1}{2}})|\\ &\leq C(\|\xi^{n+1}\|_{0,\infty} + \|\zeta^{n+1}\|_{0,\infty})\|\delta^{n+1}\|\cdot\|\phi^{n+\frac{1}{2}}\|\\ &\leq CH^{-1}(\|\xi^{n+1}\| + \|\zeta^{n+1}\|)\|\delta^{n+1}\|\cdot\|\phi^{n+\frac{1}{2}}\|,\\ |(I_3, \phi^{n+\frac{1}{2}})| &= |(\kappa(u^{n+1})(\boldsymbol{z}^{n+1} - \Pi_H\boldsymbol{z}^{n+1}), \phi^{n+\frac{1}{2}})|\\ &\leq C\|\delta^{n+1}\|\cdot\|\phi^{n+\frac{1}{2}}\|.\end{aligned}$$

Hence, by (4.43), we conclude that

$$|(I,\phi^{n+\frac{1}{2}})| \leq C\left[\|\zeta^{n+1}\| + \|\xi^{n+1}\| + H^{-1}(\|\xi^{n+1}\| + \|\zeta^{n+1}\|)\|\delta^{n+1}\| + \|\delta^{n+1}\|\right]\|\phi^{n+\frac{1}{2}}\|. \tag{4.59}$$

Summing (4.56) over time levels, and using (4.57)-(4.59), we derive

$$\begin{aligned}&\|\xi^{n+1}\|^2-\|\xi^0\|^2+\|\phi^{n+1}\|^2-\|\phi^0\|^2-\Delta t\left[(\nabla\cdot\phi^{n+1},\xi^{n+1})-(\nabla\cdot\phi^0,\xi^0)\right]\\ &\leq 2\Delta t\sum_{i=0}^{n}(\|\partial_t\zeta^{i+\frac{1}{2}}\|+\|\beta_2^i\|)\|\xi^{i+\frac{1}{2}}\|+C\Delta t\sum_{i=0}^{n}[\|\zeta^{i+1}\|+\|\xi^{i+1}\|\\ &+H^{-1}(\|\xi^{i+1}\|+\|\zeta^{i+1}\|)\|\delta^{i+1}\|+\|\delta^{i+1}\|]\|\phi^{i+\frac{1}{2}}\|.\end{aligned} \tag{4.60}$$

After imposing the initial conditions (2.24) and (2.25) in (4.60), we have

$$\begin{aligned}&\|\xi^{n+1}\|^2+\|\phi^{n+1}\|^2-\Delta t\left(\nabla\cdot\phi^{n+1},\xi^{n+1}\right)\\ &\leq 2\Delta t\sum_{i=0}^{n}(\|\partial_t\zeta^{i+\frac{1}{2}}\|+\|\beta_2^i\|)\|\xi^{i+\frac{1}{2}}\|+C\Delta t\sum_{i=0}^{n}[\|\zeta^{i+1}\|+\|\xi^{i+1}\|\\ &+H^{-1}(\|\xi^{i+1}\|+\|\zeta^{i+1}\|)\|\delta^{i+1}\|+\|\delta^{i+1}\|]\|\phi^{i+\frac{1}{2}}\|.\end{aligned}$$

Similar to (3.41), we have

$$\begin{aligned}\Delta t(\nabla\cdot\phi^{n+1},\xi^{n+1}) &\leq \Delta t\|\nabla\cdot\phi^{n+1}\|\cdot\|\xi^{n+1}\| \leq \Delta t C_0 H^{-1}\|\phi^{n+1}\|\cdot\|\xi^{n+1}\|\\ &\leq \frac{\Delta t C_0}{2H}\left(\|\phi^{n+1}\|^2+\|\xi^{n+1}\|^2\right)\\ &< \|\phi^{n+1}\|^2+\|\xi^{n+1}\|^2.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\|\xi^{n+1}\|^2+\|\phi^{n+1}\|^2 \leq \Delta t\sum_{i=0}^{n}(\|\partial_t\zeta^{i+\frac{1}{2}}\|\\ &+\|\beta_2^i\|)\|\xi^{i+\frac{1}{2}}\|+C\Delta t\sum_{i=0}^{n}[\|\zeta^{i+1}\|+\|\xi^{i+1}\|\\ &+H^{-1}(\|\xi^{i+1}\|+\|\zeta^{i+1}\|)\|\delta^{i+1}\|\\ &+\|\delta^{i+1}\|]\|\phi^{i+\frac{1}{2}}\| \leq C\Delta t\|\xi\|_{l^\infty(L^2)}\sum_{i=0}^{n}(\|\partial_t\zeta^{i+\frac{1}{2}}\|+\|\beta_2^i\|)\\ &+C\Delta t\|\phi\|_{l^\infty(L^2)}\sum_{i=0}^{n}[\|\zeta^{i+1}\|+\|\xi^{i+1}\|\end{aligned}$$

$$
\begin{aligned}
(4.61)\qquad & + H^{-1}(\|\xi^{i+1}\| + \|\zeta^{i+1}\|)\|\delta^{i+1}\| + \|\delta^{i+1}\|] \le \frac{1}{4}\|\xi\|^2_{l^\infty(L^2)} \\
& + C\left(\Delta t \sum_{i=0}^{n} \|\partial_t \zeta^{i+\frac{1}{2}}\|\right)^2 + C\left(\Delta t \sum_{i=0}^{n} \|\beta_2^i\|\right)^2 + \frac{1}{4}\|\phi\|^2_{l^\infty(L^2)} \\
& + C\left(\Delta t \sum_{i=0}^{n} \|\xi^i\|\right)^2 + C\left(\Delta t \sum_{i=0}^{n} \|\delta^i\|\right)^2 + C\left(\Delta t \sum_{i=0}^{n} \|\zeta^i\|\right)^2,
\end{aligned}
$$

since $\|\xi^{i+\frac{1}{2}}\| \le \|\xi\|_{l^\infty(L^2)}$ and $\|\phi^{i+\frac{1}{2}}\| \le \|\phi\|_{l^\infty(L^2)}$. Taking the supremum on n on the left-hand side of (4.61), we have

$$
\begin{aligned}
& \|\xi\|^2_{l^\infty(L^2)} + \|\phi\|^2_{l^\infty(L^2)} \\
(4.62)\qquad & \le C\left(\Delta t \sum_{i=0}^{n} \|\partial_t \zeta^{i+\frac{1}{2}}\|\right)^2 + C\left(\Delta t \sum_{i=0}^{n} \|\beta_2^i\|\right)^2 + C\left(\Delta t \sum_{i=0}^{n} \|\zeta^i\|\right)^2 \\
& + C\left(\Delta t \sum_{i=0}^{n} \|\delta^i\|\right)^2 + C\left(\Delta t \sum_{i=0}^{n} \|\xi^i\|\right)^2.
\end{aligned}
$$

In the following, we analyse the right-hand side of (4.62). A direct bound shows that

$$
(4.63)\qquad \Delta t \sum_{i=0}^{n} \|\partial_t \zeta^{i+\frac{1}{2}}\| \le C\left(H^{k+1}\|u\|_{L^\infty(H^{k+1}(\Omega))} + (\Delta t)^2 \left\|\frac{\partial^3 u}{\partial t^3}\right\|_{L^1(L^2)}\right).
$$

By (2.13), we have

$$
\begin{aligned}
\|\beta_2^i\| &\le \Delta t \sum_{i=1}^{n} \|\beta_1^i\| + \|Q_H u_1 - u_1\| + \left\| \frac{1}{2\Delta t}\int_0^{\Delta t} (\Delta t - t)^3 \frac{\partial^3 u}{\partial t^3}(t)dt \right\| \\
&\le C(\Delta t)^2 \left\|\frac{\partial^4 u}{\partial t^4}\right\|_{L^\infty(L^2)} + CH^{k+1} + C(\Delta t)^2 \left\|\frac{\partial^3 u}{\partial t^3}\right\|_{L^\infty(L^2)} \\
&\le C(H^{k+1} + (\Delta t)^2),
\end{aligned}
$$

and hence

$$
(4.64)\qquad \Delta t \sum_{i=0}^{n} \|\beta_2^i\| \le C\|\beta_2\|_{l^\infty(L^2)} \le C(H^{k+1} + (\Delta t)^2).
$$

Using (2.13), (2.14) (4.63) and (4.64) in (4.62), and applying discrete Gronwall's inequality, we know that for Δt and H sufficiently small

$$
(4.65)\qquad \|\xi\|^2_{l^\infty(L^2)} + \|\phi\|^2_{l^\infty(L^2)} \le C((\Delta t)^4 + H^{2k+2}).
$$

Finally, by (2.13), (2.14), (4.65) and the triangle inequality, we can derive (4.44). □

Now, we can prove the following theorem for the solution of the fine grid.

Theorem 4.2. *Let $(U_h^n, \boldsymbol{Z}_h^n) \in W_h \times \boldsymbol{V}_h$ be the solution of the two-grid algorithm of step 2 for solving the MFE scheme* (2.29)-(2.33)*. If $\Delta t < \frac{2h}{C_0}$, then there is a positive constant C such that*

$$||u - U_h||_{l^\infty(L^2)} + ||\boldsymbol{z} - \boldsymbol{Z}_h||_{l^\infty(L^2)} \leq C((\Delta t)^2 + h^{k+1} + H^{2k+1}), \tag{4.66}$$

where k is associated with the degree of the finite element polynomial.

Proof. Set $\rho^n = U_h^n - Q_h u^n$ and $\gamma^n = \boldsymbol{Z}_h^n - \Pi_h \boldsymbol{z}^n$. Let us first note the following error equations from (2.7)-(2.8) and (2.32)-(2.33),

$$(\partial_{tt}\rho^n, w_h) + (\nabla \cdot \gamma^n, w_h) = (\partial_{tt}\zeta^n, w_h) + (\beta_1^n, w_h),\ \forall w_h \in W_h, \tag{4.67}$$

$$(E, \boldsymbol{v}_h) - (\nabla \cdot \boldsymbol{v}_h, \rho^{n+1}) = 0,\ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h, \tag{4.68}$$

where β_1^n is defined by (4.47),

$$\begin{aligned} E =& \kappa'(u_H^{n+1})\boldsymbol{z}_H^{n+1}(U_h^{n+1} - u_H^{n+1}) + \kappa(u_H^{n+1})\boldsymbol{Z}_h^{n+1} - \kappa(u^{n+1})\boldsymbol{z}^{n+1} \\ &+ \kappa(u^{n+1})\Pi_h \boldsymbol{z}^{n+1} - \kappa(u^{n+1})\Pi_h \boldsymbol{z}^{n+1}, \end{aligned}$$

applying the Taylor expansions to $\kappa(u^{n+1})$ at u_H^{n+1}, i.e.

$$\kappa(u^{n+1}) = \kappa(u_H^{n+1}) + \kappa'(u_H^{n+1})(u^{n+1} - u_H^{n+1}) + \frac{1}{2}\kappa''(u^*)(u^{n+1} - u_H^{n+1})^2,$$

where $\kappa''(u^*)$ means $\kappa''(u)$ evaluated at a point u^* between u^{n+1} and u_H^{n+1}. Then, we have

$$\begin{aligned} E =& \kappa(u^{n+1})(\Pi_h \boldsymbol{z}^{n+1} - \boldsymbol{z}^{n+1}) - \kappa(u_H^{n+1})(\Pi_h \boldsymbol{z}^{n+1} - \boldsymbol{Z}_h^{n+1}) \\ &+ \kappa'(u_H^{n+1})(U_h^{n+1} - Q_h u^{n+1} + Q_h u^{n+1} - u^{n+1})\boldsymbol{z}_H^{n+1} \\ &+ \kappa'(u_H^{n+1})(u^{n+1} - u_H^{n+1})(\boldsymbol{z}_H^{n+1} - \Pi_h \boldsymbol{z}^{n+1}) \\ &- \frac{1}{2}\kappa''(u^*)(u^{n+1} - u_H^{n+1})^2(\Pi_h \boldsymbol{z}^{n+1} - \boldsymbol{z}^{n+1} + \boldsymbol{z}^{n+1} - \boldsymbol{z}_H^{n+1}) \\ &- \frac{1}{2}\kappa''(u^*)(u^{n+1} - u_H^{n+1})^2 \boldsymbol{z}_H^{n+1}. \end{aligned} \tag{4.69}$$

By (4.68) and (4.69), we get

$$(\kappa(u_H^{n+1})\gamma^{n+1}, \boldsymbol{v}_h) - (\nabla \cdot \boldsymbol{v}_h, \rho^{n+1}) = (F, \boldsymbol{v}_h),\ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h,$$

where

$$\begin{aligned} (F, \boldsymbol{v}_h) =& (\kappa(u^{n+1})\delta^{n+1}, \boldsymbol{v}_h) - (\kappa'(u_H^{n+1})(\rho^{n+1} - \zeta^{n+1})\boldsymbol{z}_H^{n+1}, \boldsymbol{v}_h) \\ &- (\kappa'(u_H^{n+1})(u^{n+1} - u_H^{n+1})(\boldsymbol{z}_H^{n+1} - \Pi_h \boldsymbol{z}^{n+1}), \boldsymbol{v}_h) \\ &+ (\frac{1}{2}\kappa''(u^*)(u^{n+1} - u_H^{n+1})^2(\boldsymbol{z}^{n+1} - \boldsymbol{z}_H^{n+1} - \delta^{n+1}), \boldsymbol{v}_h) \\ &+ (\frac{1}{2}\kappa''(u^*)(u^{n+1} - u_H^{n+1})^2 \boldsymbol{z}_H^{n+1}, \boldsymbol{v}_h) = \sum_{i=1}^{5} T_i. \end{aligned}$$

Let us define

$$\psi^0 = \frac{\Delta t}{2}\gamma^0, \quad \psi^n = \frac{\Delta t}{2}\gamma^0 + \Delta t \sum_{i=1}^{n} \gamma^i.$$

By the center difference operator $\partial_{tt}\varphi^n = \frac{1}{\Delta t}(\partial_t \varphi^{n+\frac{1}{2}} - \partial_t \varphi^{n-\frac{1}{2}})$, we obtain

$$\begin{aligned} &\left(\frac{\partial_t \rho^{n+\frac{1}{2}} - \partial_t \rho^{n-\frac{1}{2}}}{\Delta t}, w_h\right) + (\nabla \cdot \gamma^n, w_h) \\ &= \left(\frac{\partial_t \zeta^{n+\frac{1}{2}} - \partial_t \zeta^{n-\frac{1}{2}}}{\Delta t}, w_h\right) + (\beta_1^n, w_h),\ \forall w_h \in W_h. \end{aligned} \tag{4.70}$$

Summing over time levels of (4.70) and multiplying through by Δt, we have

$$\begin{aligned} (\partial_t \rho^{n+\frac{1}{2}} - \partial_t \rho^{\frac{1}{2}}, w_h) + (\nabla \cdot (\psi^n - \psi^0), w_h) = (\partial_t \zeta^{n+\frac{1}{2}} - \partial_t \zeta^{\frac{1}{2}}, w_h) \\ + \left(\Delta t \textstyle\sum_{i=1}^n \beta_1^i, w_h\right),\ \forall w_h \in W_h, \end{aligned}$$

since $\Delta t \sum\limits_{i=1}^{n} \gamma^i = \psi^n - \psi^0$. Similar to (4.52), we have

$$(\partial_t \rho^{n+\frac{1}{2}}, w_h) + (\nabla \cdot \psi^n, w_h) = (\partial_t \zeta^{n+\frac{1}{2}}, w_h) + (\beta_2^n, w_h),\ \forall w_h \in W_h, \tag{4.71}$$

where β_2^n is defined in (4.52). Observe that $\gamma^{n+1} = \partial_t \psi^{n+\frac{1}{2}}$, therefore, we get

$$(\kappa(u_H^{n+1})\partial_t \psi^{n+\frac{1}{2}}, \boldsymbol{v}_h) - (\nabla \cdot \boldsymbol{v}_h, \rho^{n+1}) = (F, \boldsymbol{v}_h),\ \forall \boldsymbol{v}_h \in \boldsymbol{V}_h. \tag{4.72}$$

Choosing $w_h = \rho^{n+\frac{1}{2}}$ and $\boldsymbol{v}_h = \psi^{n+\frac{1}{2}}$ in (4.71) and (4.72), adding them and multiplying by $2\Delta t$, we find that

$$\begin{aligned} &\|\rho^{n+1}\|^2 - \|\rho^n\|^2 + \|\kappa^{\frac{1}{2}}(u_H^{n+1})\psi^{n+1}\|^2 - \|\kappa^{\frac{1}{2}}(u_H^n)\psi^n\|^2 \\ &+((\kappa(u_H^n) - \kappa(u_H^{n+1}))\psi^n, \psi^n) \\ &+\Delta t(\nabla \cdot \psi^n, \rho^n) - \Delta t(\nabla \cdot \psi^{n+1}, \rho^{n+1}) \\ &= 2\Delta t(\partial_t \zeta^{n+\frac{1}{2}} + \beta_2^n, \rho^{n+\frac{1}{2}}) + 2\Delta t(F, \psi^{n+\frac{1}{2}}). \end{aligned} \tag{4.73}$$

Apply the Cauchy-Schwarz inequality, it is easy to get

$$(\partial_t \zeta^{n+\frac{1}{2}} + \beta_2^n, \rho^{n+\frac{1}{2}}) \le (\|\partial_t \zeta^{n+\frac{1}{2}}\| + \|\beta_2^n\|)\|\rho^{n+\frac{1}{2}}\|. \tag{4.74}$$

Using (2.13)-(2.15), (4.44), and the assumptions on $\kappa(u)$ and $\boldsymbol{z}$, we have

$$\begin{aligned} &|T_1| = |(\kappa(u^{n+1})\delta^{n+1}, \psi^{n+\frac{1}{2}})| \le Ch^{k+1}\|\psi^{n+\frac{1}{2}}\|, \\ &|T_2| \le |(\kappa'(u_H^{n+1})(\rho^{n+1} + \zeta^{n+1})\boldsymbol{z}_H^{n+1}, \psi^{n+\frac{1}{2}})| \end{aligned}$$

$$
\begin{aligned}
&\leq C(\|\rho^{n+1}\| + \|\zeta^{n+1}\|)\|\boldsymbol{z}_H^{n+1}\|_{0,\infty}\|\psi^{n+\frac{1}{2}}\| \\
&\leq C(\|\rho^{n+1}\| + h^{k+1})\|\psi^{n+\frac{1}{2}}\|, \\
|T_3| &= |(\kappa'(u_H^{n+1})(u^{n+1} - u_H^{n+1})(\boldsymbol{z}_H^{n+1} - \Pi_h \boldsymbol{z}^{n+1}), \psi^{n+\frac{1}{2}})| \\
&\leq C\|(u^{n+1} - u_H^{n+1})(\boldsymbol{z}_H^{n+1} - \Pi_h \boldsymbol{z}^{n+1})\| \cdot \|\psi^{n+\frac{1}{2}}\| \\
&\leq C\|u^{n+1} - u_H^{n+1}\|_{0,4}\|\boldsymbol{z}^{n+1} - \Pi_h \boldsymbol{z}^{n+1}\|_{0,4}\|\psi^{n+\frac{1}{2}}\| \\
&+ C\|u^{n+1} - u_H^{n+1}\|_{0,4}\|\boldsymbol{z}_H^{n+1} - \boldsymbol{z}^{n+1}\|_{0,4}\|\psi^{n+\frac{1}{2}}\| \\
&\leq C(\|u^{n+1} - u_H^{n+1}\|_{0,4}^2 + \|\boldsymbol{z}^{n+1} - \Pi_h \boldsymbol{z}^{n+1}\|_{0,4}^2 \\
&+ \|\boldsymbol{z}_H^{n+1} - \boldsymbol{z}^{n+1}\|_{0,4}^2)\|\psi^{n+\frac{1}{2}}\| \\
&\leq C(H^{2k+1} + h^{2k+2} + \Delta t^4)\|\psi^{n+\frac{1}{2}}\|, \qquad (4.75) \\
|T_4| &= |(\frac{1}{2}\kappa''(u^*)(u^{n+1} - u_H^{n+1})^2 \boldsymbol{z}_H^{n+1}, \psi^{n+\frac{1}{2}})| \\
&\leq C\|u^{n+1} - u_H^{n+1}\|_{0,4}^2\|\boldsymbol{z}_H^{n+1}\|_{0,\infty}\|\psi^{n+\frac{1}{2}}\| \\
&\leq CH^{2k+1}\|\psi^{n+\frac{1}{2}}\|, \\
|T_5| &= |(\frac{1}{2}\kappa''(u^*)(u^{n+1} - u_H^{n+1})^2(\boldsymbol{z}^{n+1} - \boldsymbol{z}_H^{n+1} - \delta^{n+1}), \psi^{n+\frac{1}{2}})| \\
&\leq C\|(\frac{1}{2}\kappa''(u^*)(u^{n+1} - u_H^{n+1})^2(\boldsymbol{z}^{n+1} - \boldsymbol{z}_H^{n+1} - \delta^{n+1})\| \cdot \|\psi^{n+\frac{1}{2}}\| \\
&\leq C\|u^{n+1} - u_H^{n+1}\|_{0,8}^2\|\boldsymbol{z}^{n+1} - \boldsymbol{z}_H^{n+1}\|_{0,4}\|\psi^{n+\frac{1}{2}}\| \\
&+ C\|u^{n+1} - u_H^{n+1}\|_{0,8}^2\|\delta^{n+1}\|_{0,4}\|\psi^{n+\frac{1}{2}}\| \\
&\leq C(H^{2k+1} + h^{2k+2})\|\psi^{n+\frac{1}{2}}\|.
\end{aligned}
$$

It follows from (4.75) that

$$
|(F, \psi^{n+\frac{1}{2}})| \leq C(h^{k+1} + H^{2k+1} + (\Delta t)^2 + \|\rho^{n+1}\|)\|\psi^{n+\frac{1}{2}}\|. \tag{4.76}
$$

Using (4.74) and (4.76), and summing (4.73) over time levels, we have

$$
\begin{aligned}
&\|\rho^{n+1}\|^2 + \|\psi^{n+1}\|^2 - \Delta t(\nabla \cdot \psi^{n+1}, \rho^{n+1}) \\
&\leq 2C\Delta t \sum_{i=0}^{n}(h^{k+1} + H^{2k+1} + (\Delta t)^2 + \|\rho^{i+1}\|)\|\psi^{i+\frac{1}{2}}\| \\
&+ 2\Delta t \sum_{i=0}^{n}(\|\partial_t \zeta^{i+\frac{1}{2}}\| + \|\beta_2^i\|)\|\rho^{i+\frac{1}{2}}\|,
\end{aligned}
$$

where we used $\rho^0 = 0$ and $\psi^0 = 0$ since the initial conditions (2.29) and (2.30). In the following, similarly as the proof of (4.65), we deduce that

$$
\|\rho\|_{l^\infty(L^2)}^2 + \|\psi\|_{l^\infty(L^2)}^2 \leq C((\Delta t)^4 + h^{2k+2} + H^{4k+2}). \tag{4.77}
$$

Thus, applying (2.13), (2.14), (4.77) and the triangle inequality, we can derive (4.66). □

Remark 4.1. From Theorem 4.2, we see that the optimal error estimate is $\mathcal{O}((\Delta t)^2 + h^{k+1})$ by taking $H = \mathcal{O}(h^{\frac{k+1}{2k+1}})$, which is coincide with the error result (4.44) obtained for the original MFE system (2.17)-(2.21).

5. Numerical experiments

In the section, we consider the following second-order nonlinear hyperbolic problem:

$$\begin{aligned}
&u_{tt} - \nabla\cdot(K(u)\nabla u) = f, \quad (\boldsymbol{x},t)\in\Omega\times J,\\
&u(\boldsymbol{x},t) = 0, \quad (\boldsymbol{x},t)\in\partial\Omega\times J,\\
&u(\boldsymbol{x},0) = u_0(\boldsymbol{x}),\ u_t(\boldsymbol{x},0) = u_1(\boldsymbol{x}), \quad \boldsymbol{x}\in\Omega,
\end{aligned}$$

where $\Omega = [0,1]^2$, $J = [0,1]$, $\boldsymbol{x} = (x_1,x_2)^T$,

$$\boldsymbol{K}(u) = \begin{pmatrix} 1+\sin^2(u) & 0 \\ 0 & 1+\sin^2(u) \end{pmatrix},$$

the functions f, u_0 and u_1 are chosen so that the exact solution $u(\boldsymbol{x},t) = e^t x_1 x_2(1-x_1)(1-x_2)$ or $u(\boldsymbol{x},t) = e^t\sin(\pi x_1)\sin(\pi x_2)$.

We use the Raviart-Thomas spaces (RT_1) with $k=1$. J is uniformly divided so that $\triangle t$ is a constant. We present the two-grid discretization error with coarse and fine mesh size pairs (H,h)=(1/4, 1/8), (1/9, 1/27), (1/16, 1/64) which satisfy the relation $h = H^{3/2}$. When the exact solution is chosen as $u(\boldsymbol{x},t) = e^t x_1 x_2(1-x_1)(1-x_2)$, we take the time step $\Delta t = 1.0e-3$, the error results, convergence rates and computational time of MEFM and two-grid method are demonstrated in Tabs. 1 and 2. When the exact solution is chosen as $u(\boldsymbol{x},t) = e^t\sin(\pi x_1)\sin(\pi x_2)$, we couple the time step with spatial mesh as $\Delta t = h$, the numerical results of MEFM and two-grid method are presented in Tabs. 3 and 4.

Table 1: Numerical results by MFEM with $u(\boldsymbol{x},t) = e^t x_1 x_2(1-x_1)(1-x_2)$.

h	$\|\|u-u_h\|\|$	$\|\|\boldsymbol{z}-\boldsymbol{z}_h\|\|$	Computing time (s)
1/8	1.5826e-03	4.1845e-03	1.52
1/27	1.4147e-04	3.7821e-04	16.33
1/64	2.4891e-05	6.7930e-04	70.65
Rates	2.0	2.0	

Table 2: Numerical results by two-grid method with $u(\boldsymbol{x},t)=e^t x_1 x_2(1-x_1)(1-x_2)$.

$(H\ ,\ h)$	$\|\|u-U_h\|\|$	$\|\|\boldsymbol{z}-\boldsymbol{Z}_h\|\|$	Computing time (s)
(1/4,1/8)	1.6114e-03	4.5097e-03	1.71
(1/9,1/27)	1.4475e-04	4.0349e-04	9.47
(1/16,1/64)	2.5978e-05	7.2536e-04	22.54
Rates	2.0	2.0	

Table 3: Numerical results by MFEM with $u(\boldsymbol{x},t)=e^t\sin(\pi x_1)\sin(\pi x_2)$.

$h=\Delta t$	$\|\|u-u_h\|\|$	$\|\|\boldsymbol{z}-\boldsymbol{z}_h\|\|$	Computing time (s)
1/8	3.2017e-03	9.6503e-03	0.12
1/27	2.9004e-04	8.8632e-04	3.97
1/64	5.1916e-05	1.5965e-04	19.32
Rates	2.0	2.0	

Table 4: Numerical results by two-grid method with $u(\boldsymbol{x},t)=e^t\sin(\pi x_1)\sin(\pi x_2)$.

$(H\ ,\ h=\Delta t)$	$\|\|u-U_h\|\|$	$\|\|\boldsymbol{z}-\boldsymbol{Z}_h\|\|$	Computing time (s)
(1/4,1/8)	3.5235e-03	1.1218e-02	0.19
(1/9,1/27)	3.1622e-04	9.9945e-04	2.08
(1/16,1/64)	5.6992e-05	1.8016e-04	8.46
Rates	2.0	2.0	

From the numerical results in Tabs. 1-4, we observe that the proposed two methods are of second-order accuracy, which is coincided with our theoretical analysis. Moreover, we also observe that the two-grid method spends less time than the usual MFEM. Thus, we can see that two-grid algorithm is a very effective algorithm when it comes to deal with the nonlinear problems.

6. Conclusions

In this paper, we develop a two-grid mixed finite element method for a class of nonlinear hyperbolic equation. We prove the stability and the error estimate for the two-grid scheme. It is shown theoretically and numerically that when the coarse and fine mesh sizes satisfy $h=\mathcal{O}(H^{(2k+1)/(k+1)})$, the two-grid solution can achieve the same accuracy as the mixed finite element solution.

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