# Chaotic dynamics of the Duffing-Holms model with external excitation

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**Abstract.** The dynamics of the Duffing-Holms model are researched, and the critical conditions for chaos of the model with external excitation are obtained using Melnikov method. The expression of Melnikov function is given. The results show that the criteria obtained for chaos motion in the sense of the Smale horseshoe is consistent with that obtained by the numerical simulation. Research shows the Melnikov function is an effective analytical method to judge the occurrence of chaotic motion.
**Keywords:** financial market, Duffing Holms model, chaos, Melnikov method.

## 1. Introduction

Financial market is a complex economic system. Financial risk can be divided into endogenous financial risk and exogenous financial risk. Endogenous financial risk refers to the financial risk generated in the process of commercial financial transactions. It mainly includes market risk, credit risk, liquidity risk and operational risk, etc., which are generally manifested as non systematic risks. Exogenous financial risk refers to the financial risk generated in the process of non-commercial financial transactions. It mainly includes currency risk, legal risk, policy risk, social and economic environment change risk, and most of them show systemic financial risk. From the characteristics and properties of financial risk, financial risk has the characteristics of uncertainty, universality, diffusion, concealment and suddenness. All these indicate that the financial system and

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its evolution have obvious characteristics of chaos. Chaos is a harmful form of movement, which may lead to the system out of control and cause the system to collapse completely. Chaos theory can provide a new method and idea for solving financial crisis and related problems.

Considering the real financial system as a discrete dynamic system and describing it with a nonlinear chaotic dynamic model, it can accurately reflect the operation and law of the financial system. In 1988, Ramsay [1] used Duffing-Holmes equation to test the fluctuation of M2 money in the US financial market from January 1959 to November 1987, and the stock fluctuation in the US commodity market from July 1962 to August 1985, the sufficient evidence of the existence of chaos is obtained. Therefore, Duffing-Holmes model has become an effective tool to study financial chaos. G. H. Zhou [2] uses Duffing-Holmes model to discuss the conditions and mechanism of financial system risk and some specific measures of risk control, and discusses the application of chaos theory in financial system. H. X. Yao, T. L. Shi [3] obtains the sufficient conditions of system hope bifurcation by using harmonic balance method and bifurcation theory. Using periodic excitation method and constant external excitation method, the chaotic behavior of the Duffing-Holms system can be effectively controlled to a stable periodic orbit or equilibrium point. Combined with the financial market, the relationship between the occurrence of chaos and financial crisis is explained. New order parameters are explored of the Duffing-Holms model in [4]. Xu, et al. studied the interaction effect among several financial factors in a financial system using mathematical models [5]. Their investigation showed that such model displays rich dynamical behaviours including chaos. Synchronization of the model were considered in their work. Zhang et al. studied a 4D chaos financial system [6]. Liao, et al. used a system of differential equations to model the evolution of a financial system and study its complexity [7].

Melnikov function is an effective analytical method for theoretical prediction of chaotic motions in nonlinear systems [8, 9]. This method can be used to analyze and judge whether Smale chaos occurs. Compared with the traditional numerical simulation methods of chaotic motion [10-15], Melnikov method can give analytical conditions for existence of chaotic motions.

In this paper, the dynamics of the Duffing-Holms model is investigated, and the critical conditions for chaos of the model with external excitation are obtained by Melnikov method. The expression of Melnikov function is given. Research shows the Melnikov function method is an effective analytical method to judge the occurrence of chaotic motion. The research results can better guide the government departments to make some strategic adjustments and avoid financial problems.

## 2. Duffing-Holms model and the equilibrium analysis

Duffing-Holms model is a nonlinear equation with oscillation introduced by Duffing in 1918. It is considered as a typical example of chaotic phenomena, and has been widely used in financial markets. Duffing-Holms model can accurately describe the various states of a complex system under different conditions [16],

$$\ddot{x} + \varepsilon\delta\dot{x} + ax + bx^3 = \varepsilon f \cos\omega t. \tag{1}$$

Where $x$ is the state of the financial market, $\dot{x}$ is the rate at which the state of the financial market changes, $\ddot{x}$ is the acceleration of changes in the state of the financial market, $\delta$ represents the government's ability to prevent financial risks, $\varepsilon$ is the control parameter of the policy, $f$ is the speculative disturbance parameter, $\omega$ is the self regulating ability of financial market order, $a, b$ are the coefficients of a cubic function, $a < 0, b > 0$.

Take the state variable of Eq. (1), let $(u, v) = (x, \dot{x})$ and reduce Eq. (1) to the first order differential equations

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -au - bu^3 - \varepsilon\delta v + \varepsilon f \cos(\omega\tau). \end{cases} \tag{2}$$

Because the exciting force $f$ is small, $-\varepsilon\delta v + \varepsilon f \cos(\omega\tau)$ is considered as disturbance terms. When $\varepsilon = 0$, Eq. (2) is called a conservative undisturbed system,

$$\begin{cases} \dot{u} = v, \\ \dot{v} = -au - bu^3. \end{cases} \tag{3}$$

When $a > 0$, $b > 0$, Eq. (3) has a unique equilibrium point $P_0(0,0)$. The characteristic roots of the linearized system are a pair of pure imaginary roots $\lambda_1 = \sqrt{a}\mathrm{i}$, , $\lambda_2 = -\sqrt{a}\mathrm{i}$ and the equilibrium point is a center. When $b < 0$, there are three equilibrium points $P_0\,(0,0)$, $P_1(\sqrt{-a/b}, 0)$, $P_2(-\sqrt{-a/b}, 0)$. $P_0(0,0)$ is a center, the characteristic roots of the linearized system corresponding to $P_0(0,0)$ are a pair of pure imaginary roots $\lambda_1 = \sqrt{a}\mathrm{i}$, $\lambda_2 = -\sqrt{a}\mathrm{i}$. For the other two non-zero equilibrium points $P_1(\sqrt{-a/b}, 0)$, $P_2(-\sqrt{-a/b}, 0)$, the characteristic roots of the linearized system are a pair of real roots, so the two non-zero equilibrium points are saddle points. When $a > 0$, the time process diagram $(t-u, t-v)$ and the phase diagram $(u-v)$ of the conservative undisturbed Eq. (3) are shown in Fig. 1-3. The system appears a period-1 motion. When $a < 0$, the time process diagram $(t-u, t-v)$ and the phase diagram $(u-v)$ of the conservative undisturbed Eq. (3) are shown in Fig. 4-6. We find that when the speculative disturbance parameter $f$ is zero, the financial system is in a stable state.

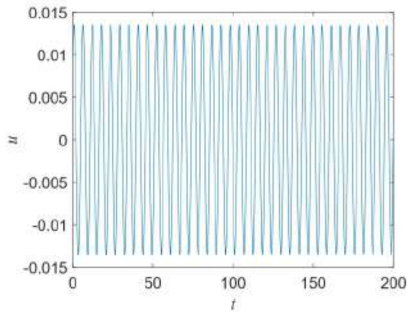



Figure 1: The time process diagram $(t-u)$.

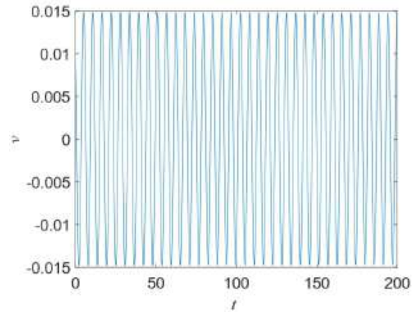



Figure 2: The time process diagram $(t-v)$.

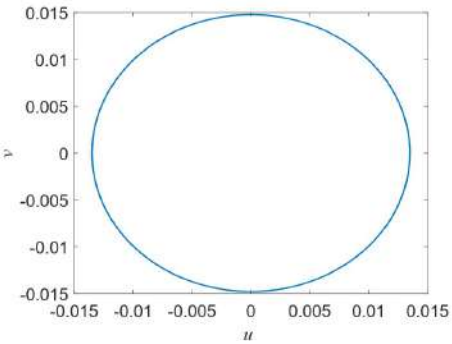



Figure 3: Phase diagram $(u-v)$ of the unperturbed system.

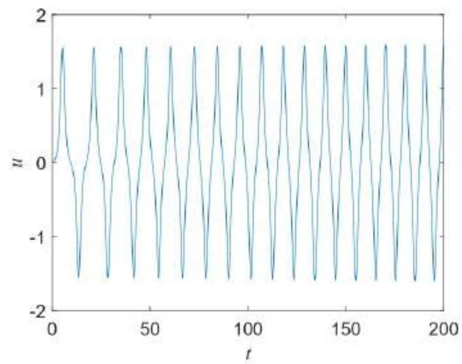



Figure 4: The time process diagram $(t-u)$.

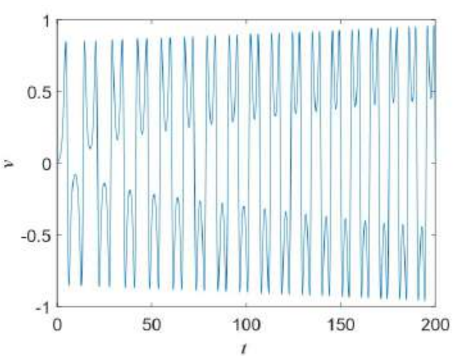



Figure 5: The time process diagram $(t-v)$.

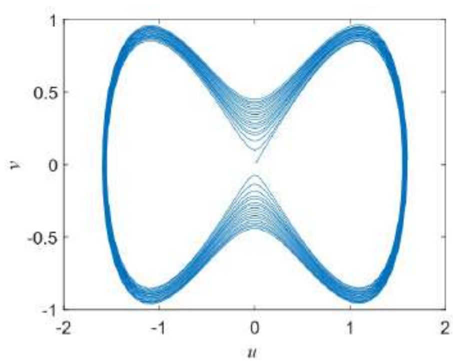



Figure 6: Phase diagram $(u-v)$ of the unperturbed system.

When $a < 0$, the homoclinic orbit of Hamilton system corresponding to Eq. (3) is

$$2\left(v^2 + au^2\right) + bu^4 = 4h. \tag{4}$$

According Eq. (4), the homoclinic orbits of the hyperbolic saddle points $P_1(\sqrt{-a/b}, 0)$, $P_2(-\sqrt{-a/b}, 0)$ of Eq. (3) are obtained,

$$q_0^{\pm}(\tau) : \begin{cases} u_0(\tau) = \pm\sqrt{-2a/b}\operatorname{sech}(\sqrt{-a}\tau), \\ v_0(\tau) = \pm a\sqrt{2/b}\operatorname{sech}(\sqrt{-a}\tau)\operatorname{th}(\sqrt{-a}\tau). \end{cases} \tag{5}$$

Where $(+)$ represents the positive axis part of the homoclinic orbits and $(-)$ represents the negative axis part of the homoclinic orbits.

## 3. Melnikov method for chaotic motion analysis of the system

When $\varepsilon = 0$, the homoclinic orbits of Eq. (2) appear under some range of parameters $a, b$. For the perturbed system, as $\varepsilon \neq 0$, the cross section homoclinic may appear. The quasi Hamiltonian system of Eq. (2) can be written

$$\dot{x} = f(x) + \varepsilon g(x, \tau). \tag{6}$$

Where $x = \begin{pmatrix} u \\ v \end{pmatrix} \in R^2$, $f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}$, $g = \begin{pmatrix} g_1(x,\tau) \\ g_2(x,\tau) \end{pmatrix}$, and $g(x, \tau)$ is a periodic function. Melnikov integral is constructed to determine the distance between stable and unstable manifolds, which is defined as

$$M_{\pm}(\tau_0) = \int_{-\infty}^{+\infty} f(q_0^{\pm}(\tau)) \wedge g(q_0^{\pm}(\tau), \tau + \tau_0) \exp[-\operatorname{trace} Df(q_0^{\pm}(s))ds]d\tau. \tag{7}$$

Where trace $Df(q_0^{\pm}(s)$ is the trace of the matrix $Df(q_0^{\pm}(s))$ and "$\wedge$" denotes the Possion symbol, which is defined as

$$f \wedge g = f_1 g_2 - f_2 g_1. \tag{8}$$

Since the unperturbed system is a Hamiltonian system, the trace $Df(q_0^{\pm}(s) \equiv 0$, Eq. (7) can be reduced,

$$M_{\pm}(\tau_0) = \int_{-\infty}^{+\infty} f(q_0^{\pm}(\tau)) \wedge g(q_0^{\pm}(\tau), \tau + \tau_0) d\tau. \tag{9}$$

Let

$$\begin{aligned} &f_1 = v,\ g_1 = 0, \\ &f_2 = -au - bu^3, \\ &g_2 = -\delta v + f\cos(\omega\tau). \end{aligned} \tag{10}$$

The Melnikov function corresponding to Eq. (2) is

$$
\begin{aligned}
M_{\pm}(\tau_0) &= \int_{-\infty}^{+\infty} v_0^{\pm}(\tau)\left(-\delta v_0^{\pm}(\tau) + f\cos\left(\omega\left(\tau+\tau_0\right)\right)\right)d\tau \\
&= -\delta\int_{-\infty}^{+\infty}\left[v_0^{\pm}(\tau)\right]^2 d\tau + f\int_{-\infty}^{+\infty} v_0^{\pm}(\tau)\sin\left(\omega\tau_0\right)\sin(\omega\tau)d\tau \\
&= -\delta B + f\sin\left(\omega\tau_0\right)A.
\end{aligned} \tag{11}
$$

The Melnikov integral can be calculated

$$
A = \int_{-\infty}^{+\infty} v_0^{\pm}(\tau)\sin(\omega\tau)d\tau, \quad B = \int_{-\infty}^{+\infty}\left[v_0^{\pm}(\tau)\right]^2 d\tau. \tag{12}
$$

According to Melnikov's theory [9, 16], if Melnikov's function has a simple zero point which does not depend on $\varepsilon$, there exists $t_0$ such that $M_{\pm}(t_0) = 0$. For a sufficiently small $\varepsilon$, on the Poincaré mapping of Eq. (2), there is a chaotic motion in the sense of Smale horseshoe. For a certain frequency $\omega$, if

$$
f/\delta > B/A. \tag{13}
$$

then Eq. (2) appears a Smale horseshoe type chaotic motion.

## 4. Numerical integration method for calculating Melnikov function

From the above, in order to get the threshold value $|B/A|$, the calculation of the Melnikov integral is very important. Although the analytical expression of the non perturbed homoclinic orbit can be obtained, it is difficult to get the Melnikov integral analytically, so it is necessary to use the numerical integration method to calculate it. Now we calculate the Melnikov integral by the methods in [17, 18]. The idea of this method is that the time variable $t$ is a function of the state variable $u$ of the homoclinic orbit. The Melnikov integral of the time variable $t$ can be transferred to the integral of the state variable $u$, and then it can be solved by the numerical calculation.

If $\tau > 0$, according to Eq. (4), we obtain,

$$
\frac{du}{d\tau} = \mp\sqrt{h - au^2 - \frac{b}{2}u^4}. \tag{14}
$$

On the homoclinic orbit $q_0^{\pm}$, for Eq. (10), we separate variables, and integrate both sides, then we obtain,

$$
\tau = \mp\int_{u_{1,2}}^{u}\frac{d\xi}{\sqrt{h - a\xi^2 - \frac{b}{2}\xi^4}}. \tag{15}
$$

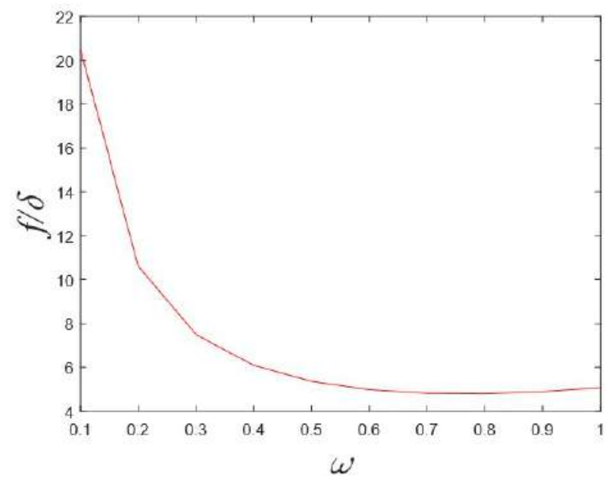



Figure 7: The time process diagram $(t-v)$.

It follows from Eq. (14, 15), and Eq. (12),

$$A = 2\int_0^{+\infty} v_0^{\pm}(\tau)\sin(\omega\tau)d\tau = 2\int_{u_{1,2}}^{u} \sin(\omega\tau)d\tau$$

(16)

$$= \mp\int_{u_{1,2}}^{u} \sin\left(\omega\frac{d\xi}{\sqrt{h-a\xi^2-\frac{b}{2}\xi^4}}\right) du.$$

(17) $$B{=}2\int_0^{+\infty}\left[v_0^{\pm}(\tau)\right]^2 d\tau{=}2\int_{u_{1,2}}^{u} v_0^{\pm}(\tau)du{=}\mp 2\int_{u_{1,2}}^{u}\left(h{-}au^2{-}\frac{b}{2}u^4\right)du.$$

When the frequency $\omega$ of the applied force is in the interval $[0,1]$, the complex Simpson formula is used to integrate $u$ in 1000 steps and $u$ in 500 steps, then the $A$ and $B$ values corresponding to each value $\omega$ can be obtained. We can get the Melnikov threshold value with $\omega$. When the ratio of the speculative disturbance parameter $f$ and the government's ability to prevent financial risks $\delta$ is greater than the value of $|B/A|$, the chaotic motion of the system appears in the sense of Smale horseshoe. The financial system will change from stable equilibrium to chaos, which will cause financial system shock or economic crisis.

(18) $$f/\delta > 4/(\sqrt[3]{2}\pi\omega\sec h(\pi\omega/2)).$$

## 5. Numerical simulation of the system

When $\omega \neq 0$, when the market is disturbed by changing external force, the system will gradually lose its stability, generate bifurcation and finally generate chaos. The bifurcation diagrams of $u$ and $v$ with the change of external force f are shown in Fig. 8-9, as $\omega = 1.5$. It shows the complex dynamic behavior of Eq. (2). With the change of the ratio of the speculative disturbance parameter, the stability of the state of the financial market and the rate at which the state of the financial market changes will change.

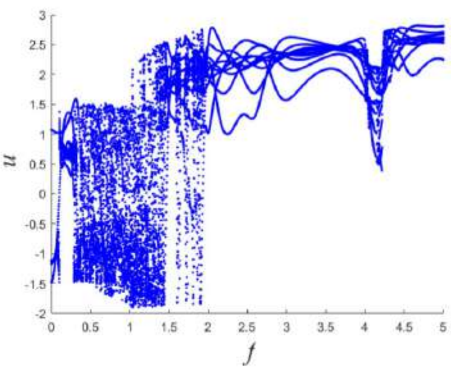



Figure 8: The bifurcation diagram $(f-u)$ of Eq. (2).

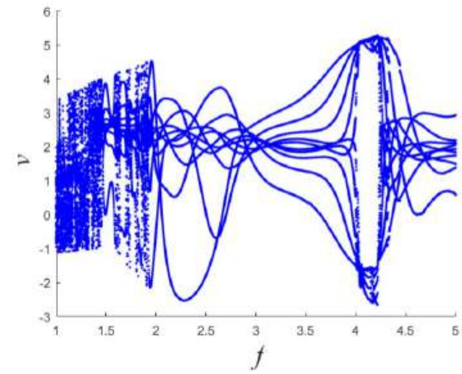



Figure 9: The bifurcation diagram $(f-v)$ of Eq. (2).

Let $a = -1.2, \varepsilon = 0.8, \delta = 0.2, b = 1, f = 1.5$, when $\omega = 1.5$, the Eq. (2) appears chaotic motion. The time process diagram $(t-u)$ and the phase diagram $(u-v)$ of the conservative undisturbed Eq. (2) are shown in Fig. 10-11.

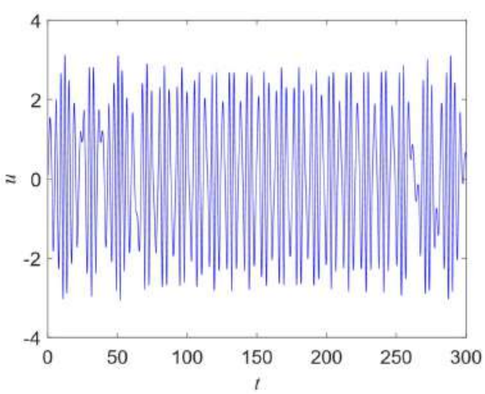



Figure 10: The time process diagram $(t-u)$ of Eq. (2).

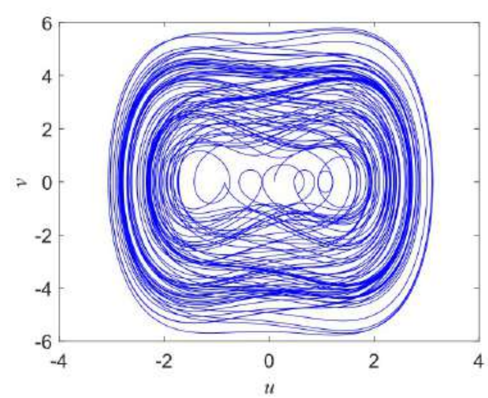



Figure 11: The phase diagram $(u-v)$ of Eq. (2).

When $\omega = 1.85$, the time process diagram $(t-u)$ and the phase diagram $(u-v)$ of the conservative undisturbed Eq. (2) are shown in Fig. 12-13. The chaotic attractor of the system on Poincaré section is shown in Fig. 14. It can be seen from the calculation results that the critical value of the parameters obtained from the numerical simulation results is consistent with the critical value determined by Melnikov method.

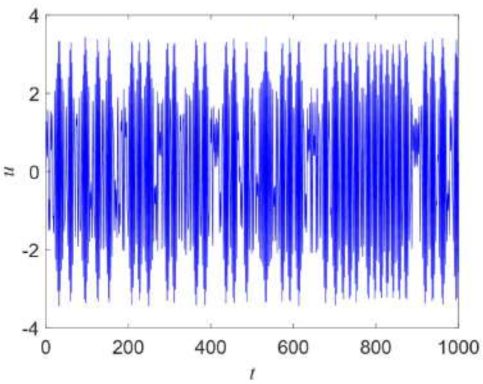



Figure 12: The time process diagram $(t-u)$ of Eq. (2).

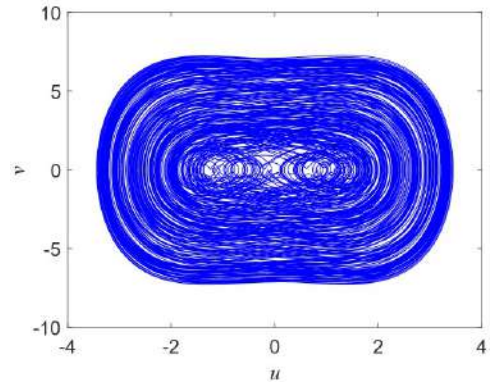
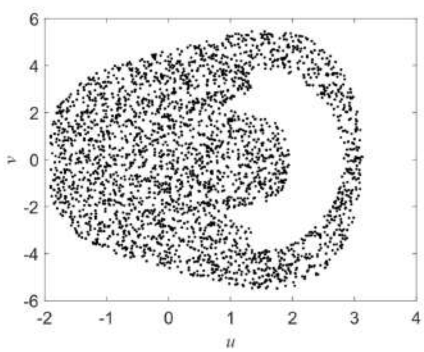
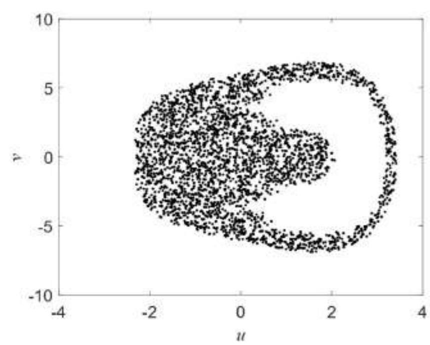



Figure 13: The phase diagram $(u-v)$ of Eq. (2).




(a) $\omega = 1.5$




(b) $\omega = 1.85$

Figure 14: Chaotic attractor on Poincaré section of Eq. (2)

When the dynamic property of chaos appears in the Duffing-Holms model, that is, the financial system appears chaos or economic crisis, the government's ability to prevent financial risks and the control parameter of the policy need to be adjusted and made to solve the current crisis.

## 6. Conclusion

The dynamics of the Duffing-Holms model are researched, and the critical conditions for chaos of the model with external excitation are obtained using Melnikov method. The results show that the criteria obtained for chaos motion by the Smale horseshoe mapping is consistent with that obtained by the numerical simulation. Research shows the Melnikov function is an effective analytical method to judge the occurrence of chaotic motion.

The research results can better guide the government departments to make strategic adjustments and avoid financial crisis. First of all, it is necessary to enhance the government's ability $\delta$ to prevent financial crisis, that is, to increase the government's foreign exchange reserves, reduce foreign debts, and improve the capital structure, so as to prevent the international financial giants from speculating on foreign exchange and stocks in a country or a region's foreign exchange market and stock market. Secondly, when $\delta$ becomes great under the interference of international speculators, the financial market will break away from the undisturbed track and enter into an unstable state. Thirdly, the financial system reform should be gradual, the policy parameter $\varepsilon$ should be adjusted, and the psychological preparation for medium and long-term governance of the financial market should be made. When a country's financial market is in an unstable or chaotic state, the government should not make sudden and significant changes to the financial system (such as the exchange rate system), but should take a small range of policy fine-tuning measures, and actively strive for a large amount of financial assistance from international financial organizations.

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