On the oscillatory behavior of a class of fourth order nonlinear damped delay differential equations with distributed deviating arguments

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Abstract. The present study concerns the oscillation of a class of fourth-order nonlinear damped delay differential equations with distributed deviating arguments. We offer a new description of oscillation of the fourth-order equations in terms of oscillation of a related well studied second-order linear differential equation without damping. Some new oscillatory criteria are obtained by using the generalized Riccati transformation, integral averaging technique and comparison principles. The effectiveness of the obtained criteria is illustrated via example.

Keywords: oscillation, fourth-order, damping term, Riccati transformation, comparison theorem, distributed deviating arguments.

1. Introduction

The purpose of this work, we are concerned with fourth-order nonlinear damped delay differential equations with distributed deviating arguments

$$(E_1) \qquad \begin{aligned} &\big(x_2(\mu)\big(x_1(\mu)\big(u''(\mu)\big)^{\alpha}\big)'\big)' + p(\mu)\Big(u''(\delta(\mu))\Big)^{\alpha} \\ &+ \int_c^d q(\mu,\varrho) f(\mu, u(g(\mu,\varrho)))d\varrho = 0, \end{aligned}$$

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where $\alpha \geq 1$ is a quotient of odd positive integers and $c < d$. Throughout this paper, we use the following assumptions:

$$\begin{cases} x_1, x_2, p, \delta \in C\big(I, [0, \infty)\big) \text{ and } x_1, x_2 > 0, \text{ where } I = [\mu_0, +\infty); \\ q, g \in C[I \times [c, d], [0, \infty)), \delta(\mu) \leq \mu, \\ \lim_{\mu \to +\infty} \delta(\mu) = \infty, \ g(\mu, \varrho) \text{ is a nondecreasing} \\ \text{function for } \varrho \in [c, d] \text{ satisfying } g(\mu, \varrho) \leq \mu \text{ and } \lim_{\mu \to +\infty} g(\mu, \varrho) = \infty; \\ f \in C(\mathbb{R}, \mathbb{R}), \text{ there exists a contacts } k_1 > 0 \text{ such that } f(\mu, u(\mu))/u^\beta \geq k_1. \end{cases}$$

We define the operators,

$$L^{[0]}u = u,\ L^{[1]}u = u',\ L^{[2]}u = x_1\big(\big(L^{[0]}u\big)''\big)^\alpha,\ L^{[3]}u = x_2\big(L^{[2]}u\big)', L^{[4]}u = (L^{[3]}u)'.$$

By a solution to (E_1), we mean a function $u(\mu)$ in $C^2[T_u, \infty)$ for which $L^{[2]}u$, $L^{[4]}u$ is in $C^1[T_u, \infty)$ and (E_1) is satisfied on some interval $[T_u, \infty)$, where $T_u \geq \mu_0$. We consider only solutions $u(\mu)$ for which $\sup\{|u(\mu)| : \mu \geq T\} > 0$ for all $T \geq T_u$. A solution of (E_1) is called oscillatory if it is neither eventually positive nor eventually negative on $[T_u, \infty)$ and otherwise, it is said to be nonoscillatory. The equation itself is termed oscillatory if all its solutions oscillate.

We define

$$\begin{aligned} A_1(\mu_1, \mu) &= \int_{\mu_1}^{\mu} x_1^{-1/\alpha}(s) ds, \\ A_2(\mu_1, \mu) &= \int_{\mu_1}^{\mu} x_2^{-1}(s) ds, \\ A_3(\mu_1, \mu) &= \int_{\mu_1}^{\mu} \Big(\big(x_1(s)\big)^{-1} A_2(\mu_1, s)\Big)^{1/\alpha} ds, \\ A_4(\mu_1, \mu) &= \int_{\mu_1}^{\mu} \int_{\mu_1}^{u} \Big(\big(x_1(s)\big)^{-1} A_2(\mu_1, s)\Big)^{1/\alpha} ds\, du, \end{aligned}$$

for $\mu_0 \leq \mu_1 \leq \mu < \infty$ and assume that

$$A_1(\mu_1, \mu) \to \infty, \quad A_2(\mu_1, \mu) \to \infty \quad \text{as} \quad \mu \to \infty. \tag{1}$$

In mathematical representations of numerous physical, chemical phenomena and biological, fourth-order differential equations are very commonly encountered [1, 3]. Applications involve, for example, problems with elasticity, structural deformation or soil settlement. Questions related to the presence of oscillatory and nonoscillatory solutions play an important role in mechanical and engineering problems [5]. Many authors have extensively studied the problem of the oscillation of fourth (higher) order differential equations, including many techniques for obtaining oscillatory criteria for fourth (higher) order differential equations. Several studies have had very interesting results related to oscillatory properties of solutions of neutral differential equations and damped delay differential equations with/without distributed deviating arguments [4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20].

Dzurina et al. [8] presented oscillation results for a fourth-order equation

$$\big(r_3(\mu)\big(r_2(\mu)(r_1(\mu)y'(\mu))\big)'\big)' + p(\mu)y'(\mu) + q(\mu)y(\tau(\mu)) = 0.$$

More precisely, the existing literature does not provide any criteria for the oscillation of Eq. (E_1). Inspired by the above papers, in this paper, using suitable Riccati type transformation, integral averaging condition, and comparison method, we present some sufficient conditions which insure that any solution of Eq. (E_1) oscillates when the associated second order equation

$$(E_2) \qquad (x_2(\mu)z'(\mu))' + \frac{p(\mu)}{x_1(\delta(\mu))}z(\mu) = 0,$$

is oscillatory or nonoscillatory.

2. Basic lemmas

In this section, we state and prove some Lemmas that are frequently used in the remainder of this paper.

Lemma 2.1 ([9])**.** *Assume that (E_2) is nonoscillatory. If Eq. (E_1) has a nonoscillatory solution $u(\mu)$ on I, $\mu_1 \geq \mu_0$, then there exists a $\mu_2 \in I$ such that $u(\mu)L^{[2]}u(\mu) > 0$ or $u(\mu)L^{[2]}u(\mu) < 0$ for $\mu \geq \mu_2$.*

Lemma 2.2. *If Eq. (E_1) has a nonoscillatory solution $u(\mu)$ which satisfies $u(\mu)L^{[2]}u(\mu) > 0$ in Lemma 2.1 for $\mu \geq \mu_1 \geq \mu_0$. Then,*

$$(2) \qquad L^{[2]}u(\mu) > A_2(\mu_1,\mu)\, L^{[3]}u(\mu), \quad \mu \geq \mu_1,$$

$$(3) \qquad L^{[1]}u(\mu) > A_3(\mu_1,\mu)\left(L^{[3]}u(\mu)\right)^{1/\alpha}, \quad \mu \geq \mu_1,$$

and

$$(4) \qquad u(\mu) > A_4(\mu_1,\mu)\left(L^{[3]}u(\mu)\right)^{1/\alpha}, \quad \mu \geq \mu_1.$$

Proof. If Eq. (E_1) has a non-oscillatory solution u. We assume that there exists a $\mu_1 \geq \mu_0$ such that $u(\mu) > 0$ and $u(g(\mu,\varrho)) > 0$ for $\mu \geq \mu_1$. From Eq. (E_1), we have

$$L^{[4]}u(\mu) = -\Big(\frac{p(\mu)}{x_1(\delta(\mu))}\Big)L^{[2]}u(\delta(\mu)) - k_1\int_c^d q(\mu,\varrho)u^{\beta}(g(\mu,\varrho))d\varrho \leq 0,$$

and $L^{[3]}u(\mu)$ is non increasing on I, we get

$$L^{[2]}u(\mu) \geq \int_{\mu_1}^{\mu}\big(L^{[2]}u(s)\big)'\,ds = \int_{\mu_1}^{\mu}(x_2(s))^{-1}L^{[3]}u(s)\,ds \geq A_2(\mu_1,\mu)\,L^{[3]}u(\mu),$$

this implies that

$$u''(\mu) \geq \Big(L^{[3]}u(\mu)\Big)^{1/\alpha}\Big((x_1(\mu))^{-1}A_2(\mu_1,\mu)\Big)^{1/\alpha}.$$

Now, twice integrating above from μ_1 to μ and using $L^{[3]}u(\mu) \leq 0$, we find

$$u'(\mu) \geq \Big(L^{[3]}u(\mu)\Big)^{1/\alpha}\int_{\mu_1}^{\mu}\Big((x_1(s))^{-1}A_2(\mu_1,s)\Big)^{1/\alpha}ds$$

and

$$u(\mu) \geq \Big(L^{[3]}u(\mu)\Big)^{1/\alpha}\int_{\mu_1}^{\mu}\int_{\mu_1}^{u}\Big((x_1(s))^{-1}A_2(\mu_1,s)\Big)^{1/\alpha}ds\,du \quad \text{for } \mu \leq \mu_1.$$

□

Lemma 2.3 ([11])**.** *Let $\xi \in C^1(I,\mathbb{R}^+)$, $\xi(\mu) \leq \mu$, $\xi'(\mu) \geq 0$ and $G(\mu) \in C(I,\mathbb{R}^+)$ for $\mu \geq \mu_0$. Assume that $y(\mu)$ is a bounded solution of second order delay differential equation*

$$(E_3) \qquad \big(x_2(\mu)\,y'(\mu))' - \Theta(\mu)\,y(\xi(\mu)) = 0.$$

If

$$(5) \qquad \limsup_{\mu\to\infty}\int_{\xi(\mu)}^{\mu}\Theta(s)\,A_2(\xi(\mu),\,\xi(s))\,ds > 1$$

or

$$(6) \qquad \limsup_{\mu\to\infty}\int_{\xi(\mu)}^{\mu}\Big(\big(x_2(\mu)\big)^{-1}\int_{u}^{\mu}\Theta(s)\,ds\Big)du > 1,$$

where $x_2(\mu)$ is as in (E_1). Then the solutions of (E_3) are oscillatory.

3. Oscillation-comparison principle method

In this section, we shall establish some oscillation criteria for Eq. (E_1). For convenience, we denote

$$Q(\mu) = \Big(\frac{p(\mu)}{x_1(\delta(\mu))}\Big)A_2(\mu_1,\delta(\mu)), \quad \psi(\mu) = \exp\Big(\int_{\mu_1}^{\mu}Q(s)ds\Big),$$
$$\widetilde{q}(\mu,\varrho) = \int_{c}^{d}q(\mu,\varrho)\,d\varrho, \quad \Theta^*(\mu) = k_1\,\widetilde{q}(\mu,\varrho)\Big(A_4(\mu_1,g(\mu,d))\Big)^{\beta}.$$

Theorem 3.1. *Assume $\alpha \geq \beta$ and the conditions* (1) *hold, Eq.* (E_2) *is nonoscillatory. Suppose there exists a $\xi \in C^1(I,\mathbb{R})$ such that*

$$g(\mu,\varrho) \leq \xi(\mu) \leq \delta(\mu) \leq \mu, \quad \xi'(\mu) \geq 0 \quad \textit{for } \mu \geq \mu_1,$$

and (5) *or* (6) *holds with*

$$\Theta(\mu) = \ell_* \, k_1 \widetilde{q}(\mu,\varrho) g^{\beta}(\mu,d)\big(A_1(\xi(\mu), g(\mu,d))\big)^{\beta} - \frac{p(\mu)}{x_1(\delta(\mu))} \geq 0, \quad \mu \geq \mu_1,$$

for constant $\ell_* > 0$. *Moreover, suppose that every solution of the first-order delay equation*

$$z'(\mu) + \psi^{1-\frac{\beta}{\alpha}}(g(\mu,d))\,\Theta^*(\mu)\, z^{\frac{\beta}{\alpha}}(g(\mu,d)) = 0. \tag{7}$$

Then every solution of Eq. (E_1) *is oscillatory.*

Proof. Let Eq. (E_1) has a nonoscillatory solution $u(\mu)$. Assume that, there exists a $\mu \geq \mu_1$ such that $u(\mu) > 0$ and $u(g(\mu,\varrho)) > 0$ for some $\mu \geq \mu_0$. From Lemma 2.1, $u(\mu)$ has the conditions either $L^{[2]}u(\mu) > 0$ or $L^{[2]}u(\mu) < 0$ for $\mu \geq \mu_1$.

Assume $u(\mu)$ has the condition $L^{[2]}u(\mu) > 0$, for $\mu \geq \mu_1$, then one can easily see that $L^{[3]}u(\mu) > 0$ for $\mu \geq \mu_1$. We can choose $\mu_2 \geq \mu_1$ such that $g(\mu,\varrho) \geq \mu_1$ for $\mu \geq \mu_2$, $g(\mu,\varrho) \to \infty$ as $\mu \to \infty$ and we have (4),

$$u(g(\mu,d)) > A_4(\mu_1, g(\mu,d))\left(L^{[3]}u(g(\mu,d))\right)^{1/\alpha}, \quad \mu \geq \mu_2. \tag{8}$$

By substituting (2), (8) in Eq. (E_1) and $L^{[3]}u(\mu)$ is decreasing, then

$$\begin{aligned}\Big(L^{[3]}u(\mu)\Big)' + \Big(\frac{p(\mu)}{x_1(\delta(\mu))}\Big)L^{[3]}u(\mu)A_2(\mu_1,\delta(\mu))\\ +k_1\,\widetilde{q}(\mu,\varrho)\Big(A_4(\mu_1, g(\mu,d))\Big)^{\beta}\Big(L^{[3]}u(g(\mu,d))\Big)^{\beta/\alpha} \leq 0.\end{aligned} \tag{9}$$

Take $\phi = L^{[3]}u$, we have

$$\phi'(\mu) + Q(\mu)\phi(\mu) + \Theta^*(\mu)\phi^{\frac{\beta}{\alpha}}(g(\mu,d)) \leq 0 \tag{10}$$

or

$$\Big(\psi(\mu)\,\phi(\mu)\Big)' + \psi(\mu)\Theta^*(\mu)\phi^{\frac{\beta}{\alpha}}(g(\mu,d)) \leq 0, \quad \text{for } \mu \geq \mu_2. \tag{11}$$

Next, setting $z = \psi\,\phi > 0$ and $\psi(g(\mu,d)) \leq \phi(\mu)$, thus we have

$$z'(\mu) + \psi^{1-\frac{\beta}{\alpha}}(g(\mu,d))\Theta^*(\mu)z^{\frac{\beta}{\alpha}}(g(\mu,d)) \leq 0. \tag{12}$$

This means (12) is a positive for this inequality. Also, by [[2], Corollary 2.3.5], it can be seen that (3.1) has a positive solution, a contradiction.

Next, assume $u(\mu)$ has the condition $L^{[2]}u(\mu) < 0$, for $\mu \geq \mu_1$, then one can easily see that $L^{[1]}u(\mu) \geq 0$, $L^{[3]}u(\mu) > 0$ for $\mu \geq \mu_3(\geq \mu_2)$. Using monotonicity of $u'(\mu)$ and mean value property of differentiation there exists a $\theta \in (0,1)$ such that

$$u(\mu) \geq \theta\,\mu\,u'(\mu), \quad \text{for } \mu \geq \mu_3. \tag{13}$$

Set $w(\mu) = L^{[1]}u(\mu)$, then $w'(\mu) = u''(\mu) < 0$. Using (13) in Eq. (E_1) we get

$$\big(x_2(\mu)\big(x_1(\mu)\big[w'(\mu)\big]^{\alpha}\big)'\big)' + p(\mu)(w'(\delta(\mu)))^{\alpha} + k_1\,(\mu\theta)^{\beta}\,\widetilde{q}(\mu,\varrho)w^{\beta}(g(\mu,d)) \leq 0,$$

and so $\big(x_1(\mu)\big[w'(\mu)\big]^{\alpha}\big) < 0$, we have $\big(x_1(\mu)\big[w'(\mu)\big]^{\alpha}\big)' > 0$ for $\mu \geq \mu_3$.

Now, for $v \geq u \geq \mu_3$, we get

$$\begin{aligned} w(u) > w(u) - w(v) &= -\int_u^v -x_1^{-1/\alpha}(\tau)(x_1(\tau)(w'(\tau))^{\alpha})^{1/\alpha}d\tau \\ &\geq x_1^{1/\alpha}(v)(-w'(v)))\left(\int_u^v x_1^{-1/\alpha}(\tau)d\tau\right) \\ &= x_1^{1/\alpha}(v)(-w'(v))A_1(u,v). \end{aligned}$$

Taking $u = \xi(\mu)$ and $v = g(\mu, d)$, we obtain

$$\begin{aligned} w(g(\mu,d)) &> A_1(g(\mu,d),\xi(\mu))\big(x_1^{1/\alpha}(\xi(\mu))(-w'(\xi(\mu)))\big) \\ &= A_1(g(\mu,d),\xi(\mu))\ y(\xi(\mu)), \end{aligned}$$

where $y(\mu) = x_1^{1/\alpha}(\xi(\mu))(-w'(\xi(\mu))\big) > 0$ for $\mu \geq \mu_3$. From Eq. (E_1), we have that $y(\mu)$ is decreasing and $g(\mu,d) \leq \xi(\mu) \leq \delta(\mu) \leq \mu$, we get

$$\begin{aligned} &(x_2(\mu)z'(\mu))' + \frac{p(\mu)}{x_1(\delta(\mu))}z(\delta(\mu)) \\ &\quad \geq k_1\,(\theta g(\mu,d))^{\beta}\,\widetilde{q}(\mu,\varrho)A_1(g(\mu,d),\xi(\mu))z^{\frac{\beta}{\alpha}-1}(\xi(\mu))z(\xi(\mu)). \end{aligned}$$

Since z is decreasing and $\alpha \geq \beta$, there exists a constant ℓ such that $z^{\frac{\beta}{\alpha}-1}(\mu) \geq \ell$ for $\mu \geq \mu_3$. Thus, we obtain

$$(x_2(\mu)z'(\mu))' \geq \Big(\ell\,k_1\,(\theta\,g(\mu,d))^{\beta}\,\widetilde{q}(\mu,\varrho)A_1(g(\mu,d),\xi(\mu)) - \frac{p(\mu)}{x_1(\delta(\mu))}\Big)z(\xi(\mu)).$$

Proceeding the rest of the proof in Lemma (2.3), we arrive at the required conclusion, and so is omitted. □

4. Oscillation-Riccati method

This section deals with some oscillation criteria for Equation Eq. (E_1) by using Ricatti Method.

Theorem 4.1. *Assume $\alpha \geq \beta$ and the conditions* (1) *hold, Eq.* (E_2) *is nonoscillatory. Suppose there exists η, $\xi \in C^1(I,\mathbb{R})$ such that $g(\mu,\varrho) \leq \xi(\mu) \leq \delta(\mu) \leq \mu$, $\xi'(\mu) \geq 0$ and $\eta > 0$ for $\mu \geq \mu_1$ with*

$$\limsup_{\mu\to\infty} \int_{\mu_5}^{\mu}\left(k_1\,\eta(s)\,\widetilde{q}(s,\varrho) - \frac{A^2(s)}{4B(s)}\right)ds = \infty \text{ for all } \mu_1 \in I, \tag{14}$$

where, for $\mu \geq \mu_1$,

$$A(\mu) = \frac{\eta'(\mu)}{\eta(\mu)} - \frac{p(\mu)}{x_1(\delta(\mu))} A_2(\mu_1, \delta(\mu)) \tag{15}$$

and

$$B(\mu) = \frac{\beta\, \ell_2^{\beta-\alpha}\, g'(\mu, d)}{\eta(\mu)} \Big(A_4(\mu_1, g(\mu, d))\Big)^{\beta-1} \Big(A_3(\mu_1, g(\mu, d))\Big)^{1/\alpha}, \tag{16}$$

also (5) *or* (6) *holds with* $\Theta(\mu)$ *as in Theorem 3.1. Then, every solution of Eq.* (E_1) *is oscillatory.*

Proof. Let Eq. (E_1) has a nonoscillatory solution $u(\mu)$. Assume that, there exists a $\mu \geq \mu_1$ such that $u(\mu) > 0$ and $u(g(\mu, \varrho)) > 0$ for some $\mu \geq \mu_0$. From Lemma 2.1, $u(\mu)$ has the conditions either $L^{[2]}u(\mu) > 0$ or $L^{[2]}u(\mu) < 0$ for $\mu \geq \mu_1$. If condition $L^{[2]}u(\mu) < 0$ holds, the proof is follows from Theorem 3.1. Next, if condition $L^{[2]}u(\mu) > 0$ holds. Define

$$\omega(\mu) = \eta(\mu) \frac{L^{[3]}u(\mu)}{u^{\beta}(g(\mu, d))}, \quad \mu \in I, \tag{17}$$

then $\omega(\mu) > 0$ for $\mu \geq \mu_1$. From (4) and $L^{[4]}u(\mu) < 0$, we have

$$\begin{aligned} \omega(\mu) &= \eta(\mu) \frac{L^{[3]}u(\mu)}{u^{\beta}(g(\mu, d))} \leq \eta(\mu) \frac{L^{[3]}u(g(\mu, d))}{u^{\beta}(g(\mu, d))} \\ &\leq \eta(\mu)(A_4(\mu_1, g(\mu, d)))^{-\alpha} u^{\alpha-\beta}(g(\mu, d)), \end{aligned} \tag{18}$$

for $\mu \geq \mu_1$. From (3) and definition $L^{[2]}u(\mu)$, we find

$$\begin{aligned} u'(g(\mu, d)) &= L^{[1]}u(g(\mu, d)) \geq A_3(\mu_1, g(\mu, d))(L^{[3]}u(\delta(\mu)))^{1/\alpha} \\ &\geq A_3(\mu_1, g(\mu, d))(L^{[3]}u(g(\mu, d)))^{1/\alpha}. \end{aligned}$$

Then

$$\begin{aligned} \frac{u'(g(\mu, d))}{u(g(\mu, d))} &\geq \left(\frac{A_3(\mu_1, g(\mu, d))}{\eta(\delta(\mu))}\right)^{1/\alpha} \frac{\eta^{1/\alpha}(\delta(\mu))(L^{[3]}u(\mu))^{1/\alpha}}{u^{\beta/\alpha}(g(\delta(\mu), d))} u^{\beta/\alpha-1}(g(\delta(\mu), d)) \\ &= \left(\frac{A_3(\mu_1, g(\mu, d))}{\eta(\mu)}\right)^{1/\alpha} \omega^{1/\alpha}(\mu) u^{\beta/\alpha-1}(g(\delta(\mu), d)). \end{aligned} \tag{19}$$

Also, since there exists a constant ℓ_1 and $\mu_2 \geq \mu_1$ such that for $L^{[3]}u(\mu) \leq L^{[3]}u(\mu_2) = \ell_1$. Therefore,

$$\begin{aligned} L^{[2]}u(\mu) = L^{[2]}u(\mu_2) + \int_{\mu_2}^{\mu} (L^{[2]}u(s))' ds &\leq L^{[2]}u(\mu_2) + \ell_1 \int_{\mu_2}^{\mu} \frac{ds}{x_2(s)} \\ = L^{[2]}u(\mu_2) + \ell_1 A_2(\mu_2, \mu) &= \left[\frac{L^{[2]}u(\mu_2)}{A_2(\mu_2, \mu)} + \ell_1\right] A_2(\mu_2, \mu) \\ &\leq \left[\frac{L^{[2]}u(\mu_2)}{A_2(\mu_2, \mu_3)} + \ell_1\right] A_2(\mu_2, \mu) = \ell_1^{*} A_2(\mu_2, \mu), \end{aligned} \tag{20}$$

holds for all $\mu \geq \mu_2$, where $\ell_1^* = \ell_1 + \frac{L^{[2]}u(\mu_1)}{A_2(\mu_2,\mu_3)}$, this implies that,

$$u'(\mu) = u'(\mu_3) + \int_{\mu_3}^{\mu} u''(s)ds \leq u'(\mu_3) + \int_{\mu_3}^{\mu} \left(\frac{\ell_1^* A_2(\mu_2, s)}{x_1(s)} \right)^{1/\alpha} ds$$
$$= u(\mu_3) + \left(\ell_1^*\right)^{1/\alpha} A_3(\mu_3, \mu) = \ell_2 \, A_3(\mu_3, \mu),$$

holds for all $\mu \geq \mu_3 (\geq \mu_2)$, where $\ell_2 = \frac{u(\mu_2)}{A_3(\mu_3,\mu_4)} + (\ell_1^*)^{1/\alpha}$. Then

$$u(\mu) = u(\mu_4) + \int_{\mu_4}^{\mu} u'(s)ds \leq u(\mu_4) + \int_{\mu_4}^{\mu} \Big(\ell_2 \, A_3(\mu_3, s)\Big) ds$$
$$= u(\mu_4) + \ell_2 \, A_4(\mu_4, \mu) = \ell_2^* \, A_4(\mu_4, \mu), \tag{21}$$

holds for all $\mu \geq \mu_4 (\geq \mu_3)$,where $\ell_2^* = \frac{u(\mu_4)}{A_4(\mu_4,\mu_1)} + \ell_2$. Further

$$u^{\beta/\alpha - 1}(g(\mu, d)) \geq \left(\ell_2^*\right)^{\beta/\alpha - 1} \big(A_4(\mu_4, g(\mu, d))\big)^{\beta/\alpha - 1}, \quad \mu \geq \mu_4. \tag{22}$$

By using (21) in (18), we obtain

$$\omega(\mu) \leq \left(\ell_2^*\right)^{\alpha - \beta} \eta(\mu) \left(A_4(\mu_1, g(\mu, d))\right)^{-\beta}, \tag{23}$$

and hence

$$\omega^{\frac{1}{\alpha} - 1}(\mu) \leq \left(\ell_2^*\right)^{(\alpha - \beta)(\frac{1}{\alpha} - 1)} \eta^{\frac{1}{\alpha} - 1}(\mu) \left(A_4(\mu_1, g(\mu, d))\right)^{-\beta(\frac{1}{\alpha} - 1)}. \tag{24}$$

Now differentiating (17), we get

$$\omega'(\mu) = \frac{\eta'(\mu)}{\eta(\mu)} \omega(\mu) + \frac{L^{[4]}u(\mu)}{L^{[3]}u(\mu)} \omega(\mu) - \beta g'(\mu, d) \frac{u'(g(\mu, d))}{u(g(\mu, d))} \omega(\mu). \tag{25}$$

Using Eq. (E_1), (2) in (25), we have

$$\begin{aligned} \omega'(\mu) &\leq \Big[\frac{\eta'(\mu)}{\eta(\mu)} - \frac{p(\mu)}{x_1(g(\mu, d))} \, A_2(\mu_4, g(\mu, d))\Big] \omega(\mu) \\ &\quad - k_1 \eta(\mu) \widetilde{q}(\mu, \varrho) - \beta g'(\mu) \frac{u'(g(\mu, d))}{u(g(\mu, d))} \omega(\mu) \\ &\leq A(\mu)\omega(\mu) - k_1 \eta(\mu) \widetilde{q}(\mu, \varrho) - \beta g'(\mu) \frac{u'(g(\mu, d))}{u(g(\mu, d))} \omega(\mu). \end{aligned} \tag{26}$$

By using (19), (22) and (25) in (26), we have

$$\begin{aligned} \omega'(\mu) &\leq A(\mu)\omega(\mu) - k_1 \eta(\mu) \widetilde{q}(\mu, \varrho) \\ &\quad - \frac{\beta \, \ell_2^{\beta - \alpha} \, g'(\mu)}{\eta(\mu)} \Big(A_4(\mu_1, g(\mu, d))\Big)^{\beta - 1} \Big(A_3(\mu_1, g(\mu, d))\Big)^{1/\alpha} \omega^2(\mu) \\ &= A(\mu)\omega(\mu) - k_1 \eta(\mu) \widetilde{q}(\mu, \varrho) + B(\mu)\omega^2(\mu) \qquad (27) \\ &= -k_1 \eta(\mu) \widetilde{q}(\mu, \varrho) + \left[\sqrt{B(\mu)} \, \omega(\mu) - \frac{1}{2} \frac{A(\mu)}{\sqrt{B(\mu)}} \right]^2 + \frac{1}{4} \frac{A^2(\mu)}{B(\mu)} \\ &\leq -k_1 \eta(\mu) \widetilde{q}(\mu, \varrho) + \frac{1}{4} \frac{A^2(\mu)}{B(\mu)}. \qquad (28) \end{aligned}$$

Integrating (28) from $\mu_5(>\mu_4)$ to μ gives

$$\int_{\mu_5}^{\mu}\left(k_1\,\eta(s)\,\widetilde{q}(s,\varrho)-\frac{1}{4}\frac{A^2(s)}{B(s)}\right)ds\le\omega(\mu_5), \tag{29}$$

which contradicts (14). □

Corollary 4.1. *Assume $\alpha\ge\beta$ and the conditions* (1) *hold, Eq.* (E_2) *is nonoscillatory. Suppose there exists η, $\xi\in C^1(I,\mathbb{R})$ such that $g(\mu,\varrho)\le\xi(\mu)\le\delta(\mu)\le\mu$, $\xi'(\mu)\ge 0$ and $\eta>0$ for $\mu\ge\mu_1$ such that the function $A(\mu)\le 0$,*

$$\limsup_{\mu\to\infty}\int_{\mu_5}^{\mu}\Big(\eta(s)\,\widetilde{q}(s,\varrho)\Big)ds=\infty \text{ for all } \mu_1\in I, \tag{30}$$

where $A(\mu)$ is defined in (15)*, also* (5) *or* (6) *holds with $\Theta(\mu)$ as in Theorem 3.1. Then every solution of Eq. (E_1) is oscillatory.*

Next, we examine the oscillation results of solutions of (E_1) by Philos-type. Let $\mathbb{D}_0=\{(\mu,s):a\le s<\mu<+\infty\}$, $\mathbb{D}=\{(\mu,s):a\le s\le\mu<+\infty\}$ the continuous function $H(\mu,s)$, $H:\mathbb{D}\to\mathbb{R}$ belongs to the class function $\mathbb{R}$

(i) $H(\mu,\mu)=0$ for $\mu\ge\mu_0$ and $H(\mu,s)>0$ for $(\mu,s)\in\mathbb{D}_0$,

(ii) H has a continuous and non-positive partial derivative on $\mathbb{D}_0$ with respect to the second variable such that

$$-\frac{\partial H(\mu,s)}{\partial s}=h(\mu,s)[H(\mu,s)]^{1/2},$$

for all $(\mu,s)\in\mathbb{D}_0$.

Theorem 4.2. *Assume $\alpha\ge 1$ and the conditions* (1) *hold, Eq.* (E_2) *is nonoscillatory. Suppose there exists η, $\xi\in C^1(I,\mathbb{R})$ such that $g(\mu,\varrho)\le\xi(\mu)\le\delta(\mu)\le\mu$, $\xi'(\mu)\ge 0$, $\eta>0$ and $H(\mu,s)\in\mathbb{R}$ for $\mu\ge\mu_1$ with*

$$\limsup_{\mu\to\infty}\frac{1}{H(\mu,\mu_5)}\int_{\mu_5}^{\mu}\left(k_1\,\eta(s)\,\widetilde{q}(s,\varrho)H(\mu,s)-\frac{P^2(\mu,s)}{4B(s)}\right)ds=\infty, \tag{31}$$

for all $\mu_1\in I$, where $P(\mu,s)=h(\mu,s)-A(s)\sqrt{H(\mu,s)}$ and $A(\mu)$, $B(\mu)$ are defined in Theorem 4.1, also (5) *or* (6) *holds with $\Theta(\mu)$ as in Theorem 3.1. Then every solution of Eq. (E_1) is oscillatory.*

Proof. Let Eq. (E_1) has a nonoscillatory solution $u(\mu)$. Assume that, there exists a $\mu\ge\mu_1$ such that $u(\mu)>0$ and $u(g(\mu,\varrho))>0$ for some $\mu\ge\mu_0$. Proceeding as in the proof of Theorem 4.1, we obtain the inequality (27), i.e.,

$$\omega'(\mu)\le A(\mu)\omega(\mu)-k_1\eta(\mu)\widetilde{q}(\mu,\varrho)+B(\mu)\omega^2(\mu),$$

and so,

$$\begin{aligned}\int_{\mu_5}^{\mu} H(\mu,s)\eta(s)\widetilde{q}(s,\varrho)ds &\leq \int_{\mu_5}^{\mu} H(\mu,s)[-\omega'(s)+A(s)\omega(s)-B(s)\omega^2(s)]ds\\ &= -H(\mu,s)\Big[\omega(s)\Big]_{\mu_5}^{\mu} + \int_{\mu_5}^{\mu}\Big[\frac{\partial H(\mu,s)}{\partial s}\omega(s)\\ &\quad +H(\mu,s)\Big[A(s)\omega(s)-B(s)\omega^2(s)\Big]\Big]ds\\ &= H(\mu,\mu_5)\omega(\mu_5) - \int_{\mu_5}^{\mu}\Big[\omega^2(s)B(s)H(\mu,s)\\ &\quad +\omega(s)\Big(h(\mu,s)\sqrt{H(\mu,s)}-H(\mu,s)A(s)\Big)\Big]ds\\ &\leq H(\mu,\mu_5)\omega(\mu_5)+\int_{\mu_5}^{\mu}\frac{P^2(\mu,s)}{4B(s)}ds,\end{aligned}$$

which contradicts to (31). The rest of the proof is similar to that of Theorem 4.1 and hence is omitted. □

5. Examples

Below, we present a example to show application of the main results.

Example 5.1. For $\mu \geq 1$, consider fourth order differential equation

$$(32)\quad \big(1/2\mu\big(9e^{-\mu}(\mu)\big(u''(\mu)\big)\big)'\big)' + 36e^{-s/2}u^{(ii)}(\frac{\mu}{2}) + \int_1^2 \frac{\mu}{3}u(\varrho, 36e^{\mu/3})d\varrho = 0.$$

Here, $x_1 = 9e^{-\mu}$, $x_2 = 1/2\mu$, $\alpha = \beta = 1$, $p(\mu) = 36e^{-s/2}$, $q(\mu,\varrho) = \mu/3$ and $\delta(\mu) = \mu/2$, $g(\mu,\varrho) = \mu/3$. Now Pick $\eta(\mu) = 36e^{\mu/3}$, we obtain

$$\begin{aligned} A_1(\mu_1,\mu) &= \int_1^{\mu}(9e^s)^{-1}ds = 9(e^{\mu}-e),\\ A_2(\mu_1,\mu) &= \int_1^{\mu}2s\,ds = \mu^2-1 = (\mu+1)(\mu-1),\\ A_3(\mu_1,\mu/3) &= \int_1^{\mu/3}(9e^s)^{-1}(s^2-1)ds = e^{\mu/3}(\mu-3)^2,\\ \widetilde{q}(s,\varrho) &= \frac{s}{3}\int_1^2 d\varrho = s/3,\end{aligned}$$

$A^2(s) = \frac{(3\mu^2-5)^2}{9}$ and $B(s) = \frac{(s-3)^2}{36}$. Now,

$$\begin{aligned}&\limsup_{\mu\to\infty}\int_2^{\mu}\left(k_1\,\eta(s)\,\widetilde{q}(s,\varrho) - \frac{A^2(s)}{4B(s)}\right)ds\\ &= \limsup_{\mu\to\infty}\int_2^{\mu}\Big(12k_1\,s\,e^{s/3} - \Big(\frac{3s^2-5}{s-3}\Big)^2\Big)ds \to \infty \text{ as } \mu\to\infty,\end{aligned}$$

and all hypotheses of Theorem 4.1 are satisfied, so every solution of (32) is oscillatory.

6. Conclusions

It is clear that the form of problem Eq. (E_1) is more general than all the problems considered in the study. In this paper, using the suitable Riccati type transformation, integral averaging condition, and comparison method, we offer some oscillatory properties which ensure that any solution of Eq. (E_1) oscillates under assumption of $A_1(\mu_1, \mu) \to \infty$, $A_2(\mu_1, \mu) \to \infty$ as $\mu \to \infty$. Also, it would be useful to extend oscillation criteria of Eq. (E_1) under the condition of $A_1(\mu_1, \mu) < \infty$, $A_2(\mu_1, \mu) < \infty$ as $\mu \to \infty$. In addition, we can consider the oscillation of Eq. (E_1) when equation Eq. (E_2) is oscillatory, and we can try to get some oscillation criteria of Eq. (E_1) if $p(t) < 0$ in the future work.

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