Bivariate extension of λ-hybrid type operators

Md. Heshamuddin*
Department of Natural and Applied Sciences
School of Science and Technology
Glocal University
Saharanpur-247121
India
muhammadhishaam1607@gmail.com

Amit Kumar
Department of Mathematics
Hansraj College
Delhi University
New Delhi-110007

Nadeem Rao
Department of Mathematics
University Center for Research and Development
Chandigarh University
Mohali-140413, Punjab
India
nadeemrao1990@gmail.com

Mohd Shadab
Department of Mathematics
School of Basic and Applied Sciences
Lingaya's Vidyapeeth (Deemed to be University)
Faridabad-121002, Haryana
India

Anshul Srivastava
Department of Mathematics
Dr. Akhilesh Das Gupta Institute of Technology and Management
GGSIPU New Delhi-110053
India
anshul_sriv@rediffmail.com

Abstract. In this manuscript, we develop a bi-variate extension of hybrid type operators. We discuss the order of approximation via modulus of continuity, Peetre's K-functional,the rate of convergence, Lipschitz maximal functions and Voronovskaja type result. In addition to this, we investigate global approximation results. In the last section, we study the approximation properties of the operators in Bögel-spaces in terms of mixed-modulus of continuity.

∗. Corresponding author

Keywords: hybrid type operators, modulus of continuity, Peetre's K-functional, Voronovskaja type result, mixed-modulus of continuity, Bögel functions.

1. Introduction

Approximation theory is an important part of mathematical analysis where the main purpose of investigation is to approximate a delicate, difficult and sophisticated function with the help of simple and smooth function. Karl Weierstrass (1885) developed an elegant theorem called as Weierstrass approximation theorem [25] which is widely known and accepted that all types of algebraic polynomial in the category of continuous real valued function on closed interval are dense. Among these, S. N. Bernstein (1912)[4] introduced the polynomials via binomial distribution to give the simplest and easiest proof of this celebrated theorem as follows:

$$B_n(f;x)=\sum_{\nu=0}^{n}p_{n,\nu}(x)f\left(\frac{\nu}{n}\right),n\in\mathbb{N}, \tag{1}$$

where $p_{n,\nu}(x)=\binom{n}{k}x^{\nu}(1-x)^{n-\nu}$ and $f\in[0,1]$. He established that $B_n(f;x)\rightrightarrows f$ for each $f\in C[0,1]$ where $\rightrightarrows$ holds for uniform convergence. Szász [1] generalized the operators defined by (1) on unbounded interval, i.e. on $[0,\infty)$ as

$$S_n(f;x)=e^{-nx}\sum_{\nu=0}^{\infty}\frac{(nx)^{\nu}}{\nu!}f\left(\frac{\nu}{n}\right),\ n\in\mathbb{N}. \tag{2}$$

Several generalizations studied for (2) to yield the convergence properties by these sequences on $[0,\infty)$. Operators (1) and (2) are limited for continuous functions only. Durrmeyer [2] suggested for an integral modification of Bernstein operators (1) on an interval [0,1] to study the approximation properties for Lebsgue integrable functions given by

$$D_n(f;x)=\sum_{\nu=0}^{n}p_{n,\nu}(x)\int_0^1 p_{n,\nu}(t)f(t)dt. \tag{3}$$

With the help of the Bézier bases and shape parameter $\lambda\in[1,-1]$, Cai et. al. [14] obtained a generalization of classical Bernstein operators. In the sequence, Cai [10], Srivastava et al. [23] and Ozger ([18], [19]) constructed Stancu, Shurer and Kantorovich variants of λ-Bernstein operators. Motivated with the idea of λ-Bernstein polynomials, Acu et al. [26] introduced a new family of modified U_m^{ρ} operators and the operator is denoted by $U_{m,\lambda}^{\rho}$. Recently, Rao et al [6], introduced a new sequence of Hybrid type operators as:

$$A_{n,\alpha}^{*}(f;x) = \sum_{k=0}^{\infty}P_{n,k}^{\alpha}(x)\frac{n^{k+\lambda+1}}{\Gamma(k+\lambda+1)}\int_0^{\infty}t^{k+\lambda+1}e^{-nt}dt \tag{4}$$

where

$$P_{n,k}^{\alpha}(x) = \frac{x^{k-1}}{(1+x)^{n+k-1}}\left\{\frac{\alpha x}{1+x}\binom{n+k-1}{k} - (1-\alpha)(1+x)\binom{n+k-3}{k-2}\right.$$
$$\left. + \quad (1-\alpha)x\binom{n+k-1}{k}\right\}$$

with $\binom{n-3}{-2} = \binom{n-2}{-1} = 0$, and the gamma function

$$\Gamma n = \int_0^\infty x^{n-1}e^{-x}dx, \quad \Gamma z = (z-1)\Gamma(z-1) = (z-1)!.$$

2. Construction of bivariate extension of λ-hybrid type operators and their basic estimates

Let $\mathcal{I}^2 = \{(u_1,u_2) : 0 \leq u_1 < \infty,\ 0 \leq u_2 < \infty\}$ and $C\left(\mathcal{I}^2\right)$ be the class of all continuous functions on $\mathcal{I}^2$ equipped with the norm

$$\|g\|_{C(\mathcal{I}^2)} = \sup_{(u_1,u_2)\in\mathcal{I}^2} |g(u_1,u_2)|.$$

Then, for all $f \in C\left(\mathcal{I}^2\right)$ and $n_1, n_2 \in \mathbb{N}$, we construct a new sequences of bi-variate extension of λ-Hybrid type operators as follow:

$$B_{n_1,n_2}^{\alpha}(f;u_1,u_2) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} P_{{}_{1,n_1,k_1}}(u_1)P_{{}_{2,n_2,k_2}}(u_2)$$
$$\cdot\int_0^\infty\int_0^\infty G_{{}_1}^*(u_1)G_{{}_2}^*(u_2)f(t_1,t_2)dt_1dt_2. \tag{5}$$

where

$$P_{i,n,k}^*(u_i) = \frac{{u_i}^{k_i-1}}{(1+u_i)^{n_i+k_\nu-1}}\left\{\frac{\alpha u_i}{1+u_i}\binom{n_i+k_i-1}{k_i}\right.$$
$$\left. -(1-\alpha)(1+u_i)\binom{n_i+k_i-3}{k_i-2} + (1-\alpha)u_1\binom{n_i+k_i}{k_i}\right\}$$

and $\mathcal{G}_i^*(u_i) = \frac{{n_i}^{k_i+\lambda_i+1}}{\Gamma(k_i+\lambda_i+1)}\int_0^\infty {t_i}^{k_i+\lambda_i+1}e^{-n_it_i}dt_i$ for $i = 1,2$.

Lemma 2.1 ([6])**.** *For the operators defined by* (4) *and* $e_i(x) = x^i, i \in \{0,1,2\}$, *test function, we have the following identities:*

$$A_{n,\alpha}^*(e_0;x) = 1,$$
$$A_{n,\alpha}^*(e_1;x) = x + \frac{2}{n}(\alpha-1)x + \frac{\lambda+1}{n},$$
$$A_{n,\alpha}^*(e_2;x) = x^2\left(1+\frac{4\alpha-3}{n}\right) + x\left(\frac{2\lambda+3}{n} + \frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{n^2}\right)$$
$$+ \frac{\lambda^2+3\lambda+2}{n^2},$$

where $n \in \mathbb{N}$, and $\alpha \in [-1,1]$.

Lemma 2.2 ([6])**.** *Let $\eta_j(x) = (t-x)^j$, $j \in \{0,1,2\}$ be the central moments. Then for the operator $A^*_{n,\alpha}(.;.)$, given by (4), we have the following equalities:*

$$\begin{aligned} A^*_{n,\alpha}(\eta_0;x) &= 1, \\ A^*_{n,\alpha}(\eta_1;x) &= \frac{2(\lambda-1)x}{n} + \frac{\lambda+1}{n}, \\ A^*_{n,\alpha}(\eta_2;x) &= O\left(\frac{1}{n}\right)\left(x^2+x+1\right). \end{aligned}$$

Lemma 2.3. *Let $e_{i,j} = {u_1}^i {u_2}^j$. Then, for the operator $B^{\alpha}_{n_1,n_2}(.;.)$, we have*

$$\begin{aligned} &B^{\alpha}_{n_1,n_2}(e_{0,0};u_1,u_2) = 1, \\ &B^{\alpha}_{n_1,n_2}(e_{1,0};u_1,u_2) = u_1 + \frac{2}{n_1}(\alpha-1)u_1 + \frac{\lambda+1}{n_1}, \\ &B^{\alpha}_{n_1,n_2}(e_{0,1};u_1,u_2) = u_2 + \frac{2}{n_2}(\alpha-1)u_2 + \frac{\lambda+1}{n_2}, \\ &B^{\alpha}_{n_1,n_2}(e_{1,1};u_1,u_2) = \left(u_1\frac{2}{n_1}(\alpha-1)u_1 + \frac{\lambda+1}{n_1}\right)\left(u_2 + \frac{2}{n_2}(\alpha-1)u_2 + \frac{\lambda+1}{n_2}\right), \\ &B^{\alpha}_{n_1,n_2}(e_{2,0};u_1,u_2) = u_1^2\left(1+\frac{4\alpha-3}{n_1}\right)+u_1\left(\frac{2\lambda+3}{n_1}+\frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{n_1^2}\right) \\ &\qquad + \frac{\lambda^2+3\lambda+2}{n_1^2}, \\ &B^{\alpha}_{n_1,n_2}(e_{0,2};u_1,u_2) = u_2^2\left(1+\frac{4\alpha-3}{n_2}\right)+u_2\left(\frac{2\lambda+3}{n_2}+\frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{n_2^2}\right) \\ &\qquad + \frac{\lambda^2+3\lambda+2}{n_2^2}. \end{aligned}$$

Proof. In the light of lemma (2.1) and linearly property, we have

$$\begin{aligned} B^{\alpha}_{n_1,n_2}(e_{0,0};u_1,u_2) &= B^{\alpha}_{n_1,n_2}(e_0;u_1,u_2)B^{\alpha}_{n_1,n_2}(e_0;u_1,u_2), \\ B^{\alpha}_{n_1,n_2}(e_{1,0};u_1,u_2) &= B^{\alpha}_{n_1,n_2}(e_1;u_1,u_2)B^{\alpha}_{n_1,n_2}(e_0;u_1,u_2), \\ B^{\alpha}_{n_1,n_2}(e_{0,1};u_1,u_2) &= B^{\alpha}_{n_1,n_2}(e_0;u_1,u_2)B^{\alpha}_{n_1,n_2}(e_1;u_1,u_2), \\ B^{\alpha}_{n_1,n_2}(e_{1,1};u_1,u_2) &= B^{\alpha}_{n_1,n_2}(e_1;u_1,u_2)B^{\alpha}_{n_1,n_2}(e_1;u_1,u_2), \\ B^{\alpha}_{n_1,n_2}(e_{2,0};u_1,u_2) &= B^{\alpha}_{n_1,n_2}(e_2;u_1,u_2)B^{\alpha}_{n_1,n_2}(e_0;u_1,u_2), \\ B^{\alpha}_{n_1,n_2}(e_{0,2};u_1,u_2) &= B^{\alpha}_{n_1,n_2}(e_0;u_1,u_2)B^{\alpha}_{n_1,n_2}(e_2;u_1,u_2), \end{aligned}$$

which proves Lemma (2.3). □

Lemma 2.4. *Let $\Psi^{u_1,u_2}_{i,j}(t,s) = \eta_{i,j}(t,s) = (t-u_1)^i(s-u_2)^j$, $\quad i,j \in \{0,1,2\}$ be the central moments. Then from the operators $B^{\alpha}_{n_1,n_2}(.;.)$ defined by (5) satisfies* *the following identities*

$$\begin{aligned}
B^{\alpha}_{n_1,n_2}(\eta_{0,0};u_1,u_2) &= 1,\\
B^{\alpha}_{n_1,n_2}(\eta_{1,0};u_1,u_2) &= \frac{2(\lambda-1)u_1}{n_1}+\frac{\lambda+1}{n_1},\\
B^{\alpha}_{n_1,n_2}(\eta_{0,1};u_1,u_2) &= \frac{2(\lambda-1)u_1}{n_2}+\frac{\lambda+1}{n_2},\\
B^{\alpha}_{n_1,n_2}(\eta_{1,1};u_1,u_2) &= \frac{2(\lambda-1)u_1}{n_1}+\frac{\lambda+1}{n_1}\frac{2(\lambda-1)u_1}{n_2}+\frac{\lambda+1}{n_2},\\
B^{\alpha}_{n_1,n_2}(\eta_{2,0};u_1,u_2) &= O\left(\frac{1}{n}\right)\left(u_1^2+u_1+1\right),\\
B^{\alpha}_{n_1,n_2}(\eta_{0,2};u_1,u_2) &= O\left(\frac{1}{n}\right)\left(u_2^2+u_2+1\right).
\end{aligned}$$

Proof. Using lemma (2.2) and linearly property, we have

$$\begin{aligned}
B^{\alpha}_{n_1,n_2}(\eta_{0,0};u_1,u_2) &= B^{\alpha}_{n_1,n_2}(\eta_0;u_1,u_2)B^{\alpha}_{n_1,n_2}(\eta_0;u_1,u_2),\\
B^{\alpha}_{n_1,n_2}(\eta_{1,0};u_1,u_2) &= B^{\alpha}_{n_1,n_2}(\eta_1;u_1,u_2)B^{\alpha}_{n_1,n_2}(\eta_0;u_1,u_2),\\
B^{\alpha}_{n_1,n_2}(\eta_{0,1};u_1,u_2) &= B^{\alpha}_{n_1,n_2}(\eta_0;u_1,u_2)B^{\alpha}_{n_1,n_2}(\eta_1;u_1,u_2),\\
B^{\alpha}_{n_1,n_2}(\eta_{1,1};u_1,u_2) &= B^{\alpha}_{n_1,n_2}(\eta_1;u_1,u_2)B^{\alpha}_{n_1,n_2}(\eta_1;u_1,u_2),\\
B^{\alpha}_{n_1,n_2}(\eta_{2,0};u_1,u_2) &= B^{\alpha}_{n_1,n_2}(\eta_2;u_1,u_2)B^{\alpha}_{n_1,n_2}(\eta_0;u_1,u_2),\\
B^{\alpha}_{n_1,n_2}(\eta_{0,2};u_1,u_2) &= B^{\alpha}_{n_1,n_2}(\eta_0;u_1,u_2)B^{\alpha}_{n_1,n_2}(\eta_2;u_1,u_2),
\end{aligned}$$

which proves Lemma 2.4. □

Lemma 2.5. *For all $u_1, u_2 \in \mathcal{I}^2$ and sufficiently large $n_1, n_2 \in \mathbb{N}$ the operators $H^*_{n_1,n_2}(.\ ;\ .)$ satisfy following*

(1) $B^{\alpha}_{n_1,n_2}(\Psi^{2,0}_{u_1,u_2};u_1,u_2) = O\left(\frac{1}{n_1}\right)(u_1+1)^2 \le C_1(u_1+1)^2$ *as* $n_1, n_2 \to \infty$,

(2) $B^{\alpha}_{n_1,n_2}(\Psi^{0,2}_{u_1,u_2};u_1,u_2) = O\left(\frac{1}{n_2}\right)(u_2+1)^2 \le C_2(u_2+1)^2$ *as* $n_1, n_2 \to \infty$,

(3) $B^{\alpha}_{n_1,n_2}(\Psi^{4,0}_{u_1,u_2};u_1,u_2) = O\left(\frac{1}{n_1^2}\right)(u_1+1)^4 \le C_3(u_1+1)^4$ *as* $n_1, n_2 \to \infty$,

(4) $B^{\alpha}_{n_1,n_2}(\Psi^{0,4}_{u_1,u_2};u_1,u_2) = O\left(\frac{1}{n_2^2}\right)(u_2+1)^4 \le C_4(u_2+1)^4$ *as* $n_1, n_2 \to \infty$.

3. Some approximation results in weighted space and their degree of convergence

Let φ be weight function such that $\varphi(u_1,u_2) = 1 + u_1^2 + u_2^2$ and satisfying $B_{\varphi}\left(\mathcal{I}^2\right) = \{g :\mid g(u_1,u_2) \mid\le C_g\varphi(u_1,u_2),\quad C_g > 0\}$, where $B_{\varphi}\left(\mathcal{I}^2\right)$ is the set of all bounded function on $\mathcal{I}^2 = [0,\infty)\times[0,\infty)$. Suppose $C^{(m)}\left(\mathcal{I}^2\right)$ be the m-times continuously differentiable functions defined on $\mathcal{I}^2 = \{(u_1,u_2) \in \mathcal{I}^2 : u_1, u_2 \in [0,\infty)\}$. The equipped norm on B_φ defined by $\| g \|_\varphi = \sup_{u_1,u_2 \in \mathcal{I}^2} \frac{|g(u_1,u_2)|}{\varphi(u_1,u_2)}$. Moreover, we have classified here some classes of function as follows:

$$C_\varphi^m\left(\mathcal{I}^2\right) = \left\{ g : g \in C_\varphi\left(\mathcal{I}^2\right); \quad \text{such that} \lim_{(u_1,u_2)\to\infty} \frac{g(u_1,u_2)}{\varphi(u_1,u_2)} = k_g < \infty \right\},$$

$$C_\varphi^0\left(\mathcal{I}^2\right) = \left\{ f : f \in C_\varphi^m\left(\mathcal{I}^2\right); \quad \text{such that} \lim_{(u_1,u_2)\to\infty} \frac{g(u_1,u_2)}{\varphi(u_1,u_2)} = 0 \right\},$$

$$C_\varphi\left(\mathcal{I}^2\right) = \{g : g \in B_\varphi \cap C_\varphi\left(\mathcal{I}^2\right)\}.$$

Suppose $\omega_\varphi(g;\delta_1,\delta_2)$ is the weighted modulus of continuity for all $g \in C_\varphi^0\left(\mathcal{I}^2\right)$ and $\delta_1, \delta_2 > 0$, defined by

$$\omega_\varphi(g;\delta_1,\delta_2) = \sup_{(u_1,u_2)\in[0,1]} \sup_{0\le|\theta_1|\le\delta_1, 0\le|\theta_2|\le\delta_2} \frac{| g(u_1+\theta_1, u_2+\theta_2) - g(u_1,u_2) |}{\varphi(u_1,u_2)\varphi(\theta_1,\theta_2)}. \tag{6}$$

For any $\eta_1, \eta_2 > 0$ one has

$$\omega_\varphi(g;\eta_1\delta_1,\eta_2\delta_2) \le 4(1+\eta_1)(1+\eta_2)(1+\delta_1^2)(1+\delta_2^2)\omega_\varphi(g;\delta_1,\delta_2),$$

$$| g(t,s) - g(u_1,u_2) | \le \varphi(u_1,u_2)\varphi\left(| t-u_1 |, | s-u_2 |\right)\omega_\varphi\left(g; | t-u_1 |, | s-u_2 |\right)$$

$$\le (1+u_1^2+u_2^2)(1+(t-u_1)^2)(1+(s-u_2)^2)\omega_\varphi\left(g; | t-u_1 |, | s-u_2 |\right).$$

Theorem 3.1. *Let $g \in C_\varphi^0\left(\mathcal{I}^2\right)$, then for sufficiently large $n_1, n_2 \in \mathbb{N}$ operator $B_{n_1,n_2}^\alpha(.;.)$ satisfying the inequality*

$$\frac{| B_{n_1,n_2}^\alpha(g;u_1,u_2) - g(u_1,u_2) |}{(1+u_1^2+u_2^2)}$$

$$\le \Psi_{u_1,u_2}\left(1 + O\left(n_1^{-1}\right)\right)\left(1 + O\left(n_2^{-1}\right)\right)\omega_\varphi\left(g; O\left(n_1^{-\frac{1}{2}}\right), \left(n_2^{-\frac{1}{2}}\right)\right),$$

where $\Psi_{u_1,u_2} = (1+(u_1+1)+C_1(u_1+1)^2+\sqrt{C_3}(u_1+1)^3)(1+(u_2+1)+C_2(u_2+1)^2+\sqrt{C_4}(u_2+1)^3)$ and $C_1, C_2, C_3, C_4 > 0$.

Proof. For all $\delta_{n_1}, \delta_{n_2} > 0$ we have

$$| g(t,s) - g(u_1,u_2) | \le 4(1+u_1^2+u_2^2)\left(1+(t-u_1)^2\right)\left(1+(s-u_2)^2\right)$$

$$\times \left(1 + \frac{| t-u_1 |}{\delta_{n_1}}\right)\left(1 + \frac{| s-u_2 |}{\delta_{n_2}}\right)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_\varphi\left(g;\delta_{n_1},\delta_{n_2}\right)$$

$$= 4(1+u_1^2+u_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)$$

$$\times \left(1 + \frac{| t-u_1 |}{\delta_{n_1}} + (t-u_1)^2 + \frac{1}{\delta_{n_1}} \, | t-u_1 | \, (t-u_1)^2\right)$$

$$\times \left(1 + \frac{| s-u_2 |}{\delta_{n_2}} + (s-u_2)^2 + \frac{| s-u_2 |}{\delta_{n_2}}(s-u_2)^2\right)\omega_\varphi\left(g;\delta_{n_1},\delta_{n_2}\right).$$

Applying $B^{\alpha}_{n_1,n_2}(.;.)$ both the sides and then using Cauchy-Schwarz inequality,

$$\begin{aligned}
&| B^{\alpha}_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2) | \leq B^{\alpha}_{n_1,n_2}\left(| g(.,.)-g(u_1,u_2) |;u_1,u_2\right) 4(1+u_1^2+u_2^2)\\
&\times B^{\alpha}_{n_1,n_2}\left(1+\frac{| t-u_1 |}{\delta_{n_1}}+(t-u_1)^2+\frac{1}{\delta_{n_1}} | t-u_1 | (t-u_1)^2;u_1,u_2\right)\\
&\times B^{\alpha}_{n_1,n_2}\left(1+\frac{| s-u_2 |}{\delta_{n_2}}+(s-u_2)^2+\frac{| s-u_2 |}{\delta_{n_2}}(s-u_2)^2;u_1,u_2\right)\\
&\times (1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_{\varphi}(g;\delta_{n_1},\delta_{n_2})\\
&= 4(1+u_1^2+u_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_{\varphi}(g;\delta_{n_1},\delta_{n_2})\\
&\times \Bigg(1+\frac{1}{\delta_{n_1}}B^{\alpha}_{n_1,n_2}(| t-u_1 |;u_1,u_2)+B^{\alpha}_{n_1,n_2}((t-u_1)^2;u_1,u_2)\\
&+\frac{1}{\delta_{n_1}}B^{\alpha}_{n_1,n_2}(| t-u_1 | (t-u_1)^2;u_1,u_2\Bigg)\\
&\times \Bigg(1+\frac{1}{\delta_{n_2}}B^{\alpha}_{n_1,n_2}(| s-u_2 |;u_1,u_2)+B^{\alpha}_{n_1,n_2}((s-u_2)^2;u_1,u_2)\\
&+\frac{1}{\delta_{n_2}}B^{\alpha}_{n_1,n_2}(| s-u_2 | (s-u_2)^2;u_1,u_2\Bigg);
\end{aligned}$$

$$\begin{aligned}
&| B^{\alpha}_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2) | \leq 4(1+u_1^2+u_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_{\varphi}(g;\delta_{n_1},\delta_{n_2})\\
&\times \Bigg[1+\frac{1}{\delta_{n_1}}\sqrt{B^{\alpha}_{n_1,n_2}((t-u_1)^2;u_1,u_2)}+B^{\alpha}_{n_1,n_2}((t-u_1)^2;u_1,u_2)\\
&+\frac{1}{\delta_{n_1}}\sqrt{B^{\alpha}_{n_1,n_2}((t-u_1)^2;u_1,u_2)}\sqrt{B^{\alpha}_{n_1,n_2}((t-u_1)^4;u_1,u_2)}\Bigg]\\
&\times \Bigg[1+\frac{1}{\delta_{n_2}}\sqrt{B^{\alpha}_{n_1,n_2}((s-u_2)^2;u_1,u_2)}+B^{\alpha}_{n_1,n_2}((s-u_2)^2;u_1,u_2)\\
&+\frac{1}{\delta_{n_2}}\sqrt{B^{\alpha}_{n_1,n_2}((s-u_2)^2;u_1,u_2)}\sqrt{B^{\alpha}_{n_1,n_2}((s-u_2)^4;u_1,u_2)}\Bigg].
\end{aligned}$$

In view of Lemma 2.5 and choose $\delta_{n_1}=O(n_1^{-\frac{1}{2}})$ and $\delta_{n_2}=O(n_2^{-\frac{1}{2}})$, then

$$\begin{aligned}
&| B^{\alpha}_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2) | \leq 4(1+u_1^2+u_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_{\varphi}(g;\delta_{n_1},\delta_{n_2})\\
&\times \Bigg[1+\frac{1}{\delta_{n_1}}\sqrt{O\left(\frac{1}{n_1}\right)(u_1+1)^2}+O\left(\frac{1}{n_1}\right)(u_1+1)^2\\
&+\frac{1}{\delta_{n_1}}\sqrt{O\left(\frac{1}{n_1}\right)(u_1+1)^2}\sqrt{O\left(\frac{1}{n_1}\right)(u_1+1)^4}\Bigg]\\
&\times \Bigg[1+\frac{1}{\delta_{n_2}}\sqrt{O\left(\frac{1}{n_2}\right)(u_2+1)^2}+O\left(\frac{1}{n_2}\right)(u_2+1)^2\\
&+\frac{1}{\delta_{n_2}}\sqrt{O\left(\frac{1}{n_2}\right)(u_2+1)^2}\sqrt{O\left(\frac{1}{n_2}\right)(u_2+1)^4}\Bigg]
\end{aligned}$$

$$\mid B^{\alpha}_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2)\mid\leq 4(1+u_1^2+u_2^2)(1+\delta_{n_1}^2)(1+\delta_{n_2}^2)\omega_{\varphi}(g;\delta_{n_1},\delta_{n_2})$$
$$\times\left[1+(u_1+1)+C_1(u_1+1)^2+\sqrt{C_2}(u_1+1)^3\right]\left[1+(u_2+1)\right.$$
$$\left.+C_3(u_2+1)^2+\sqrt{C_4}(u_2+1)^3\right].$$

Which completes the proof. □

Lemma 3.1 ([38, 39])**.** *For the positive sequence of operators* $\{L_{n_1,n_2}\}_{n_1,n_2\geq 1}$*, which acting* $C_{\varphi}\to B_{\varphi}$ *defined earlier then there exists some positive* K *such that*

$$\parallel L_{n_1,n_2}(\varphi;u_1,u_2)\parallel_{\varphi}\leq K.$$

Theorem 3.2 ([38, 39])**.** *For the positive sequence of operators* $\{L_{n_1,n_2}\}_{n_1,n_2\geq 1}$ *acting* $C_{\varphi}\to B_{\varphi}$ *defined earlier satisfying the following conditions*

$$\begin{aligned}
(1)\quad \lim_{n_1,n_2\to\infty}\parallel L_{n_1,n_2}(1;u_1,u_2)-1\parallel_{\varphi} &= 0,\\
(2)\quad \lim_{n_1,n_2\to\infty}\parallel L_{n_1,n_2}(t;u_1,u_2)-u_1\parallel_{\varphi} &= 0,\\
(3)\quad \lim_{n_1,n_2\to\infty}\parallel L_{n_1,n_2}(s;u_1,u_2)-u_2\parallel_{\varphi} &= 0,\\
(4)\quad \lim_{n_1,n_2\to\infty}\parallel L_{n_1,n_2}((t^2+s^2);u_1,u_2)-(u_1^2+u_2^2)\parallel_{\varphi} &= 0.
\end{aligned}$$

Then, for all $g\in C^0_{\varphi}$*,*

$$\lim_{n_1,n_2\to\infty}\parallel L_{n_1,n_2}g-g\parallel_{\varphi}=0$$

and there exists another function $f\in C_{\varphi}\setminus C^0_{\varphi}$*, such that*

$$\lim_{n_1,n_2\to\infty}\parallel L_{n_1,n_2}f-f\parallel_{\varphi}\geq 1.$$

Theorem 3.3. *If* $g\in C^0_{\varphi}\left(\mathcal{I}^2\right)$ *then, we have*

$$\lim_{n_1,n_2\to\infty}\parallel B^{\alpha}_{n_1,n_2}(g)-g\parallel_{\varphi}=0.$$

Proof.

$$\parallel B^{\alpha}_{n_1,n_2}(\varphi;u_1,u_2)\parallel_{\varphi}=\sup_{(u_1,u_2)\in\mathcal{I}^2}\frac{\mid B^{\alpha}_{n_1,n_2}(1+u_1^2+u_2^2;u_1,u_2)\mid}{1+u_1^2+u_2^2}$$
$$=1+\sup_{(u_1,u_2)\in\mathcal{I}^2}\left[\frac{1}{1+u_1^2+u_2^2}\left|\left(1+B^{\alpha}_{n_1,n_2}(u_1^2;u_1,u_2)+B^{\alpha}_{n_1,n_2}(u_2^2;u_1,u_2)\right)\right|\right]$$

$$\begin{aligned}\| B^{\alpha}_{n_1,n_2}(\varphi;u_1,u_2)\|_{\varphi} &= 1+\left|1+\frac{4\alpha-3}{n_1}\right|\sup_{(u_1,u_2)\in\mathcal{I}^2}\frac{u_1^2}{1+u_1^2+u_2^2}\\ &+\left|\frac{2\lambda+3}{n_1}+\frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{n_1^2}\right|\sup_{(u_1,u_2)\in\mathcal{I}^2}\frac{u_1}{1+u_1^2+u_2^2}\\ &+\left|\frac{\lambda^2+3\lambda+2}{n_1^2}\right|\sup_{(u_1,u_2)\in\mathcal{I}^2}\frac{1}{1+u_1^2+u_2^2}\\ &+\left|1+\frac{4\alpha-3}{n_2}\right|\sup_{(u_1,u_2)\in\mathcal{I}^2}\frac{u_2^2}{1+u_1^2+u_2^2}\\ &+\left|\frac{2\lambda+3}{n_2}+\frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{n_2^2}\right|\sup_{(u_1,u_2)\in\mathcal{I}^2}\frac{u_2}{1+u_1^2+u_2^2}\\ &+\left|\frac{\lambda^2+3\lambda+2}{n_2^2}\right|\sup_{(u_1,u_2)\in\mathcal{I}^2}\frac{1}{1+u_1^2+u_2^2},\end{aligned}$$

$$\begin{aligned}\| B^{\alpha}_{n_1,n_2}(\varphi;u_1,u_2)\|_{\varphi} &\le 1+\left|1+\frac{4\alpha-3}{n_1}\right|+\left|\frac{2\lambda+3}{n_1}+\frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{n_1^2}\right|\\ &+\left|\frac{\lambda^2+3\lambda+2}{n_1^2}\right|+\left|1+\frac{4\alpha-3}{n_2}\right|\\ &+\left|\frac{2\lambda+3}{n_2}+\frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{n_2^2}\right|+\left|\frac{\lambda^2+3\lambda+2}{n_2^2}\right|.\end{aligned}$$

Now, for all $n_1, n_2 \in \mathbb{N}\setminus\{1,2\}$, there exists a positive constant K such that

$$\| B^{\alpha}_{n_1,n_2}(\varphi;u_1,u_2)\|_{\varphi}\le K.$$

Therefore, in order to prove Theorem 3.3 it is sufficient from the Lemma 3.1 and Theorem 3.2. Thus we arrive at the prove of Theorem 3.3. □

For any $g\in C(\mathcal{I}^2)$ and $\delta>0$ modulus of continuity of order second is given by

$$\omega\left(g;\delta_{n_1},\delta_{n_2}\right)=\sup\{\mid g(t,s)-g(u_1,u_2)\mid : (t,s),(u_1,u_2)\in\mathcal{I}^2\}$$

with $\mid t-u_1\mid\le\delta_{n_1}$, $\mid s-u_2\mid\le\delta_{n_2}$ with the partial modulus of continuity defined as:

$$\omega_1\left(g;\delta\right)=\sup_{0\le u_2\le 1}\ \sup_{|x_1-x_2|\le\delta}\{\mid g(x_1,u_2)-g(x_2,u_2)\mid\},$$

$$\omega_2\left(g;\delta\right)=\sup_{0\le u_1\le 1}\ \sup_{|u_1-u_2|\le\delta}\{\mid g(u_1,u_1)-g(u_1,u_2)\mid\}.$$

Theorem 3.4. *For any $g\in C(\mathcal{I}^2)$, we have*

$$\mid B^{\alpha}_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2)\mid\le 2\Big(\omega_1(g;\delta_{u_1,n_1})+\omega_2(g;\delta_{n_2,u_2})\Big).$$

Proof. In order to give the prove of Theorem 3.4, in general we use well-known Cauchy-Schwarz inequality. Thus, we see that

$$\begin{aligned}
&\mid B^{\alpha}_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2)\mid \leq B^{\alpha}_{n_1,n_2}\left(\mid g(t,s)-g(u_1,u_2)\mid ;u_1,u_2\right)\\
&\leq B^{\alpha}_{n_1,n_2}\left(\mid g(t,s)-g(u_1,s)\mid ;u_1,u_2\right)\\
&+B^{\alpha}_{n_1,n_2}\left(\mid g(u_1,s)-g(u_1,u_2)\mid ;u_1,u_2\right)\\
&\leq B^{\alpha}_{n_1,n_2}\left(\omega_1(g;\mid t-u_1\mid);u_1,u_2\right)+B^{\alpha}_{n_1,n_2}\left(\omega_2(g;\mid s-u_2\mid);u_1,u_2\right)\\
&\leq \omega_1(g;\delta_{n_1})\left(1+\delta_{n_1}^{-1}B^{\alpha}_{n_1,n_2}(\mid t-u_1\mid;u_1,u_2)\right)\\
&+\omega_2(g;\delta_{n_2})\left(1+\delta_{n_2}^{-1}B^{\alpha}_{n_1,n_2}(\mid s-u_2\mid;u_1,u_2)\right)\\
&\leq \omega_1(g;\delta_{n_1})\left(1+\frac{1}{\delta_{n_1}}\sqrt{B^{\alpha}_{n_1,n_2}((t-u_1)^2;u_1,u_2)}\right)\\
&+\omega_2(g;\delta_{n_2})\left(1+\frac{1}{\delta_{n_2}}\sqrt{B^{\alpha}_{n_1,n_2}((s-u_2)^2;u_1,u_2)}\right).
\end{aligned}$$

If we choose $\delta^2_{n_1}=\delta^2_{n_1,u_1}=B^{\alpha}_{n_1,n_2}((t-u_1)^2;u_1,u_2)$ and $\delta^2_{n_2}=\delta^2_{n_2,u_2}=B^{\alpha}_{n_1,n_2}((s-u_2)^2;u_1,u_2)$, then we easily to reach our desired results. □

Here, we find convergence in terms of the Lipschitz class for bivariate function. For $M>0$ and $\rho_1,\rho_2\in[0,1]$, Lipschitz maximal function space on $E\times E\subset\mathcal{I}^2$ defined by

$$\begin{aligned}
\mathcal{L}_{\rho_1,\rho_2}(E\times E) &= \Big\{g:\sup(1+t)^{\rho_1}(1+s)^{\rho_2}\left(g_{\rho_1,\rho_2}(t,s)-g_{\rho_1,\rho_2}(u_1,u_2)\right)\\
&\leq M\frac{1}{(1+u_1)^{\rho_1}}\frac{1}{(1+u_2)^{\rho_2}}\Big\},
\end{aligned}$$

where g is continuous and bounded on $\mathcal{I}^2$, and

$$g_{\rho_1,\rho_2}(t,s)-g_{\rho_1,\rho_2}(u_1,u_2)=\frac{\mid g(t,s)-g(u_1,u_2)\mid}{\mid t-u_1\mid^{\rho_1}\mid s-u_2\mid^{\rho_2}};\quad (t,s),(u_1,u_2)\in\mathcal{I}^2. \tag{7}$$

Theorem 3.5. *Let $g\in\mathcal{L}_{\rho_1,\rho_2}(E\times E)$, then for any $\rho_1,\rho_2\in[0,1]$, there exists $M>0$ such that*

$$\begin{aligned}
&\mid B^{\alpha}_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2)\mid\\
&\leq M\Bigg\{\left((d(u_1,E))^{\rho_1}+\left(\delta^2_{n_1,u_1}\right)^{\frac{\rho_1}{2}}\right)\left((d(u_2,E))^{\rho_2}+\left(\delta^2_{n_2,u_2}\right)^{\frac{\rho_2}{2}}\right)\\
&+(d(u_1,E))^{\rho_1}(d(u_2,E))^{\rho_2}\Bigg\},
\end{aligned}$$

where δ_{n_1,u_1} and δ_{n_2,u_2} defined by Theorem 3.4.

Proof. Take $\mid u_1-x_0\mid=d(u_1,E)$ and $\mid u_2-y_0\mid=d(u_2,E)$. For any $(u_1,u_2)\in\mathcal{I}^2$, and $(x_0,y_0)\in E\times E$ we let $d(u_1,E)=\inf\{\mid u_1-u_2\mid : u_2\in E\}$. Thus, we can write here

$$\mid g(t,s)-g(u_1,u_2)\mid\leq M\mid g(t,s)-g(x_0,y_0)\mid+\mid g(x_0,y_0)-g(u_1,u_2)\mid. \tag{8}$$

Apply $B^{\alpha}_{n_1,n_2}$, we obtain

$$\begin{aligned}
&\mid B^{\alpha}_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2)\mid\leq B^{\alpha}_{n_1,n_2}(\mid g(u_1,u_2)-g(x_0,y_0)\mid\\
&+\mid g(x_0,y_0)-g(u_1,u_2)\mid)\\
&\leq MB^{\alpha}_{n_1,n_2}(\mid t-x_0\mid^{\rho_1}\mid s-y_0\mid^{\rho_2};u_1,u_2)\\
&+M\mid u_1-x_0\mid^{\rho_1}\mid u_2-y_0\mid^{\rho_2}.
\end{aligned}$$

For all $A,B\geq 0$ and $\rho\in[0,1]$ we know inequality $(A+B)^{\rho}\leq A^{\rho}+B^{\rho}$, thus

$$\mid t-x_0\mid^{\rho_1}\leq\mid t-u_1\mid^{\rho_1}+\mid u_1-x_0\mid^{\rho_1},$$

$$\mid s-y_0\mid^{\rho_1}\leq\mid s-u_2\mid^{\rho_2}+\mid u_2-y_0\mid^{\rho_2}.$$

Therefore,

$$\begin{aligned}
\mid B^{\alpha}_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2)\mid &\leq MB^{\alpha}_{n_1,n_2}(\mid t-u_1\mid^{\rho_1}\mid s-u_2\mid^{\rho_2};u_1,u_2)\\
&+M\mid u_1-x_0\mid^{\rho_1}B^{\alpha}_{n_1,n_2}(\mid s-u_2\mid^{\rho_2};u_1,u_2)\\
&+M\mid u_2-y_0\mid^{\rho_2}B^{\alpha}_{n_1,n_2}(\mid t-u_1\mid^{\rho_1};u_1,u_2)\\
&+2M\mid u_1-x_0\mid^{\rho_1}\mid u_2-y_0\mid^{\rho_2}B^{\alpha}_{n_1,n_2}(\mu_{0,0};u_1,u_2).
\end{aligned}$$

On apply Hölder inequality on $B^{\alpha}_{n_1,n_2}$, we get

$$\begin{aligned}
&B^{\alpha}_{n_1,n_2}(\mid t-u_1\mid^{\rho_1}\mid s-u_2\mid^{\rho_2};u_1,u_2)\\
&=\mathcal{U}^{\alpha_1}_{n_1,k}(\mid t-u_1\mid^{\rho_1};u_1,u_2)\,\mathcal{V}^{\alpha_2}_{n_2,l}(\mid s-u_2\mid^{\rho_2};u_1,u_2)\\
&\leq\left(B^{\alpha}_{n_1,n_2}(\mid t-u_1\mid^{2};u_1,u_2)\right)^{\frac{\rho_1}{2}}\left(B^{\alpha}_{n_1,n_2}(\mu_{0,0};u_1,u_2)\right)^{\frac{2-\rho_1}{2}}\\
&\times\left(B^{\alpha}_{n_1,n_2}(\mid s-u_2\mid^{2};u_1,u_2)\right)^{\frac{\rho_2}{2}}\left(B^{\alpha}_{n_1,n_2}(\mu_{0,0};u_1,u_2)\right)^{\frac{2-\rho_2}{2}}.
\end{aligned}$$

Thus, we can obtain

$$\begin{aligned}
&\mid B^{\alpha}_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2)\mid\\
&\leq M\left(\delta^2_{n_1,u_1}\right)^{\frac{\rho_1}{2}}\left(\delta^2_{n_2,u_2}\right)^{\frac{\rho_2}{2}}+2M\,(d(u_1,E))^{\rho_1}\,(d(u_2,E))^{\rho_2}\\
&+M\,(d(u_1,E))^{\rho_1}\left(\delta^2_{n_2,u_2}\right)^{\frac{\rho_2}{2}}+L\,(d(u_2,E))^{\rho_2}\left(\delta^2_{n_1,u_1}\right)^{\frac{\rho_1}{2}}.
\end{aligned}$$

We have complete the proof. □

Theorem 3.6. *If $g\in C'(\mathcal{I}^2)$, then for all $(u_1,u_2)\in\mathcal{I}^2$, operator $B^{\alpha}_{n_1,n_2}$ satisfying*

$$\mid B^{\alpha}_{n_1,n_2}(g;u_1,u_2)-g(u_1,u_2)\mid\leq\parallel g_{u_1}\parallel_{C(\mathcal{I}^2)}\left(\delta^2_{n_1,u_1}\right)^{\frac{1}{2}}+\parallel g_{u_2}\parallel_{C(\mathcal{I}^2)}\left(\delta^2_{n_2,u_2}\right)^{\frac{1}{2}},$$

where δ_{n_1,u_1} and δ_{n_2,u_2} are defined by Theorem 3.4.

Proof. Let $g \in C'(\mathcal{I}^2)$, and for any fixed $(u_1, u_2) \in \mathcal{I}^2$ we have

$$g(t,s) - g(u_1,u_2) \;=\; \int_{u_1}^{t} g_\varrho(\varrho, s)\mathrm{d}\varrho + \int_{u_2}^{s} g_\mu(u_1,\mu)\mathrm{d}\mu.$$

On apply $B^{\alpha}_{n_1,n_2}$

$$\begin{aligned} & B^{\alpha}_{n_1,n_2}\left(g(t,s);u_1,u_2\right) - g(u_1,u_2) \\ (9) \qquad & = B^{\alpha}_{n_1,n_2}\left(\int_{u_1}^{t} g_\varrho(\varrho,s)\mathrm{d}\varrho; u_1,u_2\right) + B^{\alpha}_{n_1,n_2}\left(\int_{u_2}^{s} g_\mu(u_1,\mu)\mathrm{d}\mu; u_1,u_2\right). \end{aligned}$$

From the sup-norm on $\mathcal{I}^2$ we can see that

$$(10) \qquad \left| \int_{u_1}^{t} g_\varrho(\varrho,s)\mathrm{d}\varrho \right| \leq \int_{u_1}^{t} \left| g_\varrho(\varrho,s)\mathrm{d}\varrho \right| \leq \| g_{u_1} \|_{C(\mathcal{I}^2)} | t-u_1 |,$$

$$(11) \qquad \left| \int_{u_2}^{s} g_\mu(u_1,\mu)\mathrm{d}\mu \right| \leq \int_{u_2}^{s} \left| g_\mu(u_1,\mu)\mathrm{d}\mu \right| \leq \| g_{u_2} \|_{C(\mathcal{I}^2)} | s-u_2 | .$$

In the view of (9), (10) and (11) we can obtain

$$\begin{aligned} & | B^{\alpha}_{n_1,n_2}\left(g(u_1,u_2);u_1,u_2\right) - g(u_1,u_2) | \\ & \leq B^{\alpha}_{n_1,n_2}\left(\left|\int_{u_1}^{t} g_\varrho(\varrho,s)\mathrm{d}\varrho\right|; u_1,u_2\right) \\ & + B^{\alpha}_{n_1,n_2}\left(\left|\int_{u_2}^{s} g_\mu(u_1,\mu)\mathrm{d}\mu\right|; u_1,u_2\right) \\ & \leq \| g_{u_1} \|_{C(\mathcal{I}^2)} B^{\alpha}_{n_1,n_2}\left(| t-u_1 |; u_1,u_2\right) \\ & + \| g_{u_2} \|_{C(\mathcal{I}^2)} B^{\alpha}_{n_1,n_2}\left(| s-u_2 |; u_1,u_2\right) \\ & \leq \| g_{u_1} \|_{C(\mathcal{I}^2)} \left(B^{\alpha}_{n_1,n_2}((t-u_1)^2;u_1,u_2) B^{\alpha}_{n_1,n_2}(1;u_1,u_2)\right)^{\frac{1}{2}} \\ & + \| g_{u_2} \|_{C(\mathcal{I}^2)} \left(B^{\alpha}_{n_1,n_2}((s-u_2)^2;u_1,u_2) B^{\alpha}_{n_1,n_2}(1;u_1,u_2)\right)^{\frac{1}{2}} \\ & = \| g_{u_1} \|_{C(\mathcal{I}^2)} \left(\delta^2_{n_1,u_1}\right)^{\frac{1}{2}} + \| g_{u_2} \|_{C(\mathcal{I}^2)} \left(\delta^2_{n_2,u_2}\right)^{\frac{1}{2}} . \end{aligned}$$

□

Theorem 3.7. *For any $f \in C(\mathcal{I}^2)$, if we define an auxiliary operator such that*

$$\begin{aligned} R^{\alpha_1,\alpha_2}_{n_1,n_2}(f;u_1,u_2) &= B^{\alpha}_{n_1,n_2}(g;u_1,u_2) + f(u_1,u_2) \\ & - f\left(\mathcal{U}^{\alpha_1}_{n_1,k}(\mu_{1,0};u_1,u_2), \mathcal{V}^{\alpha_2}_{n_2,l}(\mu_{0,1};u_1,u_2)\right), \end{aligned}$$

where, from Lemma 2.4, $\mathcal{U}^{\alpha_1}_{n_1,k}(\mu_{1,0};u_1,u_2) = \frac{2(\lambda-1)u_1}{n_1} + \frac{\lambda+1}{n_1}, n_1 > 1$ and

$$\mathcal{V}^{\alpha_2}_{n_2,l}(\mu_{0,1};u_1,u_2) = \frac{2(\lambda-1)u_1}{n_2} + \frac{\lambda+1}{n_2}, n_2 > 1.$$

Then, for all $g \in C'(\mathcal{I}^2)$, $R^{\alpha_1,\alpha_2}_{n_1,n_2}$ satisfying

$$\begin{aligned} R^{\alpha_1,\alpha_2}_{n_1,n_2}(g;t,s) - g(u_1,u_2) \quad &\leq \quad \Bigg\{\delta^2_{n_1,u_1} + \delta^2_{n_2,u_2} + \left(\frac{2(\lambda-1)u_1}{n_1} + \frac{\lambda+1}{n_1} - u_1\right)^2 \\ &+ \quad \left(\frac{2(\lambda-1)u_1}{n_2} + \frac{\lambda+1}{n_2} - u_2\right)^2\Bigg\} \parallel g \parallel_{C^2(\mathcal{I}^2)} . \end{aligned}$$

Proof. In the light of operator $R^{\alpha_1,\alpha_2}_{n_1,n_2}(f;u_1,u_2)$ and Lemma 2.4, we obtain $R^{\alpha_1,\alpha_2}_{n_1,n_2}(1;u_1,u_2) = 1$, $R^{\alpha_1,\alpha_2}_{n_1,n_2}(t-u_1;u_1,u_2) = 0$ and $R^{\alpha_1,\alpha_2}_{n_1,n_2}(s-u_2;u_1,u_2) = 0$. For any $g \in C'(\mathcal{I}^2)$ the Taylor series give us:

$$\begin{aligned} g(t,s) - g(u_1,u_2) \quad &= \quad \frac{\partial g(u_1,u_2)}{\partial u_1}(t-u_1) + \int_{u_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,u_2)}{\partial \lambda^2}\mathrm{d}\lambda \\ &+ \quad \frac{\partial g(u_1,u_2)}{\partial u_2}(s-u_2) + \int_{u_2}^{s}(s-\psi)\frac{\partial^2 g(u_1,\psi)}{\partial \psi^2}\mathrm{d}\psi. \end{aligned}$$

On apply $R^{\alpha_1,\alpha_2}_{n_1,n_2}$, we see that

$$\begin{aligned} &R^{\alpha_1,\alpha_2}_{n_1,n_2}\left(g(t,s);u_1,u_2\right) - R^{\alpha_1,\alpha_2}_{n_1,n_2}\left(g(u_1,u_2)\right) \\ &= R^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\int_{u_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,u_2)}{\partial \lambda^2}\mathrm{d}\lambda;u_1,u_2\right) \\ &+ R^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\int_{u_2}^{s}(s-\psi)\frac{\partial^2 g(u_1,\psi)}{\partial \psi^2}\mathrm{d}\psi;u_1,u_2\right) \\ &= B^{\alpha}_{n_1,n_2}\left(\int_{u_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,u_2)}{\partial \lambda^2}\mathrm{d}\lambda;u_1,u_2\right) \\ &+ B^{\alpha}_{n_1,n_2}\left(\int_{u_2}^{s}(s-\psi)\frac{\partial^2 g(u_1,\psi)}{\partial \psi^2}\mathrm{d}\psi;u_1,u_2\right) \\ &- \int_{u_1}^{\frac{2(\lambda-1)u_1}{n_1}+\frac{\lambda+1}{n_1}}\left(\frac{2(\lambda-1)u_1}{n_1} + \frac{\lambda+1}{n_1} - \lambda\right)\frac{\partial^2 g(\lambda,u_2)}{\partial \lambda^2}\mathrm{d}\lambda \\ &- \int_{u_2}^{\frac{2(\lambda-1)u_1}{n_2}+\frac{\lambda+1}{n_2}}\left(\frac{2(\lambda-1)u_1}{n_2} + \frac{\lambda+1}{n_2} - \psi\right)\frac{\partial^2 g(u_1,\psi)}{\partial \psi^2}\mathrm{d}\psi. \end{aligned}$$

From hypothesis we easily obtain

$$\begin{aligned} &\left|\int_{u_1}^{t}(t-\lambda)\frac{\partial^2 g(\lambda,u_2)}{\partial \lambda^2}\mathrm{d}\lambda\right| \leq \int_{u_1}^{t}\left|(t-\lambda)\frac{\partial^2 g(\lambda,u_2)}{\partial \lambda^2}\mathrm{d}\lambda\right| \leq \parallel g \parallel_{C^2(\mathcal{I}^2)} (t-u_1)^2, \\ &\left|\int_{u_2}^{s}(s-\psi)\frac{\partial^2 g(u_1,\psi)}{\partial \psi^2}\mathrm{d}\psi\right| \leq \int_{u_2}^{s}\left|(s-\psi)\frac{\partial^2 g(u_1,\psi)}{\partial \psi^2}\mathrm{d}\psi\right| \leq \parallel g \parallel_{C^2(\mathcal{I}^2)} (s-u_2)^2, \\ &\left|\int_{u_1}^{\frac{2(\lambda-1)u_1}{n_1}+\frac{\lambda+1}{n_1}}\left(\frac{2(\lambda-1)u_1}{n_1} + \frac{\lambda+1}{n_1} - \lambda\right)\frac{\partial^2 g(\lambda,u_2)}{\partial \lambda^2}\mathrm{d}\lambda\right| \end{aligned}$$

$$\leq \| g \|_{C^2(\mathcal{I}^2)} \left(\frac{2(\lambda-1)u_1}{n_1} + \frac{\lambda+1}{n_1} - u_1 \right)^2$$

$$\left| \int_{u_2}^{\frac{2(\lambda-1)u_1}{n_2}+\frac{\lambda+1}{n_2}} \left(\frac{2(\lambda-1)u_1}{n_2} + \frac{\lambda+1}{n_2} - \psi \right) \frac{\partial^2 g(u_1,\psi)}{\partial \psi^2} \mathrm{d}\psi \right|$$

$$\leq \| g \|_{C^2(\mathcal{I}^2)} \left(\frac{2(\lambda-1)u_1}{n_2} + \frac{\lambda+1}{n_2} - u_2 \right)^2.$$

Thus,

$$\begin{aligned} &| R^{\alpha_1,\alpha_2}_{n_1,n_2}(g;t,s) - g(u_1,u_2) | \\ &\leq \Bigg\{ B^{\alpha}_{n_1,n_2}((t-u_1)^2;u_1,u_2) + B^{\alpha}_{n_1,n_2}((s-u_2)^2;u_1,u_2) \\ &+ \left(\frac{2(\lambda-1)u_1}{n_1} + \frac{\lambda+1}{n_1} - u_1 \right)^2 \\ &+ \left(\frac{2(\lambda-1)u_1}{n_2} + \frac{\lambda+1}{n_2} - u_2 \right)^2 \Bigg\} \| g \|_{C^2(\mathcal{I}^2)} . \end{aligned}$$

We complete the proof of desired Theorem 3.7. □

4. Some approximation results in Bögel space

Take any function $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ for a real compact intervals of $\mathcal{I}_1 \times \mathcal{I}_2$. For all (t,s), $(u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ suppose $\Delta^*_{(t,s)} g(u_1,u_2)$ denotes the bivariate mixed difference operators defined as follows:

$$\Delta^*_{(t,s)} g(u_1,u_2) = g(t,s) - g(t,u_2) - g(u_1,s) + g(u_1,u_2).$$

If at any point $(u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ the function $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ defined on $\mathcal{I}_1 \times \mathcal{I}_2$, then $\lim_{(t,s)\to(u_1,u_2)} \Delta^*_{(t,s)} g(u_1,u_2) = 0$. If set of all the space of all Bögel-continuous(B-continuous) denoted by $C_B(\mathcal{I}_1 \times \mathcal{I}_2)$ on $(u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ and be defined such that $C_B(\mathcal{I}_1 \times \mathcal{I}_2) = \{g,$ such that $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ is $g,\ B-$bounded on $\mathcal{I}_1 \times \mathcal{I}_2\}$. Next, the Bögel-differentiable function on $(u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ be $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ and limit exists finite defined by

$$\lim_{(t,s)\to(u_1,u_2),\, t\neq u_1,\, s\neq u_2} \frac{1}{(t-u_1)(s-u_2)} \left(\Delta^*_{(t,s)} \right) = D_B g(u_1,u_2) < \infty.$$

Let the classes of all Bögel-differentiable function denoted by $D_\varphi g(u_1,u_2)$ and be $D_\varphi(\mathcal{I}_1 \times \mathcal{I}_2) = \{g,$ such that $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$ is g, B$-$differentiable on $\mathcal{I}_1 \times \mathcal{I}_2\}$. Suppose the function g is B-bounded on D and be $g : \mathcal{I}_1 \times \mathcal{I}_2 \to \mathbb{R}$, then for all (t,s), $(u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ there exists positive constant M such that

$| \Delta^*_{(t,s)} g(u_1,u_2) | \leq M$. The classes of all B-continuous function is called a B-bounded on $\mathcal{I}_1 \times \mathcal{I}_2$, whene $\mathcal{I}_1 \times \mathcal{I}_2$ is compact subset. Let $B_\varphi(\mathcal{I}_1 \times \mathcal{I}_2)$ denote the classes of all B-bounded function defined on $\mathcal{I}_1 \times \mathcal{I}_2$ which equipped the norm on B as $\| g \|_B = \sup_{(t,s),(u_1,u_2)\in \mathcal{I}_1\times\mathcal{I}_2} | \Delta^*_{(t,s)} g(u_1,u_2) |$. As we know to approximate the degree for a set of all B-continuous function on positive linear operators, it is essential to use the properties of mixed-modulus of continuity. So we let for all (t,s), $(u_1,u_2) \in \mathcal{I}_1 \times \mathcal{I}_2$ and $g \in B_\varphi(\mathcal{I}_{\alpha_n})$, the mixed-modulus of continuity of function g bt $\omega_B : [0,\infty) \times [0,\infty) \to \mathbb{R}$ and be defined such as:

$$\omega_B(g;\delta_1,\delta_2) = \sup\{\Delta^*_{(t,s)} g(u_1,u_2) :| t-u_1 | \leq \delta_1,\ | s-u_2 | \leq \delta_2\}.$$

For any $\mathcal{I}^2 = [0,1]\times[0,1]$, we suppose the classes of all B-continuous function defined on $\mathcal{I}^2$ denoted by $C_\varphi(\mathcal{I}^2)$. Moreover, let set of all ordinary continuous function defined on $\mathcal{I}^2$ be $C(\mathcal{I}^2)$. For further details on space of Bögel functions related to this paper we propose the article [35, 36].

Let $(u_1,u_2) \in \mathcal{I}^2$ and $n_1, n_2 \in \mathbb{N}$ then for all $g \in C(\mathcal{I}^2)$ we define the GBS type operators for the positive linear operators $B^\alpha_{n_1,n_2}$. Thus we suppose

$$K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(t,s);u_1,u_2) = B^\alpha_{n_1,n_2}\Big(g(t,u_2) + g(u_1,s) - g(t,s);u_1,u_2\Big). \tag{12}$$

More precisely, the generalized GBS operator for bivariate function is defined as follows:

$$\begin{aligned} &K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(t,s);u_1,u_2) \\ &= \sum_{k,l=0}^{\infty} \mathcal{Q}^*_1(n_1,u_1)\mathcal{Q}^*_2(n_2,u_2)\int_0^\infty\int_0^\infty \mathcal{P}^*_1(n_1,u_1)P^*_2(n_2,u_2), \end{aligned} \tag{13}$$

where $P_{u_1,u_2}(t,s) = \big(g(t,u_2) + g(u_1,s) - g(t,s)\big)$.

Theorem 4.1. *For all $g \in C_\varphi(\mathcal{I}^2)$, it follows that*

$$| K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(t,s);u_1,u_2) - g(u_1,u_2) | \leq 4\omega_B\left(g;\delta_{n_1,u_1},\delta_{n_2,u_2}\right),$$

where δ_{n_1,u_1} and δ_{n_2,u_2} are defined by Theorem 3.4.

Proof. Let $(t,s),(u_1,u_2) \in \mathcal{I}^2$. For all $n_1,n_2 \in \mathbb{N}$ and $\delta_{n_1},\delta_{n_2} > 0$, it follows that

$$\begin{aligned} | \Delta^*_{(u_1,u_2)} g(t,s) | &\leq \omega_B\left(g;| t-u_1 | \, | s-u_2 |\right) \\ &\leq \left(1+\frac{t-u_1}{\delta_{n_1}}\right)\left(1+\frac{s-u_2}{\delta_{n_2}}\right)\omega_B\left(g;\delta_{n_1},\delta_{n_2}\right). \end{aligned}$$

From (12) and well-known Cauchy-Schwarz inequality, we easily conclude that

$$\begin{aligned}
&\mid K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(t,s);u_1,u_2) - g(u_1,u_2) \mid \leq B^{\alpha}_{n_1,n_2}\Big(\mid \Delta^*_{(u_1,u_2)} g(t,s) \mid; u_1,u_2\Big) \\
&\leq \Bigg(B^{\alpha}_{n_1,n_2}(\phi_{0,0};u_1,u_2) + \frac{1}{\delta_{n_1}}\left(B^{\alpha}_{n_1,n_2}((t-u_1)^2;u_1,u_2)\right)^{\frac{1}{2}} \\
&+ \frac{1}{\delta_{n_2}}\left(B^{\alpha}_{n_1,n_2}((s-u_2)^2;u_1,u_2)\right)^{\frac{1}{2}} \\
&+ \frac{1}{\delta_{n_1}}\left(B^{\alpha}_{n_1,n_2}((t-u_1)^2;u_1,u_2)\right)^{\frac{1}{2}} \\
&\times \frac{1}{\delta_{n_2}}\left(B^{\alpha}_{n_1,n_2}((s-u_2)^2;u_1,u_2)\right)^{\frac{1}{2}} \Bigg)\omega_B\left(g;\delta_{n_1},\delta_{n_2}\right).
\end{aligned}$$

In the view of Theorem 3.4 we easily get our results. □

If we let $x=(t,s),\ y=(u_1,u_2)\in\mathcal{I}^2$, then the Lipschitz function in terms of B-continuous functions defined by

$$Lip^{\xi}_{M} = \left\{ g \in C(\mathcal{I}^2) : \mid \Delta^*_{(u_1,u_2)} g(x,y) \mid \leq M \parallel x-y \parallel^{\xi} \right\},$$

where M is a positive constant, $0<\xi\leq 1$, and Euclidean norm $\parallel x-y \parallel = \sqrt{(t-u_1)^2+(s-u_2)^2}$.

Theorem 4.2. *For all $g\in Lip^{\xi}_{M}$ operator $K^{\alpha_1,\alpha_2}_{n_1,n_2}$ satisfying*

$$\mid K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(x,y);u_1,u_2) - g(u_1,u_2) \mid \leq M\{\delta^2_{n_1,u_1}+\delta^2_{n_2,u_2}\}^{\frac{\xi}{2}},$$

where δ_{n_1,u_1} and δ_{n_2,u_2} are defined by Theorem 3.4.

Proof. We easily see that

$$\begin{aligned}
K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(x,y);u_1,u_2) &= B^{\alpha}_{n_1,n_2}(g(u_1,y)+g(x,u_2)-g(x,s);u_1,u_2) \\
&= B^{\alpha}_{n_1,n_2}\Big(g(u_1,u_2) - \Delta^*_{(u_1,u_2)} g(x,s);u_1,u_2\Big) \\
&= g(u_1,u_2) - B^{\alpha}_{n_1,n_2}\Big(\Delta^*_{(u_1,u_2)} g(x,s);u_1,u_2\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\mid K^{\alpha_1,\alpha_2}_{n_1,n_2}(g(x,y);u_1,u_2) - g(u_1,u_2) \mid \leq B^{\alpha}_{n_1,n_2}\Big(\mid \Delta^*_{(u_1,u_2)} g(x,y) \mid; u_1,u_2\Big) \\
&\leq M B^{\alpha}_{n_1,n_2}\Big(\parallel x-y \parallel^{\xi};u_1,u_2\Big) \\
&\leq M\{B^{\alpha}_{n_1,n_2}\left(\parallel x-y \parallel^2;u_1,u_2\right)\}^{\frac{\xi}{2}} \\
&\leq M\{B^{\alpha}_{n_1,n_2}\left((t-u_1)^2;u_1,u_2\right) + B^{\alpha}_{n_1,n_2}\left((s-u_2)^2;u_1,u_2\right)\}^{\frac{\xi}{2}}.
\end{aligned}$$

□

Theorem 4.3. *If* $g \in D_{\varphi}(\mathcal{I}^2)$ *and* $D_B g \in B(\mathcal{I}^2)$, *then*

$$\begin{aligned}
&| K^{\alpha_1,\alpha_2}_{n_1,n_2}(g; u_1, u_2) - g(u_1, u_2) | \\
&\leq C\Big\{3 \parallel D_B g \parallel_\infty + \varpi_{mixed}(D_B g; \delta_{n_1}, \delta_{n_2})\Big\}(u_1 + 1)(u_2 + 1) \\
&+ \Big\{1 + \sqrt{C_2}(u_1 + 1) + \sqrt{C_1}(u_2 + 1)\Big\} \\
&\times \varpi_{mixed}(D_B g; \delta_{n_1}, \delta_{n_2})(u_1 + 1)(u_2 + 1),
\end{aligned}$$

where δ_{n_1}, δ_{n_2} *defined by Theorem 3.4 and* C *is any positive constant.*

Proof. Suppose $\rho \in (u_1, t)$, $\xi \in (u_2, s)$ and

$$\Delta^*_{(u_1,u_2)} g(t, s) = (t - u_1)(s - u_2) D_B g(\rho, \xi),$$

$$D_B g(\rho, \xi) = \Delta^*_{(u_1,u_2)} D_B g(\rho, \xi) + D_B g(\rho, y) + D_B g(x, \xi) - D_B g(u_1, u_2).$$

For all $D_B g \in B(\mathcal{I}^2)$, it follows that

$$\begin{aligned}
&| K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(\Delta^*_{(u_1,u_2)} g(t, s); u_1, u_2\right) | = | K^{\alpha_1,\alpha_2}_{n_1,n_2}((t - u_1)(s - u_2) D_B g(\rho, \xi); u_1, u_2) | \\
&\leq K^{\alpha_1,\alpha_2}_{n_1,n_2}\left(| t - u_1 || s - u_2 || \Delta^*_{(u_1,u_2)} D_B g(\rho, \xi) |; u_1, u_2\right) \\
&+ K^{\alpha_1,\alpha_2}_{n_1,n_2}\Bigg(| t - u_1 || s - u_2 | (| D_B g(\rho, u_2) | \\
&+ | D_B g(u_1, \xi) | + | D_B g(u_1, u_2) |); u_1, u_2\Bigg) \\
&\leq K^{\alpha_1,\alpha_2}_{n_1,n_2}\Bigg(| t - u_1 || s - u_2 | \\
&\times \varpi_{mixed}(D_B g; | \rho - u_1 |, | \xi - u_2 |); u_1, u_2\Bigg) \\
&+ 3 \parallel D_B g \parallel_\infty K^{\alpha_1,\alpha_2}_{n_1,n_2}(| t - u_1 || s - u_2 |; u_1, u_2).
\end{aligned}$$

Here, ϖ_{mixed}, is mixed-modulus of continuity and it follows that

$$\begin{aligned}
&\varpi_{mixed}(D_B g; | \rho - u_1 |, | \xi - u_2 |) \\
&\leq \varpi_{mixed}(D_B g; | t - u_1 |, | s - u_2 |) \\
&\leq \left(1 + \delta^{-1}_{n_1} | t - u_1 |\right)\left(1 + \delta^{-1}_{n_2} | s - u_2 |\right) \varpi_{mixed}(D_B g; | \delta_{n_1}, \delta_{n_2}).
\end{aligned}$$

Therefore, it is obvious that

$$\begin{aligned}
&| K^*_{n_1,n_2}(g; u_1, u_2) - g(u_1, u_2) | = | \Delta^*_{(u_1,u_2)} g(t, s); u_1, u_2 | \\
&\leq 3 \parallel D_B g \parallel_\infty \left(K^{\alpha_1,\alpha_2}_{n_1,n_2}\left((t - u_1)^2 (s - u_2)^2; u_1, u_2\right)\right)^{\frac{1}{2}}
\end{aligned}$$

$$+\left(K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\mid t-u_1\mid\mid s-u_2\mid;u_1,u_2\right)\right.$$
$$\left.+\delta_{n_1}^{-1}K_{n_1,n_2}^{\alpha_1,\alpha_2}\left((t-u_1)^2\mid s-u_2\mid;u_1,u_2\right)\right)$$
$$+\delta_{n_2}^{-1}K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\mid t-u_1\mid(s-u_2)^2;u_1,u_2\right)$$
$$+\delta_{n_1}^{-1}\delta_{n_2}^{-1}K_{n_1,n_2}^{\alpha_1,\alpha_2}\left((t-u_1)^2(s-u_2)^2;u_1,u_2\right)\varpi_{mixed}\left(D_Bg;\delta_{n_1},\delta_{n_2}\right);$$

$$\mid K_{n_1,n_2}^{*}\left(g;u_1,u_2\right)-g(u_1,u_2)\mid=\mid\Delta_{(u_1,u_2)}^{*}g(t,s);u_1,u_2\mid$$
$$\leq 3\parallel D_Bg\parallel_\infty\left(K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Psi_{u_1,u_2}^{2,2};u_1,u_2\right)\right)^{\frac{1}{2}}$$
$$+\left\{\left(K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Psi_{u_1,u_2}^{2,2};u_1,u_2\right)\right)^{\frac{1}{2}}\right.$$
$$+\delta_{n_1}^{-1}\left(K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Psi_{u_1,u_2}^{4,2};u_1,u_2\right)\right)^{\frac{1}{2}}+\delta_{n_2}^{-1}\left(K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Psi_{u_1,u_2}^{2,4};u_1,u_2\right)\right)^{\frac{1}{2}}$$
$$\left.+\delta_{n_1}^{-1}\delta_{n_2}^{-1}K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Psi_{u_1,u_2}^{2,2};u_1,u_2\right)\right\}\varpi_{mixed}\left(D_Bg;\delta_{n_1},\delta_{n_2}\right).$$

Which follows that

$$\mid K_{n_1,n_2}^{*}\left(g;u_1,u_2\right)-g(u_1,u_2)\mid$$
$$=3\parallel D_Bg\parallel_\infty\left(K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Psi_{u_1,u_2}^{2,0};u_1,u_2\right)K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Psi_{u_1,u_2}^{0,2};u_1,u_2\right)\right)^{\frac{1}{2}}$$
$$+\left\{\left(K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Psi_{u_1,u_2}^{2,0};u_1,u_2\right)K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Psi_{u_1,u_2}^{0,2};u_1,u_2\right)\right)^{\frac{1}{2}}\right.$$
$$+\delta_{n_1}^{-1}\left(K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Psi_{u_1,u_2}^{4,0};u_1,u_2\right)K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Psi_{u_1,u_2}^{0,2};u_1,u_2\right)\right)^{\frac{1}{2}}$$
$$+\delta_{n_2}^{-1}\left(K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Psi_{u_1,u_2}^{2,0};u_1,u_2\right)K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Psi_{u_1,u_2}^{0,4};u_1,u_2\right)\right)^{\frac{1}{2}}$$
$$\left.+\delta_{n_1}^{-1}\delta_{n_2}^{-1}K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Psi_{u_1,u_2}^{2,0};u_1,u_2\right)K_{n_1,n_2}^{\alpha_1,\alpha_2}\left(\Psi_{u_1,u_2}^{0,2};u_1,u_2\right)\right\}$$
$$\times\varpi_{mixed}\left(D_Bg;\delta_{n_1},\delta_{n_2}\right).$$

From Lemma (2.5), we demonstrate

$$\mid K_{n_1,n_2}^{*}\left(g;u_1,u_2\right)-g(u_1,u_2)\mid\leq 3\parallel D_Bg\parallel_\infty\left(\sqrt{C_1C_2}(u_1+1)(u_2+1)\right)$$
$$+\left\{\left(\sqrt{C_1C_2}(u_1+1)(u_2+1)\right)\right.$$

$$+ \delta_{n_1}^{-1}\left(\sqrt{C_2}\sqrt{O\left(\frac{1}{n_1}\right)}(u_1+1)^2(u_2+1)\right)$$
$$+ \delta_{n_2}^{-1}\left(\sqrt{C_1}\sqrt{O\left(\frac{1}{n_2}\right)}(u_2+1)^2(u_1+1)\right)$$
$$+ \delta_{n_1}^{-1}\delta_{n_2}^{-1}\left(\sqrt{O\left(\frac{1}{n_1}\right)}\sqrt{O\left(\frac{1}{n_2}\right)}(u_1+1)(u_2+1)\right)\Bigg\}$$
$$\times\, \varpi_{mixed}\left(D_B g; \delta_{n_1}, \delta_{n_2}\right).$$

Which complete the proof of Theorem 4.3. □

5. Conclusion and remarks

These types of generalization, that is, Bivariate Szász operators is a new generalization. In this, manuscript our investigation is to generalize the Szász Durrmeyer operators based on Dunkl analogue [41] by introducing the bivariate functions. We study the bivariate properties of Szász Durrmeyer operators with the help of modulus of continuity, mixed-modulus of continuity and then find the approximation results in Peetre's K-functional, Voronovskaja type theorem and Lipschitz maximal functions for these bivariate operators. Next, we also construct the GBS type operator of these generalized operators and study approximation in Bögel continuous functions by use of mixed-modulus of continuity.

References

[1] O. Szász, *Generalization of S. Bernstein's polynomials to the infinite interval*, J. Res. Nat. Bur. Stand., 45 (1950), 239–245.

[2] J.L. Durrmeyer, *Une formule dinversion de la transformée de Laplace-applications à la théorie desmoments*, Thése de 3e cycle, Faculté des Sciences de l'Universit de Paris, 1967.

[3] T. Acar, A. Aral, S.A. Mohiuddine, *On Kantorovich modification of (p,q)-Baskakov operators*, J. Inequal. Appl., 2016, Article Id: 98 (2016).

[4] S.N. Berntein, *Démonstration du théorème de Weierstrass fondée sur le calcul de probabilités*, Commun. Soc. Math. Kharkow., 13 (1912), 1–2.

[5] S. Varma, F. Taşdelen, *Szász type operators involving Charlier polynomials*, Math. Comput. Modelling, 56 (2012), 118–122.

[6] N. Rao, M. Heshamuddin, Mohd Shadab, *Hybrid Operators of Summation-Integral type using shape parameter$-\alpha$*, communicated, 2020.

[7] S. Sucu, E. Ibikli, *Rate of convergence for Szász type operators including Sheffer polynomials*, Stud. Univ. Babeş-Bolyai Math., 58 (2013), 55–63.

[8] S. Sucu, *Dunkl analogue of Szász operators*, Appl. Math. Comput., 244 (2014), 42–48.

[9] M. Rosenblum, *Generalized Hermite polynomials and the Bose-like oscillator calculus*, Oper. Theory Adv. Appl., 73 (1994), 369–396.

[10] Q.B. Cai, B.Y. Lian, G. Zhou, *Approximation properties of λ-Bernstein operators*, J. Inequal. Appl. Article, 61 2018.

[11] A. Wafi, N. Rao, *Szász-Durrmeyer operator based on Dunkl analogue*, Complex Anal. Oper. Theory, 12 (2018), 1519-1536.

[12] T. Acar, A. Aral, S.A. Mohiuddine, *On Kantorovich modification of (p,q)-Bernstein operators*, Iran. J. Sci. Technol. Trans. A, Sci., 42 (2018), 1459-1464.

[13] S.A. Mohiuddine, T. Acar, A. Alotaibi, *Construction of a new family of Bernstein$-K$ antorovich operators*, Math. Methods Appl. Sci., 40 (2017), 7749-7759.

[14] Q.B. Cai, *The Bézíer variant of Kantorovich type λ-Bernstein operators*, J. Inequal. Appl. Article, 90 (2018).

[15] S.A. Mohiuddine, T. Acar, A. Alotaibi, *Durrmeyer type (p,q)-Baskakov operators preserving linear functions*, J. Math. Inequal., 12 (2018), 961-973.

[16] S.A. Mohiuddine, T. Acar, M.A. Alghamdi, *Genuine modified Bernstein-Durrmeyer operators*, J. Inequal. Appl., 2018, Article id: 104.

[17] M. Mursaleen, K.J. Ansari, A. Khan, *On (p,q)-analogue of Bernstein operators*, Appl. Math. Comput., 266 (2015), 874-882, Appl. Math. Comput., 278 (2016), 70–71.

[18] F. Ozger, *Weighted statistical approximation properties of univariate and bi-variate λ- Kantorovich operators*, Filomat, 33 (2019), 1-15.

[19] F. Ozger, " *On new bases with Schurer polynomials and corresponding results in approximation theory*, Commun. Fac. Sci. Univ. Ank. Ser. Al. Math.

[20] M. Mursaleen, M. Nasiruzzaman, A. Nurgali, *Some approximation results on Bernstein-Schurer operators defined by (p,q)-integers*, J. Inequal. Appl., 2015, Article Id: 249.

[21] M. Mursaleen, F. Khan, A. Khan, *Approximation by Kantorovich type q-Bernstein-Stancu operators*, Complex Anal. Oper. Theory., 11 (2017), 85–107.

[22] M. Mursaleen, M. Nasiruzzaman, *Some approximation properties of bivariate Bleimann-Butzer-Hahn operators based on* (p, q)*-integers*, Boll. Unione Mat. Ital..

[23] H.M. Srivastava, F. Ozger, S.A. Mohiuddine, *Construction of Stancu-type Bernstein operators Based on Bezier bases with shape parameter* λ, Symmetry, 11 (2019), Article 316.

[24] M. Mursaleen, M. Nasiruzzaman, N. Ashirbayev, A. Abzhapbarov, *Higher order generalization of Bernstein type operators defined by* (p, q)*-integers*, J. Comput. Anal. Appl., 25 (2018), 817-829.

[25] K.G. Weierstrass, *Über die analyishe Darstellbar keit sogenonnter willkürlincher Funtionen einerreelen Veründerlichen*, Sitz. Akad. Berl., 2 (1885), 633–693.

[26] A.M. Acu, T. Acar, V.A. Radu, *Approximation by modifed* U_n^ρ *operators*, Rev. R. Acad. Ciene. Exactas Fis. Nat. Ser. A Math., 113 (2019), 2715-2729.

[27] T. Acar, M. Mursaleen, S.A. Mohiuddine, *Stancu type* (p, q)*-Szász-Mirakyan-Baskakov operators*, Commun. Fac. Sci. Univ. Ank. Sér. A1 Math. Stat., 67 (2018), 116-128.

[28] T. Acar, A. Aral, S.A. Mohiuddine, *Approximation by bivariate* (p, q)*-Bernstein-Kantorovich operators*, Iran. J. Sci. Technol. Trans. Sci., 42 (2018), 655-662.

[29] T. Acar, A. Kajla, *Degree of approximation for bivariate generalized Bernstein type operators*, Results in Mathematics, 79 (2018) (in press).

[30] T. Acar, S.A. Mohiuddine, M. Mursaleen, *Approximation by* (p, q)*-Baskakov-Durrmeyer-Stancu operators*, Complex Anal. Oper. Theory., 12 (2018), 1453-1468.

[31] A. Ali, H. Erbay, *Parametric generalization of Baskakov operators*, Math. Commun., 24 (2019), 119-131.

[32] A. Alotaibi, M. Nasiruzzaman, M. Mursaleen, *A Dunkl type generalization of Szász operators via post-quantum calculus*, J. Inequal. Appl., 2018, Article Id: 287.

[33] A. Alotaibi, *Approximation on a class of Phillips operators generated by* q*-analogue*, J. Inequal. Appl., 2020, Article Id: 121.

[34] A. Alotaibi, M. Mursaleen, *Approximation of Jakimovski-Leviatan-Beta type integral operators via* q*-calculus.* AIMS Mathematics., 5 (2020), 3019-3034.

[35] K. Bögel, *Mehrdimensionale differentiation von Funktionen mehrerer veränderlichen*, J. Reine Angew. Math., 170 (1934), 197-217.

[36] K. Bögel, *Über die mehrdimensionale differentiation*, Jahresber. Dtsch. Math.Ver., 65 (1935), 45-71

[37] Q. Cai, B. Lian, G. Zhou, *Approximation properties of λ-Bernstein operators*, J. Inequal. Appl., 2018, Article Id: 61.

[38] A.D. Gadžiev, *Positive linear operators in weighted spaces of functions of several variables*, Izv. Akad. Nauk Azerbaidzhan. SSR Ser. Fiz.-Tekhn. Mat. Nauk, 4 (1980), 32-37.

[39] A.D. Gadžiev, H. Hacisalihoglu, *Convergence of the sequences of linear positive operators*, Ankara University, Yenimahalle, 1995.

[40] A. Khan, V. Sharma, *Statistical approximation by (p, q)-analogue of Bernstein-Stancu OPERATORS*, Ajerbaijan Journal of Mathematics, 8 (2018), 2218-6816.

[41] M. Nasiruzzaman, N. Rao, S. Wazir, R. Kumar, *Approximation on parametric extension of Baskakov-Durrmeyer operators on weighted spaces*, J. Inequal. Appl., 2019, Article Id: 103.

[42] A. Wafi, N. Rao, *Szasz–Gamma operators based on Dunkl analogue*, Iran J. Sci. Technol. Trans. Sci., 43 (2019), 213-223.

Accepted: August 16, 2021