Strongly m-system and strongly primary ideals in posets

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Abstract. In this paper, we study and establish some interesting results of strongly prime ideal and strongly m-system in posets. Also, we study the notion of strongly primary ideals in posets and show some properties of the set $\sqrt{I} = \{x : L(x)^* \cap I \neq \phi\}$ for ideal I of P.

Keywords: Posets, ideals, strongly prime ideal, strongly m-system, strongly primary ideal, minimal strongly prime ideal.

1. Introduction

Throughout this paper $(P, \leq)$ denotes a poset with smallest element 0. For basic terminology and notation for posets, we refer [8] and [9]. For $M \subseteq P$, let $L(M) = \{x \in P : x \leq m, \text{ for all } m \in M\}$ denote the lower cone of M in P and $U(M) = \{x \in P : m \leq x, \text{ for all } m \in M\}$ be the upper cone of M in P. Let $A, B \subseteq P$, we write $L(A, B)$ instead of $L(A \cup B)$ and dually for the upper cones. If $M = \{x_1, x_2, ..., x_n\}$ is finite, then we use the notation $L(x_1, x_2, ..., x_n)$ instead of $L(\{x_1, x_2, ..., x_n\})$(and dually). It is clear that for any subset A of P, we have $A \subseteq L(U(A))$ and $A \subseteq U(L(A))$. If $A \subseteq B$, then $L(B) \subseteq L(A)$ and $U(B) \subseteq U(A)$. Moreover, $LUL(A) = L(A)$ and $ULU(A) = U(A)$. Following [12], a non-empty subset I of P is called semi-ideal if $b \in I$ and $a \leq b$, then $a \in I$. A subset I of P is called ideal if $a, b \in I$ implies $L(U((a, b)) \subseteq I$ (see [8]). Following [7], for any subset X of P, $[X]$ is the smallest ideal of P containing X and $X^* = X \backslash \{0\}$. If $X = \{b\}$, then $L(b)$ is called the principle ideal of P generated by b. A proper semi-ideal (ideal) I of P is called prime if $L(a, b) \subseteq I$ implies that either $a \in I$ or $b \in I$ (see [9]). An ideal I of P is called semi-prime if $L(a, b) \subseteq I$ and $L(a, c) \subseteq I$ together imply $L(a, U(b, c))) \subseteq I$ (see [8]). Following

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[4], an ideal I of P is called strongly prime if $L(A^*, B^*) \subseteq I$ implies that either $A \subseteq I$ or $B \subseteq I$ for different proper ideals A, B of P. A non-empty subset M of P is called m-system if for any $x_1, x_2 \in M$, there exists $t \in L(x_1, x_2)$ such that $t \in M$. Following [6], a non-empty subset M of P is called strongly m-system if $A \cap M \neq \phi$ and $B \cap M \neq \phi$ imply $L(A^*, B^*) \cap M \neq \phi$ for any different proper ideals A, B of P. It is clear that an ideal I of P is strongly prime if and only if $P \backslash I$ is a strongly m- system of P and every strongly m-system of P is m-system. Following [4], an ideal I of P is called strongly semi-prime if $L(A^*, B^*) \subseteq I$ and $L(A^*, C^*) \subseteq I$ together imply $L(A^*, U(B^*, C^*)) \subseteq I$ for any different proper ideals A, B and C of P. For any semi-ideal I of P and a subset A of P, we define $\langle A, I\rangle = \{z \in P : L(a, z) \subseteq I, \text{ for all } a \in A\} = \bigcap_{a\in A} \langle a, I\rangle$ (see [4]). If $A = \{x\}$, then we write $\langle x, I\rangle$ instead of $\langle \{x\}, I\rangle$. For any ideal I of P, a strongly prime ideal Q of P is said to be a minimal strongly prime ideal of I if $I \subseteq Q$ and there is no strongly prime ideal R of P such that $I \subset R \subset Q$. The set of all strongly prime ideals of P is denoted by $Sspec(P)$ and the set of minimal strongly prime ideals of P is denoted by $Smin(P)$. For any ideal I of P, $P(I)$ and $SP(I)$ denotes the intersection of all prime semi-ideals and strongly prime ideals of P containing I respectively. It is clear from Theorem 6 of [9] and Example 1.1 of [6] that $P(I) = I$ and $SP(I) \neq I$ for any ideal I of P. Following [2], let I be a semi-ideal of P. Then, I is said to have (*) condition if whenever $L(A, B) \subseteq I$, we have $A \subseteq \langle B, I\rangle$ for any subsets A and B of P. From [8], a non empty subset F of a poset P is called semi-filter if $x \leq y$ and $x \in F$, then $y \in F$. It is clear that for any subset I of P, I is a semi-ideal of P if and only if $P \backslash I$ is a semi-filter of P. A subset F of P is called filter if for $x, y \in F$ implies $U(L(x, y)) \subseteq F$. A filter F is called prime, whenever $U(x, y) \subseteq F$ implies $x \in F$ or $y \in F$.

2. Minimal strongly prime ideals

Lemma 2.1. *Let M be a strongly m-system of P. Then, the following statements hold:*

- *(i) $P \backslash M$ satisfies the condition that $L(A^*, B^*) \subseteq P \backslash M$ implies $A \subseteq P \backslash M$ or $B \subseteq P \backslash M$ for any different proper ideals A, B of P.*
- *(ii) If $P \backslash M$ is a semi-ideal of P, then M is a prime filter of P.*
- *(iii) If $P \backslash M$ is an ideal of P, then $P \backslash M$ is a strongly prime ideal of P.*

Proof. (i) Let A and B be different proper ideals of P such that $L(A^*, B^*) \subseteq P \backslash M$. If $A \nsubseteq P \backslash M$ and $B \nsubseteq P \backslash M$, then $A \cap M \neq \phi$ and $B \cap M \neq \phi$ imply that $L(A^*, B^*) \cap M \neq \phi$, a contradiction.

(ii) Let $x, y \in M$. Then, $L(x) \cap M \neq \phi$ and $L(y) \cap M \neq \phi$, there exists $t \in L(x, y) \cap M$ with $U(L(x, y)) \subseteq U(t) \subseteq M$. So, M is a filter.

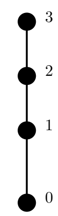
Let $U(a, b) \subseteq M$ for some $a, b \in P$. Then, $U(a) \cap M \neq \phi$ and $U(b) \cap M \neq \phi$ which imply there exists $a_1 \in U(a) \cap M$ and $b_1 \in U(b) \cap M$ such that

$L(L(a_1)^*, L(b_1)^*) \cap M \neq \phi$, so $L(L(a)^*, L(b)^*) \cap M \neq \phi$. Thus, $L(L(a)^*, L(b)^*) \nsubseteq P \backslash M$. By (i), we have $a \in M$ and $b \in M$. So, M is a prime filter.

(iii) It is trivial from (i). □

The following example shows the condition "$P \backslash M$ is an ideal of P "is not superficial in Lemma 2.1 (iii).

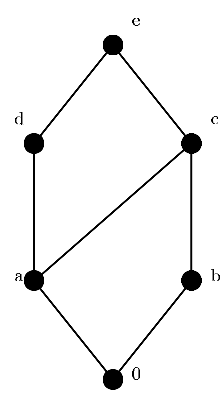
Example 2.2. Consider $P = \{0, 1, 2, 3\}$ and define a relation $\leq$ on P as follows.



Then, $(P, \leq)$ is a poset and $M = \{1, 2\}$ is a strongly m-system of P, but $P \backslash M$ is not an ideal of P. □

The below example shows that every prime filter of P need not to be strongly m-system of P in general.

Example 2.3. Consider $P = \{0, a, b, c, d, e\}$ and define a relation $\leq$ on P as follows.



Then, $(P, \leq)$ is a poset and $F = \{b, c, e\}$ is a prime filter of P, but not strongly m-system as $A = \{0, b\}$ and $B = \{0, a, b, c\}$ are the ideals of P with $A \cap F \neq \phi$ and $B \cap F \neq \phi$, but $L(A^*, B^*) \cap F = \phi$. □

In the papers [10], [11] and [13], authors related the concept of minimal prime ideal over an ideal I and the maximal multiplicative system disjoint from I in rings, semigroups and lattices. Following the above papers, we have some interesting results in posets.

Theorem 2.4. *Let I be an ideal of P. If $P \backslash I$ is a maximal strongly m-system of P, then I is a minimal strongly prime of P.*

Proof. Let I be an ideal of P such that $P\backslash I$ is a maximal strongly m-system of P. Then, I is strongly prime ideal. If J is a strongly prime ideal of P such that $J \subset I$, then $P\backslash I \subset P\backslash J$, a contradiction to the maximality of $P\backslash I$. □

Example 2.5. Let $n \in Z^+\backslash\{0,1\}$ and ρ be the "less than or equal "relation on set of integers. Then, $P_n = \{a : a \text{ is an integer and } a\rho n\}$ is a poset and $I_n = \{a : a\rho(n-1)\}$ is a minimal strongly prime ideal of P_n. Here $P_n\backslash I_n$ is not a maximal strongly m-system of P_n as $P_n\backslash I_n$ is contained in a strongly m-system $P_n\backslash\{0\}$ of P_n.

The above example shows that the converse of Theorem 2.4 is not true in general, but we have the following.

Theorem 2.6. *Let I be an ideal of P. If the complement of every strongly m-system of P is a semi-ideal of P and I is minimal strongly prime ideal, then $P\backslash I$ is a maximal strongly m-system of P.*

Proof. Let I be a minimal strongly prime ideal of P. Then, $P\backslash I$ is a strongly m-system of P. If there exists a strongly m-system M of P such that $P\backslash I \subset M$. Then, $P\backslash M \subset I$. We now prove $P\backslash M$ is an ideal of P. Let $x, y \in P\backslash M$ and $L(U(x,y)) \nsubseteq P\backslash M$. Then, there exists $t \in L(U(x,y)) \cap M$ with $U(x,y) \subseteq U(t) \subseteq M$ which implies that $U(x) \cap M \neq \phi$ and $U(y) \cap M \neq \phi$, there exists $t_1 \in U(x) \cap M$ and $t_2 \in U(y) \cap M$ such that $t_1, t_2 \in M$. Since M is strongly m-system, we have $L(L(t_1)^*, L(t_2)^*) \cap M \neq \phi$ which implies $L(L(x)^*, L(y)^*) \cap M \neq \phi$. Thus $L(L(x)^*, L(y)^*) \nsubseteq P\backslash M$. By Lemma 2.1(i), we have $x \in M$ and $y \in M$, a contradiction. So, $P\backslash M$ is an ideal of P. By Lemma 2.1(iii), we have $P\backslash M$ is a strongly prime ideal of P, a contradiction to the minimality of I. □

As a consequence of above theorem, we have the following.

Corollary 2.7. *Let M be a strongly m-system of P. If M is a semi-filter of P, then $P\backslash M$ is an ideal of P.*

Theorem 2.8. *Let $I \neq 0$ be an ideal of P satisfies $(*)$ condition and M be a strongly m-system of P. If M is semi-filter, then the following are equivalent:*

(i) M is a maximal strongly m-system of P with respect to $M \cap I = \phi$.

(ii) $P\backslash M$ is a minimal strongly prime ideal of P containing I.

(iii) For a strongly prime ideal $P\backslash M$ containing I, for each $x \in P\backslash M$, there exists $t \in U(x)$ and $y \in M$ such that $L(L(t)^*, L(y)^*) \subseteq I$.

Proof. (i)⇒(ii) It follows from Corollary 2.7 and Theorem 2.4, $P\backslash M$ is a minimal strongly prime ideal of P containing I.

(ii)⇒(iii) It is trivial from Theorem 2.2 of [3].

(iii)⇒(i) From (iii), we have M is a strongly m-system of P with $M \cap I = \phi$.

Suppose N is a strongly m-system of P such that $N \cap I = \phi$ and $M \subset N$. Then, there exists $a \in N \backslash M$, $y \in M$ and $t \in U(a)$ such that $L(L(t)^*, L(y)^*) \subseteq I$ which implies $L(y)^* \subseteq \langle L(t)^*, I\rangle \subseteq \langle L(a)^*, I\rangle$. So, $L(L(a)^*, L(y)^*) \subseteq I$. Since $y, a \in N$ and N is strongly m-system, we have $L(L(a)^*, L(y)^*) \cap N \neq \phi$ which implies $I \cap N \neq \phi$, a contradiction. □

Theorem 2.9. *Let I be an ideal of P and M be a strongly m-system of P such that $M \cap I = \phi$. Then, there exists a maximal strongly m-system N containing M with $N \cap I = \phi$.*

Proof. It follows from Theorem 2.1 of [3]. □

Lemma 2.10. *Let P be a poset and $r \in P$. If $P\backslash U(r)$ satisfies $(*)$ condition, then $U(r)$ is a strongly m-system of P.*

Proof. Let A and B be different proper ideals of P such that $A \cap U(r) \neq \phi$ and $B \cap U(r) \neq \phi$. Suppose $L(A^*, B^*) \cap U(r) = \phi$. Then, $L(A^*, B^*) \subseteq P\backslash U(r)$ and $B^* \subseteq \langle A^*, P\backslash U(r)\rangle = \bigcap_{a \in A^*} \langle a, P\backslash U(r)\rangle \subseteq \langle q, P\backslash U(r)\rangle \subseteq \langle r, P\backslash U(r)\rangle$ for some $q \in A \cap U(r)$. Since $U(r)$ is a m-system of P, then $P\backslash U(r)$ is a prime semi-ideal of P. By Theorem 20 of [8], we have $B^* \subseteq \langle r, P\backslash U(r)\rangle = P\backslash U(r)$, a contradiction. □

For any subset X of P, we define $V^{'}(X) = \{Q \in Smin(P) : X \subseteq Q\}$ and $D^{'}(X) = Smin(P)\backslash V^{'}(X)$.

Theorem 2.11. *Let A be a non empty subset of P and $J \neq \{0\}$ be an ideal of P. If every semi-ideal of P satisfies $(*)$ condition and every m-system of P is a semi-filter of P, then $\langle A, J\rangle = \bigcap\{Q : Q \in V^{'}(J) \cap D^{'}(A)\}$.*

Proof. Let $x \in \langle A, J\rangle$. Then, $L(a, x) \subseteq J$, for all $a \in A$. For $Q \in V^{'}(J) \cap D^{'}(A)$, there exists $a_1 \in A\backslash Q$ such that $L(L(x)^*, L(a_1)^*) \subseteq J \subseteq Q$ which implies $x \in Q$. Hence, $x \in \bigcap\{Q : Q \in V^{'}(J) \cap D^{'}(A)\}$.

Conversely, let $x \in \bigcap\{Q : Q \in V^{'}(J) \cap D^{'}(A)\}$ and $x \notin \langle A, J\rangle$. Then, $L(x, t) \nsubseteq J$ for some $t \in A$, so there exists $r \in L(x, t)\backslash J$ with $U(r) \cap J = \phi$. By Lemma 2.10, we have $U(r)$ is a strongly m-system such that $U(r) \cap J = \phi$. Then, by Theorem 2.9, there exists a maximal strongly m-system K of P containing $U(r)$ such that $K \cap J = \phi$ and, by Theorem 2.8, $P\backslash K \in V^{'}(J)$. Since $r \leq x$ and $r \in K$, we have $U(x) \subseteq U(r) \subseteq K$ which implies $x \notin \bigcap\{Q : Q \in V^{'}(J) \cap D^{'}(A)\}$, a contradiction. □

Theorem 2.12. *Let $J \neq \{0\}$ be an ideal of P. If every maximal m-system is a semi-filter of P and every semi-ideal satisfies $(*)$ condition, then J is a strongly semi-prime ideal of P.*

Proof. Let J be an ideal of P such that $L(A^*, B^*) \subseteq J$ and $L(A^*, C^*) \subseteq J$ for different proper ideals A, B, C of P. If $L(A^*, \cup(B^*, C^*)) \nsubseteq J$, then there exists $t \in L(A^*, U(B^*, C^*))\backslash J$ with $U(t) \cap J = \phi$. By Lemma 2.10 and Theorem

2.9, there exists a maximal strongly m-system K of P containing $U(t)$ of P such that $K \cap J = \phi$. Then, by Theorem 2.8, $P \backslash K \in V^{'}(J)$ which implies $L(A^*, B^*) \subseteq P \backslash K$ and $L(A^*, C^*) \subseteq P \backslash K$. Since $P \backslash K$ is strongly prime ideal, we have $A \subseteq P \backslash K$ or $B, C \subseteq P \backslash K$ which imply $L(L(a)^*, L(t)^*)) \subseteq P \backslash K$, for all $a \in A^*$ and $t \in L(U(B^*, C^*)$. Since $t \in U(t) \subseteq K$ with $t \leq a$ and K is strongly m-system, we have $L(L(a)^*, L(t)^*)) \cap K \neq \phi$, a contradiction. □

Following [6], for an ideal I and a strongly prime ideal Q of P, $I_Q = \{x \in P : L(x, y) \subseteq I \text{ for some } y \notin Q\}$.

Theorem 2.13. *Let I be a strongly prime ideal of P and $J \neq \{0\}$ be an ideal of P with $(*)$ condition. Then, the following statements are equivalent:*

(i) $I \in V^{'}(J)$.

(ii) *I contains precisely one of x or $\langle x, J \rangle$, for any $x \in P$.*

(iii) $\langle x, J \rangle \backslash I \neq \phi$, *for any* $x \in I$.

(iv) $J_I = I$.

Proof. (i)⇒(ii) Assume on the contrary that $\langle x, J \rangle \subseteq I$ for $x \in I$. Since $I \in V^{'}(J)$, we have by Theorem 2.2 of [3], for each $x \notin P \backslash I$, there exists $t \in U(x)$ and $y \in P \backslash I$ such that $L(L(t)^*, L(y)^*) \subseteq J$ which implies $L(y) \subseteq \langle L(t)^*, J \rangle \subseteq \langle L(x)^*, J \rangle \subseteq \langle x, J \rangle$. So, $y \in I$, a contradiction. If $x \notin I$, let $t \in \langle x, J \rangle$. Then, $L(L(t)^*, L(x)^*) \subseteq L(x, t) \subseteq J \subseteq I$. Since I is strongly prime ideal and $x \notin I$, we have $t \in I$.

(ii)⇒(iii) It is trivial.

(iii)⇒(iv) By the definition of J_I, we have $J_I \subseteq I$. Let $x \in I$. Then, $\langle x, J \rangle \nsubseteq I$ which implies there exists $t \in \langle x, J \rangle \backslash I$. Hence, $L(t, x) \subseteq J$ for some $t \notin I$. So, $x \in J_I$.

(iv)⇒(i) It is follows from Theorem 2.10 of [6]. □

Theorem 2.14. *Let $J \neq \{0\}$ be an ideal of P with $(*)$ condition and $I \in V^{'}(J)$. Then, $\langle \langle x, J \rangle, J \rangle \subseteq I$.*

Proof. Let $I \in V^{'}(J)$ and $x \in I$. Then, by Theorem 2.2 of [3], there exists $t \in U(x)$ and $y \in P \backslash I$ such that $L(L(t)^*, L(y)^*) \subseteq J$, so $y \in \langle L(t)^*, J \rangle \subseteq \langle L(x)^*, J \rangle \subseteq \langle x, J \rangle$. Suppose $\langle \langle x, J \rangle, J \rangle \nsubseteq I$. Then, there exists $z \in \langle \langle x, J \rangle, J \rangle \backslash I$. Now, for $y, z \in P \backslash I$, we have $L(L(z)^*, L(y)^*) \cap P \backslash I \neq \phi$ which implies $L(z, y) \cap P \backslash I \neq \phi$. Then, there exists $t \in L(y, z)$ and $t \in P \backslash I$. Since $z \in \langle \langle x, J \rangle, J \rangle$, we have $L(z, r) \subseteq J$, for all $r \in \langle x, J \rangle$ which imply $L(z, y) \subseteq J \subseteq I$, a contradiction. □

Theorem 2.15. *Let I be an ideal of P with $(*)$ condition and $M = \{x : \langle x, I \rangle = I\}$. Then, M is a strongly m-system of P.*

Proof. Let A and B be different proper ideals of P such that $A \cap M \neq \phi$ and $B \cap M \neq \phi$. Then, there exists $x \in A$ and $y \in B$ such that $x, y \in M$. Suppose $L(A^*, B^*) \cap M = \phi$. Then, for all $t \in L(A^*, B^*)$ there exists $r \in P \backslash I$ and $L(r, t) \subseteq I$ which implies $t \in \langle r, I \rangle$. So, $L(A^*, B^*) \subseteq \langle r, I \rangle$ which implies $L(A^*, B^*, r) \subseteq I$. Since I satisfies (*) condition, we have $L(B^*, r) \subseteq \langle A^*, I \rangle \subseteq \langle x, I \rangle = I$ which implies $r \in \langle B^*, I \rangle \subseteq \langle y, I \rangle = I$, a contradiction. □

Lemma 2.16. *Let I be an ideal of P. Then, $SP(I) = \{c \in P$: every strongly m-system in P which contains c has a non empty intersection with $I\}$.*

Proof. Let $H = \{c \in P$: every strongly m-system in P which contains c has a non empty intersection with $I\}$ and $c \notin H$. Then, there is a strongly m-system M of P which contains c and $M \cap I = \phi$. By Theorem 2.1 of [3], there exists a strongly prime ideal Q of P with $I \subseteq Q$ and $Q \cap M = \phi$ which implies $c \notin \cap Q_i$. So, $\cap Q_i \subseteq H$.

Conversely, let $c \notin \cap Q_i$. Then, there is a strongly prime ideal Q_i of P for some i such that $c \notin Q_i$ which implies $c \in P \backslash Q_i$ and $P \backslash Q_i$ is a strongly m-system of P. Since $P \backslash Q_i \cap I = \phi$, we have $c \notin H$. Hence, $H \subseteq \cap Q_i$. □

Theorem 2.17. *Let A and B be ideals of P. Then, the following statements hold:*

(*i*) $A \subseteq B$ *implies* $SP(A) \subseteq SP(B)$.

(*ii*) $SP(L(A^*, B^*)) = SP(A \cap B) = SP(A) \cap SP(B)$.

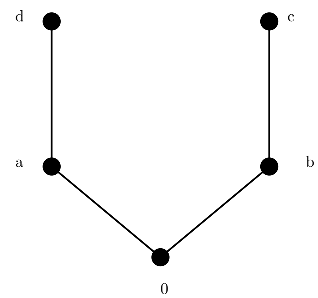
Proof. (i) It is trivial.

(ii) We have $L(A^*, B^*) \subseteq A \cap B \subseteq A$. Then, by (i), $SP(L(A^*, B^*)) \subseteq SP(A \cap B) \subseteq SP(A)$ which imply $SP(L(A^*, B^*)) \subseteq SP(A \cap B) \subseteq SP(A) \cap SP(B)$. Let $x \in SP(A) \cap SP(B)$ and K be a strongly m-system containing x. Then, by Lemma 2.16, $K \cap A \neq \phi$ and $K \cap B \neq \phi$. Since K is strongly m-system, we have $L(A^*, B^*) \cap K \neq \phi$ which implies $x \in SP(L(A^*, B^*))$. □

3. Strongly primary ideals

Theory of primary ideals played an important role in commutative ring theory. Because every ideal can be written as the intersection of finitely many primary ideals. In [1], A. Anjaneyulu developed the theory of primary ideals in arbitrary semigroup. Primary ideals in semigroup. In this section we study the notion of primary in poset. Following [1], we define $\sqrt{I} = \{x : L(x)^* \cap I \neq \phi\}$ for ideal I of P. An ideal I of P is called *primary* if $L(a, b) \subseteq I$ implies $a \in I$ or $b \in \sqrt{I}$. An ideal I of P is called *strongly primary* if $L(A^*, B^*) \subseteq I$ implies $A \subseteq I$ or $B \subseteq \sqrt{I} \cup \{0\}$ for different proper ideals A, B of P. Every strongly primary ideal of P is a primary ideal of P, and every strongly prime ideal of P is a strongly primary ideal of P. But the converse need not be true in each case in general.

Example 3.1. Consider $P = \{0, a, b, c, d, e\}$ and define a relation $\leq$ on P as follows.



Then, $(P, \leq)$ is a poset and $I = \{0, a\}$, $A = \{0, b\}$ and $B = \{0, a, d\}$ are ideals of P. Here I is a strongly primary ideal of P, but not a strongly prime as $L(A^*, B^*) \subseteq I$ with $A \nsubseteq I$ and $B \nsubseteq I$. □

Lemma 3.2. *Let A and B be ideals of P. Then, the following statements hold:*

(*i*) $A \subseteq \sqrt{A} \cup \{0\}$.

(*ii*) $\sqrt{\sqrt{A}} = \sqrt{A}$.

(*iii*) *If* $A \subseteq B$, *then* $\sqrt{A} \subseteq \sqrt{B}$.

(*iv*) $\sqrt{L(A^*, B^*)} = \sqrt{A \cap B} = \sqrt{A} \cap \sqrt{B}$.

The following theorem relates the strongly primary and strongly primness between I and $\sqrt{I}$.

Theorem 3.3. *Let I be a strongly primary ideal of P and $\sqrt{I} \cup \{0\}$ be an ideal of P. Then, $\sqrt{I} \cup \{0\}$ is a strongly prime ideal of P.*

Proof. Let A and B be different proper ideals of P such that $L(A^*, B^*) \subseteq \sqrt{I} \cup \{0\}$ and $A \nsubseteq \sqrt{I} \cup \{0\}$. Then, for all $t \in L(A^*, B^*)$, we have $t \in \sqrt{I} \cup \{0\}$ which imply $L(t)^* \cap I \neq \phi$. There exists $s \in I$ and $s \in A^*, B^*$ which imply $L(A^*, B^*) \subseteq L(s) \subseteq I$. Since I is strongly primary ideal and $A \nsubseteq I$, we have $B \subseteq \sqrt{I} \cup \{0\}$. □

The condition "$\sqrt{I} \cup \{0\}$ is an ideal of P "is not superficial in Theorem 3.3. In Example 2.3, if $I = \{0, b\}$, then $\sqrt{I} \cup \{0\} = \{0, b, c, e\}$ is not an ideal of P.

Definition 3.4. *Let Q be a strongly prime ideal of P. A strongly primary ideal I of P is said to be Q- strongly primary if $\sqrt{I} \cup \{0\} = Q$.*

Theorem 3.5. *Let $I_1, I_2, ..., I_n$ be Q-strongly primary ideals of P. Then, $\bigcap_{i=1}^{n} I_i$ is a Q-strongly primary ideal of P.*

Proof. Let $J = \bigcap_{i=1}^{n} I_i$. Then, $\sqrt{J} \cup \{0\} \subseteq \bigcap_{i=1}^{n} \sqrt{I_i} \cup \{0\}$ and $\bigcap_{i=1}^{n} \sqrt{I_i} \cup \{0\} \subseteq \sqrt{J} \cup \{0\}$ as $J \subseteq I_i \subseteq \sqrt{I_i}$. Since $I_i's$ are Q-strongly primary ideals, we have $\sqrt{J} \cup \{0\} = \bigcap_{i=1}^{n} \sqrt{I_i} \cup \{0\} = Q$. We now prove that J is a strongly primary ideal of P. Let A and B be different proper ideals of P such that $L(A^*, B^*) \subseteq J$ and $A \nsubseteq J$. Then, there is an ideal I_j of P such that $A \nsubseteq I_j$. Since $L(A^*, B^*) \subseteq J \subseteq I_j$ and I_j is strongly primary, we have $B \subseteq \sqrt{I_j} \cup \{0\} = Q = \sqrt{J} \cup \{0\}$. □

Theorem 3.6. *Let I be a strongly primary ideal of P. If I is a semi-prime ideal of P, then $\langle x, I \rangle$ is a strongly primary ideal of P for any $x \in P$.*

Proof. Let A and B be different proper ideals of P such that $L(A^*, B^*) \subseteq \langle x, I \rangle$ for any $x \in P \backslash I$. Then, $L(A^*, B^*, L(x)^*) \subseteq I$. If $L(A^*, B^*) \subseteq I$, then $A \subseteq I \subseteq \langle x, I \rangle$ or $B \subseteq \sqrt{I} \cup \{0\} \subseteq \sqrt{\langle x, I \rangle} \cup \{0\}$. If $L(A^*, B^*) \nsubseteq I$, then by Theorem 2.4 of [5], $L(A^*, L(x)^*) \subseteq I$ and $L(L(x)^*, B^*) \subseteq I$. Since I is primary and $x \notin I$, we have $A \subseteq \langle x, I \rangle \cup \{0\}$ and $B \subseteq \langle x, I \rangle \cup \{0\}$. □

Lemma 3.7. *Let I be an ideal of P and $I \subseteq Q$ for some strongly prime ideal Q of P. Then, $SP(I) \subseteq \sqrt{I} \cup \{0\}$.*

Proof. Let $x \in SP(I)$. Then, $x \in \bigcap_{I \subseteq Q_i} Q_i$, where Q_i's are strongly prime ideals of P which implies $L(Q_i) \cap I \neq \phi$ and $L(x) \cap I \neq \phi$, so $x \in \sqrt{I} \cup \{0\}$. Hence, $SP(I) \subseteq \sqrt{I} \cup \{0\}$. □

Theorem 3.8. *Let I be an ideal of P and $I \subseteq Q$ for some strongly prime ideal Q of P. Then, I is a strongly primary ideal of P.*

Theorem 3.9. *Let I be an ideal of P with $(*)$ condition and Q be a strongly prime ideal of P. If $I_Q \subseteq \sqrt{I} \cup \{0\}$, then I is strongly primary.*

Proof. Let $I_Q \subseteq \sqrt{I} \cup \{0\}$ and $L(A^*, B^*) \subseteq I$ with $A \nsubseteq I$ for different proper ideals A, B of P.
Case (i). If $I \subseteq Q$, then by Theorem 3.8, I is a strongly primary ideal of P.
Case (ii). Let $I \nsubseteq Q$. Then, there is $x \in I \backslash Q$. We now prove $B \subseteq \sqrt{I} \cup \{0\}$. Suppose not, $B \nsubseteq \sqrt{I} \cup \{0\}$. Since $I_Q \subseteq \sqrt{I} \cup \{0\}$, we have $B \nsubseteq I_Q$. Then, there exists $y \in B \backslash I_Q$ which implies $L(y, t) \nsubseteq I$, for all $t \notin Q$. In particular $L(x, y) \nsubseteq I$ which implies $B^* \nsubseteq \langle x, I \rangle = P$, a contradiction. □

Acknowledgment

The authors would like to express their gratitude to the referees for their insightful comments and suggestions for improving the paper.

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Accepted: July 30, 2021