Invariant approximation property under group passes to extensions with a finite quotient

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Abstract. Analytic properties of invariant approximation property, studies analytic techniques from operator theory that encapsulate geometric properties of a group. also we show that the invariant approximation property passes to finite extensions.

Keywords: uniform roe algebras, invariant approximation property.

1. Introduction

The purpose of this paper is to provide an illustration of an interesting and nontrivial interaction between analytic and geometric properties of a group. We provide approximation property of operator algebras associated with discrete groups. There are various notions of finite dimensional approximation properties for C^*- algebras and more generally operator algebras. Some of these (approximation properties) notations will be defined in this paper, the reader is referred to [2], [3], [4], [7], [10], [11], [12], [13], and [15] for these beautiful concepts: Haagerup discovery that the reduced C^*- algebra $\mathbb{F}_n$ has the metric approximation property, Higson and Kasparov's resolution of the Baum-connes conjecture for the Haagerup groups. We study analytic techniques from operator theory that encapsulate geometric properties of a group. The approximation properties of group C^*- algebra are everywhere; it is powerful, important, backbone of countless breakthroughs.

Roe considered the discrete group of the reduced group C^*- algebra of $C^*_r(G)$ is the fixed point algebra $\{Ad\rho(t) : t \in G\}$ acting on the uniform Roe algebra $C^*_u(G)$ [14]. A discrete group G has natural coarse structure which allows us to define the the uniform Roe algebra, $C^*_u(G)$ [14]. We say that the uniform Roe algebra, $C^*_u(G)$, is the C^*- algebra completion of the algebra of bounded operators on $\ell^2(X)$ which have finite propagation. The reduced C^*- algebra $C^*_r(G)$ is naturally contained in $C^*_u(G)$ [14]. According to [Roe] [14], G has the invariant approximation property (IAP) if

$$C^*_\lambda(G) = C^*_u(G)^G.$$

2. Preliminaries

In this section we shall establish the basic definitions and notations for the category of coarse metric spaces. Coarse geometry is the study of the large scale properties of spaces. The notion of large scale is quantified by means of a coarse structure.

Example 2.1 ([14]). Let G be a finitely generated group. Then the bounded coarse structure associated to any word metric on G is generated by the diagonals

$$\Delta_g = \{(h, hg) : h \in G\}.$$

We next recall some basic fact about uniform Roe algebra and metric property of a discrete group. Next we recall the following definitions; Let X be a discrete metric space.

Definition 2.2 ([14]). We say that discrete metric space X has *bounded geometry* if for all R there exists N in $\mathbb{N}$ such that for all $x \in X$, $|B_R(x)| < N$, where $B(x,r) = \{x \in X : d(y,x) \leq r\}$.

Definition 2.3 ([14]). A kernel $\phi : X \times X \longrightarrow \mathbb{C}$:

- is *bounded* if there, exists $M > 0$ such that $|\phi(s,t)| < M$ for all $s, t \in X$
- has *finite propagation* if there exists $R > 0$ such that $\phi(s,t) = 0$ if $d(s,t) > R$.

Let $B(X)$ be a set of bounded finite propagation kernels on $X \times X$. Each such ϕ defines a bounded operator on $\ell^2(X)$ via the usual formula for matrix multiplication

$$\phi * \zeta(s) = \sum_{r \in G} \phi(s,r)\zeta(r) \text{ for } \zeta \in \ell^2(X).$$

We shall denote the finite propagation kernels on X by $A^\infty(X)$.

Definition 2.4 ([14]). The uniform Roe algebra of a metric space X is the closure of $A^\infty(X)$ in the algebra $B(\ell^2(X))$ of bounded operators on X.

If a discrete group G is equipped with its bounded coarse structure introduced in Example 2.1, then one can associate with its uniform Roe algebra $C^*_u(G)$ by repeating the above. A discrete group G has a natural coarse structure which allows us to define the uniform Roe algebra $C^*_u(G)$. A group G can be equipped with either the left or right-invariant of the metric. A choice of one of the determines whether $C^*_\lambda(G)$ or $C^*_\rho(G)$ is a sublagebra of the uniform Roe algebra $C^*_u(G)$ of G.

Hence, any element of $\mathbb{C}[G]$ will give the finite propagation and this assignment extends to an inclusion

$$C^*_\lambda(G) \hookrightarrow C^*_u(G).$$

Next, if the metric on G is left-invariant then

$$C^*_\rho(G) \subset C^*_u(G).$$

Let d_1 be the left-invariant metric on G

$$d_1(x,y) = d_1(gx, gy) \ \forall\ g \in G.$$

Now, we choose a right invariant metric for G so that $C^*_\lambda(G) \hookrightarrow C^*_u(G)$. The right regular representation ρ gives the adjoint action on $C^*_u(G)$ defined by

$$Ad\rho(g)T = \rho(g)T\rho(g)^* = \rho(g)T\rho(g)^{-1},$$

for all $t \in G$, $T \in C^*_u(G)$. Our remarks above show that the elements of $C^*_\lambda(G)$ are invariant with respect to this action and so $C^*_\lambda(G)$ is contained in invariant subalgebra $C^*_u(G)^G$.

Lemma 2.5. *If $T \in C^*_u(G)$ has kernel $A(x,y)$, then $Ad\rho(t)T$ has kernel $A(xt,yt)$*

Proof. We have that:

$$\begin{aligned}(Ad\rho(t)T\zeta)(s) &= \rho(t)(T\rho(t)^*\zeta)(s)\\ &= T\rho(t)^*\zeta(st)\\ &= \sum_{x\in G} A(st,x)(\rho(t)^{-1}\zeta)(x)\\ &= \sum_{x\in G} A(st,x)\zeta(xt^{-1}).\end{aligned}$$

Now, $A(st,x)$ is non-zero whenever $x,y,t \in G$ such that $y = xt^{-1}$,
so $x = yt$ and we have

$$(Ad\rho(t)T\zeta)(s) = \sum_{x\in G} A(st,yt)\zeta(y)$$

Thus, $Ad\rho(t)T$ has kernel $A(st,yt)$. □

In general, if $T \in C^*_u(X)$, then $\forall\ x,y \in G$:

$$\begin{aligned}\langle Ad(\rho(t))T\delta_x, \delta_y\rangle &= \left\langle \rho(t)T\rho(t^{-1})\delta_x, \delta_y\right\rangle\\ &= \left\langle T\rho(t^{-1})\delta_x, \rho(t^{-1})\delta_y\right\rangle\\ &= \langle T\delta_{xt}, \delta_{yt}\rangle\,.\end{aligned}$$

So, the operator T is $Ad\rho-$ invariant if and only if

$$\forall\ x,y \in X\ \forall\ t \in G\ \ \langle T\delta_{xt}, \delta_{yt}\rangle = \langle T\delta_x, \delta_y\rangle\,.$$

We now define the invariant approximation: property (IAP).

Definition 2.6 ([14])**.** We say that G has the *invariant approximation property*(IAP) if

$$C^*_\lambda(G) = C^*_u(G)^G.$$

3. The IAP passes to extensions with a finite quotient

In this section, we show that the invariant approximation property passes to extensions. For details of extensions see [15] . Consider two groups H and N, and let G be an extension of H by N where $N \cong G/H$. Let

$$1 \longrightarrow H \stackrel{i}{\hookrightarrow} G \stackrel{\pi}{\longrightarrow} G/H \longrightarrow 1$$

be an exact sequence.

Let G be the set $G = H \times N$ and $i : H \longrightarrow G$ be given by $i(a) = (a, e)$ (for any $a \in H$), with $\pi : G \longrightarrow N$ given by $\pi(a, \gamma) = \gamma$ (for any $(a, \gamma) \in G$). We choose a set-theoretic cross-section $\sigma : N \longrightarrow G$, $1 \longmapsto 1$ of σ such that $\pi \circ \sigma = Id_{G/H}$. We define

$$f : N \times N \longrightarrow G$$

by

$$f(n_1, n_2) = \sigma(n_1)\sigma(n_2)\sigma(n_1 n_2)^{-1}, \ \forall \ n_1, \ n_2 \ \in \ N.$$

Let $\rho(\gamma)$ be the conjugation by $\sigma(\gamma)$ in H:

$$\rho(\gamma)(h) = \sigma(\gamma)h\sigma(\gamma)^{-1}.$$

For $\alpha \in N$,

$$Ad(\alpha) : N \longrightarrow N \text{ and } \gamma \longmapsto \alpha\gamma\alpha^{-1}.$$

Then, the function f and ρ are related as follows [5]:

$$\rho(\beta)\rho(\gamma) = Ad(f(\beta, \gamma))\rho(\beta\gamma), \tag{3.1}$$

and

$$f(\gamma_1, \gamma_2)f(\gamma_1\gamma_2, \gamma_3) = \rho(\gamma_1)f(\gamma_2, \gamma_3)f(\gamma_1, \gamma_2\gamma_3). \tag{3.2}$$

Since

$$Ad(f(\beta, \gamma))\rho(\beta\gamma) = f(\beta, \gamma)f(\beta, \gamma)^{-1}\rho(\beta\gamma) = \rho(\beta)\rho(\gamma)$$

and

$$\begin{aligned}
f(\gamma_1, \gamma_2)f(\gamma_1\gamma_2, \gamma_3) &= \sigma(\gamma_1)\sigma(\gamma_2)\sigma(\gamma_1\gamma_2)^{-1}\sigma(\gamma_1\gamma_2)\sigma(\gamma_3)\sigma(\gamma_1\gamma_2\gamma_3)^{-1} \\
&= \sigma(\gamma_1)\sigma(\gamma_2)\sigma(\gamma_3)\sigma(\gamma_1\gamma_2\gamma_3)^{-1} \\
&= \sigma(\gamma_1)1\sigma(\gamma_1)^{-1}\sigma(\gamma_1)\sigma(\gamma_2)\sigma(\gamma_3)\sigma(\gamma_1\gamma_2\gamma_3)^{-1} \\
&= \rho(\gamma_1)\sigma(\gamma_2)\sigma(\gamma_2)^{-1}\sigma(\gamma_1)\sigma(\gamma_2\gamma_3)\sigma(\gamma_1\gamma_2\gamma_3)^{-1} \\
&= \rho(\gamma_1)\sigma(\gamma_2)\sigma(\gamma_3)\sigma(\gamma_3)^{-1}\sigma(\gamma_2)^{-1}\sigma(\gamma_1)\sigma(\gamma_2\gamma_3)\sigma(\gamma_1\gamma_2\gamma_3)^{-1} \\
&= \rho(\gamma_1)\sigma(\gamma_2)\sigma(\gamma_3)\sigma(\gamma_2\gamma_3)^{-1}\sigma(\gamma_1)\sigma(\gamma_2\gamma_3)\sigma(\gamma_1\gamma_2\gamma_3)^{-1} \\
&= \rho(\gamma_1)f(\gamma_2, \gamma_3)f(\gamma_1, \gamma_2\gamma_3)
\end{aligned}$$

The set group law is given by $(h_1,\gamma_1)(h_2,\gamma_2)=(h_1\rho(\gamma_1)(h_2)f(\gamma_1,\gamma_2),\gamma_1\gamma_2)$. Let G be a group. Now, we choose a set-theoretic section in

$$1\longrightarrow H\overset{i}{\hookrightarrow}G\overset{\pi}{\longrightarrow}G/H\longrightarrow 1$$

is the same as to choose coset representatives in G/H: $r_1,\cdots,r_n$. G/H is a group, but it is not true in general that

$$r_ir_j\in R=\{\text{set of coset representatives}\}$$

since r_ir_j is product in G, there is a new product on R (which is a product on G/H). Let $r_1*r_2\in G$ such that $r_1*r_2=r \leftrightsquigarrow$ the choosen coset representatives of $[r_1r_2]$. And, also

$$(Hr_1)(Hr_2)=Hr_1r_2=H\left(r_1r_2\left(r_1*r_2\right)^{-1}\right)\left(r_1*r_2\right),$$

and $(r_1r_2(r_1*r_2)^{-1})\in H$. So, r_1*r_2 is the product in G/H. To choose coset representatives, we have a set-theoretic identification:

$$G=H\times G/H\ \ \text{(This is called \textbf{Jolissaint product})}.$$

We assume that there is a bijective

$$\phi:G\longrightarrow H\times G/H,$$

$$g=h_gr_g\longmapsto(h_g,r_g).$$

Where ϕ is a group isomorphism if $H\times G/H$ is equipped with the Jolissaint product. Coset representation, $\forall g\in G$, $\exists\ h_g\in H,\ r_g\in G/H$ such that $g=h_gr_g$ and $\forall g'\in G$, $\exists\ h_{g'}\in H,\ r_{g'}\in G/H$ such that $g'=h_{g'}r_{g'}$. Since H is normal subgroup of G, so $Hg\cong gH$. Right G action on $H\times G/H$. Consider

$$\begin{aligned} gg' &= \left(h_gr_g\right)\left(h_{g'}r_{g'}\right)\\ &= h_g\left(r_gh_{g'}r_g^{-1}\right)r_gr_{g'}\\ &= h_g\left(r_gh_{g'}r_g^{-1}\right)r_gr_{g'}\left(r_g*r_{g'}\right)^{-1}\left(r_g*r_{g'}\right),\end{aligned}$$

where $r_g*r_{g'}\in R$ and $h_g(r_gh_{g'}r_g^{-1})\in H$. $r_g*r_{g'}$ and $r_gr_{g'}$ determine the same coset, so $\exists\ s\in\ H$ such that $s(r_g*r_{g'})=r_gr_{g'}$,

$$s=r_gr_{g'}\left(r_g*r_{g'}\right)^{-1}.$$

To show that ϕ is a group homomorphism we compute:

$$\begin{aligned} gg' &\longmapsto \left\{h_g\left(r_gh_{g'}r_g^{-1}\right)r_gr_{g'}\left(r_g*r_{g'}\right)^{-1},\left(r_g*r_{g'}\right)\right\}\\ &= \left(h_g,r_g\right)*\left(h_{g'},r_{g'}\right)\end{aligned}$$

but

$$h_g \left(r_g h_{g'} r_g^{-1}\right) r_g r_{g'} \left(r_g * r_{g'}\right)^{-1} \in H.$$

Since ϕ becomes a group isomorphism $\phi : G \longrightarrow H \times G/H$, when the space on the left is equipped with the product

$$(h_g, r_g)(h_{g'}, r_{g'}) = \left\{h_g \left(r_g h_{g'} r_g^{-1}\right) r_g r_{g'} \left(r_g * r_{g'}\right)^{-1}, r_g * r_{g'}\right\}.$$

Therefore $H \times G/H$ is a group. While H is a subgroup of G, N is not subgroup of G, since, for example if $r_g = r_{g'} = e$, then $(h_g, e)(h_{g'}, e) = (h_g h_{g'}, e)$ or if $h_g = h_{g'} = e$, then

$$(e, r_g)(e, r_{g'}) = \left\{r_g r_{g'} \left(r_g * r_{g'}\right)^{-1}, r_g * r_{g'}\right\}.$$

Next, we consider the left G action on

$$G = G/H \times H \text{ (This is called } \textbf{Jolissaint product}\text{)}$$

We assume that there is a bijective $\phi : G \longrightarrow G/H \times H$. This is a group isomorphism when the right hand side is equipped with the Jolissaint product $g = r_g h_g \longmapsto (r_g, h_g)$. Coset representation, $\forall g \in G \; \exists \; h_g \in H, \; r_g \in G/H$ such that $g = r_g h_g$ and $\forall g' \in G \; \exists \; h_{g'} \in H, \; r_{g'} \in G/H$ such that $g' = r_{g'} h_{g'}$. To show that ϕ is a group homomorphism, we compute:

$$\begin{aligned} gg' &= (r_g h_g)\left(r_{g'} h_{g'}\right) \\ &= r_g r_{g'} \left(r_{g'}^{-1} h_g r_{g'}\right) h_{g'} \\ &= \left(r_g * r_{g'}\right)\left(r_g * r_{g'}\right)^{-1} r_g r_{g'} \left(r_{g'}^{-1} h_g r_{g'}\right) h_{g'}. \end{aligned}$$

We have

$$gg' \longmapsto (r_g, h_g) * \left(r_{g'}, h_{g'}\right).$$

Since ϕ becomes a group isomorphism $\phi : G \longrightarrow G/H \times H$, when the space on the right is equipped with the product

$$(r_g, h_g)(r_{g'}, h_{g'}) = \left\{\left(r_g * r_{g'}\right), \left(r_g * r_{g'}\right)^{-1} (r_g r_{g'})(r_{g'})^{-1} h_g r_{g'} h_{g'}\right\}.$$

Therefore, $G/H \times H$ is a group.

While H is a subgroup of G, G/H is not subgroup of G, since, for example if $r_g = r_g' = e$, then

$$(e, h_g)(e, h_{g'}) = \left\{(e * e), (e * e)^{-1} (ee')(e)^{-1} h_g e h_{g'}\right\} = \left\{e, h_g h_{g'}\right\}$$

or if $h_g = h_{g'} = e$, then

$$\begin{aligned} (r_g, e)(r_{g'}, e) &= \left\{\left(r_g * r_{g'}\right), \left(r_g * r_{g'}\right)^{-1} (r_g r_{g'})(r_{g'})^{-1} r_{g'}\right\} \\ &= \left\{\left(r_g * r_{g'}\right), \left(r_g * r_{g'}\right)^{-1} (r_g r_{g'})\right\}. \end{aligned}$$

G/H is a subgroup when the assignment $[r] \longmapsto r \in R \subset G$ is a group homormophsim, i.e., when $r_g * r_{g'} = r_g r_{g'}$.

Next, we show that the main result of this Chapter:

Theorem 3.1. *Let G be a discrete group. If H is a finite index normal subgroup of G with IAP, and*

$$0 \longrightarrow H \xrightarrow{i} G \longrightarrow G/H \longrightarrow 0,$$

then G has IAP.

Proof. Since $\phi : G \xrightarrow{\cong} G/H \times H$, which is a fact becomes a group isomorphism when the space on the right is equipped with **Jolissaint product** [5]. We want to understand if there is an isomorphism

$$C_u^*(G)^G \cong C_u^*(G/H \times H)^{G/H \times H}.$$

Since

$$\phi : G \xrightarrow{\cong} G/H \times H,$$

we have

$$C_\lambda^*(G) \xrightarrow{\cong} C_\lambda^*(G/H \times H).$$

We need to show that

$$C_u^*(G)^G \cong C_\rho^*(G).$$

The left coset decomposition of G

$$G = \coprod_{r \in R} rH,$$

where R is the set of left coset representatives. This space has a natural right multiplication action by H, as it preserves left cosets. R can be made into a group ($R \subset G$, a subset of G) with the $*-$ product and R is not a subgroup of G. It follows that there is a corresponding action on

$$\ell^2(G) = \bigoplus_{r \in R} \ell^2(rH),$$

where $\ell^2(rH)$ is invariant under $\rho(H)$. That is: For every $r \in R$ is the set of left coset representatives

$$\ell^2(rH) = \overline{\text{span}}\{\delta_{rh} \mid r \in R,\ h \in H\},$$

we have $s \in H, \rho(s)\delta_{rh} = \delta_{rhs} \in \ell^2(rH)$. On the other hand, the bijection ϕ gives a Hilbert spaces isomorphism $\ell^2(G) = \ell^2(G/H) \otimes \ell^2(H)$. But G/H is finite, so this is just

$$\ell^2(G) = \mathbb{C}^n \otimes \ell^2(H), n = |R| = \bigoplus_{r \in R} \ell^2(H),$$

where $\bigoplus_{r\in R}\ell^2(H)$ is the n copies of $\ell^2(H)$. The isomorphism ϕ works by means of unitary maps

$$V_r : \ell^2(rH) \longrightarrow \ell^2(H),$$
$$\delta_{rh} \longmapsto \delta_h,$$

the inverse map

$$V_r^* : \ell^2(H) \longrightarrow \ell^2(rH),$$
$$\delta_h \longmapsto \delta_{rh},$$

On $\ell^2(G)$ we can define a family of projections $P_s : \ell^2(G) \longrightarrow \ell^2(sH), s \in H$. Using the decomposition

$$G = \coprod_{r\in R} rH.$$

We can represent each function $\zeta \in \ell^2(G)$ as a linear combination $\zeta = \sum_{r\in R}\zeta_r$, where $\zeta_r \in \ell^2(rH)$ (this is understood as a subspace of $\ell^2(G)$ so that ζ_r is a function on $\ell^2(G)$ which vanishes outside rH) $P_s(\zeta) = \zeta_s$ (it seems that this works for infinite G/H as well). Note that P_s commutes with $\rho(h)$, $h \in H$ $s \in R$. So:

$$\rho(h)\zeta(t) = \sum_{r\in R}\zeta_r(th).$$

We have $(P_s\rho(h)\zeta)(t) = \rho(h)\zeta_s(t) = \zeta_s(th) = (\rho(h)P_s\zeta)(t)$. Now, take $T \in C_u^*(G)$. With respect to the decomposition

$$G = \coprod_{r\in R} rH,$$

this can be represented as

$$T = \sum_{r,r'\in R} P_rTP_{r'}$$

where $P_rTP_{r'} : \ell^2(rH) \longrightarrow \ell^2(r'H)$. In other words, T can be represented as matrix

$$\begin{pmatrix} & \vdots & \\ \cdots & P_rTP_{r'} & \cdots \\ & \vdots & \end{pmatrix}.$$

The points is that this decomposition is invariant with respect to the action of $\rho(H)$:

$$\forall\ h, h' \in H \quad P_{r'}\rho(h')T\rho(h)P_r = \rho(h')P_{r'}TP_r\rho(h).$$

Note that in particular $P_eTP_e : C_\rho^*(G) \longrightarrow C_\rho^*(G)$ and is a conditional expectation. Note also that $\forall s \in R$, the unitary operator $V_s : \ell^2(sH) \longrightarrow \ell^2(H)$ commute with $\rho(H)$

$$\rho(h')\delta_{sh} = \delta_{s(hh')} \overset{V_s}{\longmapsto} \delta_{hh'} = \rho(h')V_s\delta_{sh}.$$

We want to understand the right regular representation ρ of H in terms of the bijection $G \xrightarrow{\cong} G/H \times H$ or $G \xrightarrow{\cong} H \times G/H$. If we use left cosets of G, then

$$\phi : G \longrightarrow G/H \times H.$$

Now, we call the isomorphism $\Phi : C_u^*(G) \xrightarrow{\cong} C_u^*(G/H) \otimes C_u^*(H)$ given by

$$\Phi : T = \sum_{r,s \in R} P_r T P_s \longmapsto \sum_{r,s \in R} E_{r,s} \otimes V_r P_r T P_s V_s^*,$$

where

$$V_r : \ell^2(rH) \longrightarrow \ell^2(H)$$

and

$$P_r : \ell^2(G) = \bigoplus_{r \in R} \ell^2(rH) \longrightarrow \ell^2(rH).$$

This commutes with the action of $\rho(H)$. Note that H is a subgroup of $G/H \times H$

$$h \longmapsto (e, h).$$

We have

$$(r,h)(e,h') = ((r * e), (r * e)^{-1}(re)(e)^{-1}heh') = (r, r^{-1}rhh') = (r, hh').$$

So, (e, h') acts trivially on the first factor in $G/H \times H$. Next, we show the following important proposition, which is used for the main result (Theorem 3.1) of this Chapter.

Proposition 3.2. *The isomorphism Φ commutes with the adjoint action $Ad\rho$ of H.*

Proof. $\forall\, h \in H$

$$\begin{aligned}
\Phi(Ad\rho(h)T) &= \Phi(\sum_{r,s \in R} P_r Ad\rho(h) T P_s) \\
&= \sum_{r,s \in R} (E_{r,s} \otimes V_r P_r Ad\rho(h) T P_s V_s^*) \\
&= \sum_{r,s \in R} E_{r,s} \otimes Ad\rho(h)\,(V_r P_r T P_s V_s^*) \\
&= Ad\rho(h) \left(\sum_{r,s \in R} E_{r,s} \otimes V_r P_r T P_s V_s^* \right) \\
&= Ad\rho(h)\Phi(T).
\end{aligned}$$

$\square$

Conclusion 3.3. Since $G/H \times H$ is the right equipped with Jolissaint product [5], taking the induce action of H on both side, we have

$$C_u^*(G)^H \cong C_u^*(G/H \times H)^H \cong C_u^*(G/H)^H \otimes C_u^*(G)^H.$$

So, if H has the IAP:

$$C_u^*(G)^H \cong C_u^*(G/H) \otimes C_\lambda^*(H) = M_n(C_\lambda^*(H)),$$

then, we know

$$C_u^*(G)^G \subseteq C_u^*(G)^H \subseteq M_n(C_\lambda^*(H)).$$

Proposition 3.4. *If $T \in C_u^*(G)$ is $H-$ invariant then $\sum_{r\in R} Ad\rho(r)T$ is a $G-$ invariant .*

Proof. Take $g \in G$, such that $g = r_g h_g$, where $r_g \in R$ and $h_g \in H$. We have

$$\begin{aligned} Ad\rho(g)\left(\sum_{r\in R} Ad\rho(r)T\right) &= \sum_{r\in R} g(rTr^{-1})g^{-1} \\ &= \sum_{r\in R} (r_g h_g)\, rTr^{-1}(r_g h_g)^{-1} \\ &= \sum_{r\in R} r_g h_g r T r^{-1} h_g^{-1} r_g^{-1}. \end{aligned}$$

If we take $h_g r \in G \ \ \exists s \in R$, $h \in H$ such that $h_g r = sh$. Then

$$\begin{aligned} Ad\rho(g)\left(\sum_{r\in R} Ad\rho(r)T\right) &= \sum_{r\in R} r_g h_g r T r^{-1} h_g^{-1} r_g^{-1} \\ &= \sum_{r\in R} r_g s h T r^{-1} h^{-1} s^{-1} r_g^{-1} \\ &= \sum_{r\in R} r_g s T r^{-1} s^{-1} r_g^{-1}. \end{aligned}$$

We need to claim that $r_g s$ runs through R and $h_g r = r(r^{-1}h_g r)$. So:

$$\begin{aligned} Ad\rho(g)\left(\sum_{r\in R} Ad\rho(r)T\right) &= \sum_{r\in R} r_g h_g r T r^{-1} h_g^{-1} r_g^{-1} \\ &= \sum_{r\in R} r_g r(r^{-1}h_g r) T r^{-1} h_g^{-1} r_g^{-1} \\ &= \sum_{r\in R} r_g r T (r_g r)^{-1} \\ &= \sum_{r\in R} (r_g * r)(r_g * r)^{-1} r_g r T (r_g r)^{-1} (r_g * r)(r_g * r)^{-1} \\ &= \sum_{r\in R} (r_g * r) T (r_g * r)^{-1} \\ &= \sum_{s\in R} r_s T r_s^{-1}. \quad \square \end{aligned}$$

When we define $C_u^*(G)^G$, we consider the right action of G on $\ell^2(G)$ which induces the $Ad\rho-$ action on $C_u^*(G)$. Take $g \in G$, such that $g = r_g h_g$, where $r_g \in R$ and $h_g \in H$

$$Ad\rho(g)T{=}\rho(r_g h_g)T\rho(r_g h_g)^*{=}\rho(r_g)\rho(h_g)T\rho(h_g)^{-1}\rho(r_g)^{-1}{=}Ad\rho(r_g)(Ad\rho(h_g)T).$$

It seems that when $T \in \left(C_u^*(G)^H\right)^{G/H}$ (which still needs to be defined) then

$$Ad\rho(h_g)T = T, \text{ and } Ad\rho(r_g)\left(Ad\rho(h_g)T\right) = T.$$

So, $Ad\rho(g)T = T$. Consider $C_u^*(G)^H$. Take $r, t \in R$, $T \in C_u^*(G)^H$. We have

$$\begin{aligned} Ad\rho(rt)T &= Ad\left(\rho(r*t)(r*t)^{-1}rt\right)T \\ &= Ad\rho(r*t)\left(Ad(\rho(r*t)^{-1}rt)T\right) \\ &= Ad\rho(r*t)T. \end{aligned}$$

Conclusion 3.5. We seem to have an $R-$ action G/H on $C_u^*(G)^H$. If this is so, this could imply that

$$C_u^*(G)^G \cong \left(C_u^*(G)^H\right)^{G/H}.$$

We define $\left(C_u^*(G)^H\right)^{G/H}$:a possible action of R on $C_u^*(G)^H$. $R \subset G$, so it makes sense to consider $Ad\rho(r)T$, for any $r \in R$, $T \in C_u^*(G)$, where ρ is the right regular representation of G. Since $\rho(r)\rho(s) \neq \rho(r*s) r, s \in R$, then for $T \in C_u^*(G)^H$, we have:

$$\begin{aligned} Ad\rho(r)Ad\rho(s)T &= Ad\rho(r)\left(Ad\rho(s)T\right) \\ &= \rho(r)\left(\rho(s)T\rho(s)^{-1}\right)\rho(r)^{-1} \\ &= \rho(rs)T\rho(rs)^{-1} \\ &= \rho(r*s)\rho\left((r*s)^{-1}rs\right)T\rho\left((r*s)^{-1}rs\right)^{-1}\rho(r*s)^{-1} \\ &= Ad\rho(r*s)T. \end{aligned}$$

We obtain the following important proposition, which is used for the main result (Theorem 3.1) of this Chapter.

Proposition 3.6. *The group $(R,*) \cong G/H$ acts on $C_u^*(G)^H$, and the action is induced by the right regular representation ρ of G.*

We need to show that

$$C_u^*(G)^G \cong \left\{C_u^*(G)^H\right\}^{G/H}.$$

If $T \in \left(C_u^*(G)^H\right)^{G/H}$, then $T \in C_u^*(G)^G$. Since for every $g \in G$, such that $g = r_g h_g$ and

$$Ad\rho(r_g h_g)T = Ad\rho(r_g)Ad\rho(h_g)T = T$$

So, $\left(C_u^*(G)^H\right)^{G/H} \subseteq C_u^*(G)^G$. We also have $C_u^*(G)^G \subseteq C_u^*(G)^H$. If $T \in C_u^*(G)^G$ then $T \in C_u^*(G)^H$. Since, for every $g \in G$, $g = r_g h_g$. Then $Ad\rho(h_g)T = T$. We have $Ad\rho(r_g)(Ad\rho(h_g)T) = \rho(r_g)(\rho(h_g T\rho(h_g^{-1})\rho(r_g)^{-1} = \rho(r_g h_g)T\rho(r_g h_g)^{-1} = Ad\rho(g)T$. So, $C_u^*(G)^G \subseteq \left(C_u^*(G)^H\right)^{G/H}$. Which would give

$$C_u^*(G)^G \cong \left(C_u^*(G)^H\right)^{G/H}.$$

Next, we need to show that:

$$\left((C_u^*(G/H) \otimes C_u^*(H))^H\right)^{G/H} \cong C_u^*(G/H)^{G/H} \otimes C_u^*(H)^H.$$

We denote by P_i the projection onto $\ell^2(Hi)$;

$$P_i : \ell^2(G) \longrightarrow \ell^2(Hi).$$

For every $r \in R$, there is also a unitary isomorphism $V_i : \ell^2(Hi) \longrightarrow \ell^2(H)$, induced by the map $hi \longmapsto h, \forall h \in H$. We have

$$(P_i\rho(r))(P_i\rho(r))^* = P_i\rho(r)\rho(r)^*P_i^* = P_iP_i^* = P_i$$

and

$$\rho(s)P_i : \ell^2(Hr) \longrightarrow \ell^2(H(r * s)),$$

$$(\rho(s)P_r)^*(\rho(s)P_r) = P_r^*\rho(r)^*\rho(r)P_r = P_r^*P_r = P_r = id_{H_r}$$

we get the unitary isomorphsim $P_i\rho(r)^* : Hs \xrightarrow{\cong} Hi$, $i = s * r^{-1}$. Then

$$\rho(r)(P_iV_i^*TV_jP_j)\rho(r)^* : \ell^2(Hs) \xrightarrow{P_j\rho(r)^*} \ell^2(Hj) \xrightarrow{V_j} \ell^2(H) \xrightarrow{V_i^*} \ell^2(Hi) \xrightarrow{\rho(r)P_i} \ell^2(H(i * r)).$$

Thus

$$\rho(r)(P_iV_i^*TV_jP_j)\rho(r)^* : \ell^2(Hs) \longrightarrow \ell^2(H(i * r)).$$

We get $E_{i*r,j*r} = Ad_{\rho_{G/H}}E_{i,j}$. Then $T \otimes E_{i,j} \longmapsto T \otimes E_{i*r,j*r}$. Therefore,

$$\left((C_u^*(G/H) \otimes C_u^*(H))^H\right)^{G/H} \cong C_u^*(G/H)^{G/H} \otimes C_u^*(H)^H.$$

We know that the isomorphsim

$$\Phi : C_u^*(G) \xrightarrow{\cong} C_u^*(G/H) \otimes C_u^*(H)$$

is $H-$ equivariant so that

$$C_u^*(G)^H \cong C_u^*(G/H \times H)^H \cong C_u^*(G/H) \otimes C_u^*(H)^H.$$

The isomorphsim uses that H is a subgroup of $G/H \times H$ and acts trivially on G/H. We now need to understand the action $\rho_{G/H\times H}$ on $C_u^*(G/H) \otimes C_u^*(H)$.

We obtain the following: we want to understand the right regular representation ρ of G in terms of the bijection $G \xrightarrow{\cong} G/H \times H$ or $G \xrightarrow{\cong} H \times G/H$, we have

$$\begin{aligned} C_u^*(G)^G &\cong \left(C_u^*(G)^H\right)^{G/H} \\ &\cong \left(C_u^*(G/H) \otimes C_u^*(H)^H\right)^{G/H}. \end{aligned}$$

Taking invariants with respect to G/H.

$$\begin{aligned} \left(C_u^*(G)^H\right)^{G/H} &\cong \left((C_u^*(G/H) \otimes C_u^*(H))^H\right)^{G/H} \\ &\cong C_u^*(G/H)^{G/H} \otimes C_u^*(H)^H. \end{aligned}$$

Since H has IAP. Then

$$C_u^*(G/H)^{G/H} \otimes C_u^*(H)^H = C_u^*(G/H)^{G/H} \otimes C_\lambda^*(H).$$

Since G/H is finite group, every finite group is amenable group. Roe shows that the amenable group has IAP [14]. Thus,

$$\begin{aligned} C_u^*(G)^G &\cong \left(C_u^*(G)^H\right)^{G/H} \\ &\cong C_u^*(G/H)^{G/H} \otimes C_\lambda^*(H)^H \\ &\cong C_\lambda^*(G/H) \otimes C_\lambda^*(H). \end{aligned}$$

Next, we need to show that the following Proposition:

Proposition 3.7. *The left regular representation λ_G on $\ell^2(G)$ is isomorphic to the left regular representation $\lambda_H \otimes \lambda_{G/H}$ on $\ell^2(H) \otimes \ell^2(G/H)$.*

Proof. Let R be the set of right coset representation. We have a bijection

$$G = \coprod_{r \in R} Hr$$

which induces the Hilbert space isomorphism

$$\ell^2(G) = \coprod_{r \in R} \ell^2(Hr).$$

We denote by P_r the projection onto $\ell^2(Hr)$;

$$P_r : \ell^2(G) \longrightarrow \ell^2(Hr).$$

For every $r \in R$ there is also a unitary isomorphism $V_r : \ell^2(Hr) \longrightarrow \ell^2(H)$, induced by the map $hr \longmapsto h, \forall h \in H$. As we have seen before, the coset decomposition of G induces a bijection

$$\phi : G \xrightarrow{\cong} H \times G/H$$

and a Hilbert space isomorphism $\ell^2(G) \longrightarrow \ell^2(H) \otimes \ell^2(G/H)$. This gives a rise to the C^*- algebra isomorphism

$$\Phi : C^*_u(G) \xrightarrow{\cong} C^*_u(H) \otimes C^*_u(G/H)$$

given by

$$T \longmapsto \sum_{r',r \in R} V_{r'} P_{r'} T P_r V_r^* \otimes E_{r'r}.$$

The direct sum decomposition of $\ell^2(G)$ allows one to respect it operators in $C^*_u(G)$ as matrices of size $|R| \times |R|$ whose entries are operators

$$\ell^2(Hr) \longrightarrow \ell^2(Hr'), \text{ for } r',\ r \ \in \ R.$$

This induces an analogous matrix decomposition of element of $C^*_\lambda(G)$, and we shall now use this representation to constrict an isomorphism $\lambda_G \cong \lambda_H \otimes \lambda_{G/H}$.

We have a bijection

$$sHr \cong (sHs^{-1})sr(s*r)^{-1} \cong H(s*r),$$

$$\forall\ s, r \ \in \ R,\ \alpha(s,r) : H \longrightarrow (sHs^{-1})sr(s*r)^{-1} \in H$$
$$h \longmapsto (shs^{-1})sr(s*r)^{-1}.$$

This is a bijection, which induces a unitary isomorphism

$$U_{\alpha(s,r)} : \ell^2(H) \longrightarrow \ell^2(H)$$

given by $(U_{\alpha(s,r)}\xi)(t) = \xi(\alpha(s,r)t)$. We extend it to a map

$$H(s*r) \longrightarrow H(s*r),$$

$$h(s*r) \longmapsto (\alpha(s,r)h)(s*r).$$

We have

$$(\alpha(s,r)H)(s*r) \cong sHr,$$

where $\alpha(s,r)$ is a composition of $ad(s)$ with $\rho(sr(s*r)^{-1})$,

$$ad(s) : H \longrightarrow H$$

is a group isomorphism. And $ad(s)(h) = shs^{-1}$ and $\rho(h')(h) = hh'$. $ad(s)H$ is an isomorphism of H, while $\rho(h')$ commutes with the left action of H.

Let $g = hs \ \in \ G$, where $h \in H$, $s \in R$. When restricted to $\ell^2(Hr)$(by means of projection P_r), $\lambda_G(hs)$ can be explicitly computed as follows : Thanks to isomorphism $\ell^2(G) \longrightarrow \ell^2(H) \otimes \ell^2(G/H)$. We know that the set of linear combinations of functions on G of the form $\eta\gamma$, where $\eta \in \ell^2(H)$ and

$\gamma \in \ell^2 (G/H)$ is dense in $\ell^2(G)$. We can therefore assume that $\zeta \in \ell^2(G)$ is of the form $\zeta = \eta\gamma$. Then, for every $t \in H$ $r \in$ R and $\xi \in \ell^2(Hr)$.

$$\begin{aligned} (\lambda_G(hs)\xi(tr)) &= \xi\left(s^{-1}h^{-1}tr\right) \\ &= \xi\left(s^{-1}(h^{-1}t)ss^{-1}r(s^{-1}*r)^{-1}(s^{-1}*r)\right) \\ &= \xi\left(\alpha(s^{-1},r)(h^{-1}t)(s^{-1}*r)\right) \\ &= \eta\left(\alpha(s^{-1},r)(h^{-1}t)\right)\gamma\left((s^{-1}*r)\right). \end{aligned}$$

Now, the operator of multiplication on the left by $\alpha(s^{-1}, r) \in H$ induces a unitary isomorphism

$$U_{\alpha(s,r)} : \ell^2(H) \longrightarrow \ell^2(H)$$

given by

$$\eta \longmapsto (U_{\alpha(s^{-1},r)}\eta)(t) = \eta(\alpha(s^{-1}, r)t).$$

Thus, we have $(\lambda_G(hs)\xi(tr)) = \left(\lambda_H(h)U_{(s^{-1},r)}\eta\right)(t)(\lambda_{G/H}(s)\gamma)$. Next, we need to show that the following Lemma:

Lemma 3.8. *With the above notations* $\lambda_H(h)U_{\alpha(s,r)} = U_{\alpha(s,r)}\lambda_H(ad(s)h)$.

Proof.

$$\begin{aligned} \left(\lambda_H(h)U_{\alpha(s,r)}\zeta\right)(t) &= U_{\alpha(s,r)}\zeta(h^{-1}t) \\ &= \zeta(\alpha(s,r)(h^{-1}t)) \\ &= \zeta(s(h^{-1}t)s^{-1}sr(s*r)^{-1}) \\ &= \zeta(s(h^{-1}s^{-1})sts^{-1}sr(s*r)^{-1}) \\ &= \zeta(ad(s)(h^{-1})\alpha(s,r)(t)) \\ &= U_{\alpha(s,r)}\lambda_H\left((ad(s)h^{-1})^{-1}\zeta\right)(t) \\ &= U_{\alpha(s,r)}\lambda_H\left((ad(s)h)\zeta\right)(t). \end{aligned}$$

We have $\lambda_H(h)U_{\alpha(s^{-1},r)}\zeta = (U_{\alpha(s^{-1},r)}\lambda_H(ad(s)h))\zeta$. □

Here the Lemma:

Lemma 3.9. *The following diagram commutes:* $r, s \in R$

$$\begin{array}{ccc} \ell^2(G) & \xrightarrow[V_{s^{-1}*r}]{} & \ell^2(H) \otimes \ell^2(G/H) \\ \uparrow {\scriptstyle P_{s^{-1}*r}\lambda_G(hs)P_r} & & \uparrow {\scriptstyle \lambda_H(h)U(s^{-1},r)\otimes\lambda_{G/H}(h)} \\ \ell^2(g) & \xrightarrow[V_r]{} & \ell^2(H) \otimes \ell^2(G/H) \end{array}$$

Proof. Since we have $s, r \in R$

$$\lambda_H(h)U_{\alpha(s^{-1},r)}\zeta = (U_\alpha(s^{-1}, r)\lambda_H(ad(s)h))\zeta.$$

The following diagram commutes

$$\begin{array}{ccccc}
\ell^2(H)\otimes\ell^2(G/H) & \xrightarrow[V_{s^{-1}*r}]{} & \ell^2(H)\otimes\ell^2(G/H) & \xrightarrow[\cong]{} & \ell^2(H)\otimes\ell^2(G/H) \\
\uparrow{\scriptstyle \lambda_H(h)U(s^{-1},r)\otimes\lambda_{G/H}(h)} & & \uparrow{\scriptstyle U(s^{-1},r)\lambda_H(ad(s)h)\otimes\lambda_{G/H}(s)} & & \uparrow{\scriptstyle \lambda_H(ad(s^{-1})h)\otimes\lambda_{G/H}(s)} \\
\ell^2(H)\otimes\ell^2(G/H) & \xrightarrow[V_r]{} & \ell^2(H)\otimes\ell^2(G/H) & \xrightarrow[\cong]{} & \ell^2(H)\otimes\ell^2(G/H)
\end{array}$$

We have proved:

$$\begin{aligned}
P_{s^{-1}*r}\lambda_G(hs)P_r &\cong \lambda_H(h)U(s^{-1},r)\otimes\lambda_{G/H}(h) \\
&\cong U(s^{-1},r)\lambda_H(ad(s)h)\otimes\lambda_{G/H}(h) \\
&\cong \lambda_H(ad(s^{-1})h)\otimes\lambda_{G/H}(s). \quad \square
\end{aligned}$$

On the other hand, next we need to find $(e,s)^{-1}\in H\times G/H$: The inverse of $s\in R\cong G/H$ will be denoted by $\overline{s}$. If (e,s) and $(h,\overline{s})\in H\times G/H$: we have

$$(e,s)*(h,\overline{s}) = \left\{(shs^{-1})(s\overline{s})(s*\overline{s})^{-1},(s*\overline{s})\right\}.$$

If $(e,s)^{-1}=(h,\overline{s})$, then

$$\left\{(shs^{-1})(s\overline{s})(s*\overline{s})^{-1},(s*\overline{s})\right\}=(e,e).$$

If $s*\overline{s}=e=\overline{s}*s$, then $\overline{s}=s^{-1}t$, for some $t\in H \iff s\overline{s}=t$ and

$$(shs^{-1})t=e \iff t^{-1}=shs^{-1} \iff t=sh^{-1}s^{-1},$$

thus $\overline{s}=s^{-1}t=h^{-1}s^{-1} \iff h=(\overline{s}s)^{-1}$. Thus $(e,s)^{-1}=(h,\overline{s})=((\overline{s}s)^{-1},\overline{s})$. If (e,s) and $(h,r)\in H\times G/H$ and $\xi\in\ell^2(H)\otimes\ell^2(G/H)$:

$$\begin{aligned}
\left(\lambda_{H\times G/H}(e,s)\xi\right)(h,r) &= \xi\left((e,s)^{-1}(h',k')\right) \\
&= \xi\left(((\overline{s}s)^{-1},\overline{s})(h,r)\right) \\
&= \xi\left\{(\overline{s}s)^{-1}(\overline{s}h\overline{s}^{-1})(\overline{s}r)(\overline{s}*r)^{-1},(\overline{s}*r)\right\},
\end{aligned}$$

but $(\overline{s}h\overline{s}^{-1})(\overline{s}r)(\overline{s}*r)^{-1}$ is an automorphism of H and $s\in R\longmapsto\overline{s}s\in H$. But, then $\lambda_G(hs)\ell^2(Hr)$ will be isomorphic to $\lambda_H(h)\otimes\lambda_{G/H}(s)$ acting on $\ell^2(H)\otimes\ell^2(G/H)$ via the composition of the map ϕ with the isomorphism. We have the isomorphism $C^*_\lambda(H)\otimes C^*_\lambda(G/H)\cong C^*_\lambda(G)$. $\square$

We already proved $C^*_u(G)^G\cong C^*_\lambda(H)\otimes C^*_\lambda(G/H)$. By using Proposition 3.7, $C^*_u(G)^G\cong C^*_\lambda(H\times G/H)\cong C^*_\lambda(G)$. Therefore, G has IAP. $\square$

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