# Upper and lower estimates for products of two hyperbolic $p$-convex functions

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**Abstract.** In this study, we obtain upper and lower estimates for product of two hyperbolic $p$-convex functions, which is analogous to Hermite-Hadamard type inequalities for product of two hyperbolic $p$-convex functions.
**Keywords:** convex functions, hyperbolic p-convex functions, Hadamard's inequality.

## 1. Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considerable significant in the literature (see [7], [8]). These inequalities state that if $f : I \to \mathbb{R}$ is a convex function on the interval $I$ of real numbers and $a, b \in I$ with $a < b$, then

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)dx \le \frac{f(a)+f(b)}{2}. \tag{1}$$

Both inequalities hold in the reversed direction if f is concave. We note that Hermite-Hadamard inequality may be regarded as a refinement of the concept of convexity and it follows easily from Jensen's inequality. Over the last twenty

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years, the numerous studies have focused on to establish generalization of the Inequality (1) and to obtain new bounds for left hand side and right hand side of the Inequality (1) (see [12], [17], [6], [22], [16] and [13]).

In [20], Pachpatte established some new integral inequalities analogous to that of Hadamard's inequality given in (1) involving two convex functions.

**Theorem 1.1** ([20]). *Let $f$ and $g$ be real-valued, non-negative and convex functions on $[a,b]$. Then we have*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \le \frac{1}{3}M(a,b)+\frac{1}{6}N(a,b), \tag{2}$$

*and*

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right)-\frac{1}{3}M(a,b)-\frac{1}{6}N(a,b) \le \frac{1}{b-a}\int_a^b f(x)g(x)dx, \tag{3}$$

*where $M(a,b)=f(a)g(a)+f(b)g(b)$ and $N(a,b)=f(a)g(b)+f(b)g(a)$.*

Over the year, the generalized versions of Inequalities (2) and (3) for several convexity has been proved. For some of them please refer to ([16], [4], [14], [15], [21] and [23]). In [3], on the other hand, F. Chen proved Hermite-Hadamard type inequalities for product of two convex functions via Riemann-Liouville fractional integrals. For the other results of this topic, please refer to ([2], [18] and [19]). In [1], Mohamed S. S. Ali introduced the definition of hyperbolic p-convex functions as follows:

**Definition 1.1** ([1]). *A function $f: I \to \mathbb{R}$ is said to be sub $H$-function on $I$ or hyperbolic p-convex function, if for any arbitrary closed subinterval $[a,b]$ of $I$ the graph of $f(x)$ for $x \in [a,b]$ lies nowhere above the graph of function, determined by the equation:*

$$H(x)=H(x,a,b,f)=A\cosh px+B\sinh px,\quad p\in\mathbb{R}\setminus\{0\},$$

*where $A$ and $B$ are chosen such that $H(a)=f(a)$, and $H(b)=f(b)$.*

*Equivalently, for all $x\in[a,b]$*

$$f(x)\le H(x)=\frac{f(a)\sinh p(b-x)+f(b)\sinh p(x-a)}{\sinh p(b-a)}. \tag{4}$$

*For $x=(1-t)a+tb$, $t\in[a,b]$, the condition (4) becomes*

$$f((1-t)a+tb)\le \frac{\sinh[p(1-t)(b-a)}{\sinh[p(b-a)])}f(a)+\frac{\sinh[pt(b-a)]}{\sinh[p(b-a)]}f(b). \tag{5}$$

*If the inequality (4) holds with $'' \geq''$, then the function is called hyperbolic p-concave on $I$.*

For some properties and results concerning the class of hyperbolic $p$-convex functions (see [1], [6], [10], [12], [11], [9] and [5]).

In 2016, Mohamed S. S. Ali introduced the following Hermite-Hadamard inequality for hyperbolic p-convex functions. The analysis of this proof is depending on the geometric observation which shows that the graph of the function $f(x)$ lies between the chord $H(x)$ and the support $S_u(x)$ (see [1]). In 2018, Dragomir proved Hermite-Hadamard Inequality (1) for hyperbolic $p$-convex functions with different method (see [6]).

**Theorem 1.2** ([1])**.** *Assume that the function $f : I \to \mathbb{R}$ is hyperbolic $p$-convex on $I$ and $a, b \in I$ with $a < b$. Then, one has the inequality*

$$\frac{2}{p} f\left(\frac{a+b}{2}\right) \sinh p\left(\frac{b-a}{2}\right) \leq \int_a^b f(x)dx \leq \frac{1}{p}[f(a)+f(b)] \tanh p(\frac{b-a}{2}). \tag{6}$$

**Theorem 1.3** ([6])**.** *Assume that the function $f : I \to \mathbb{R}$ is hyperbolic $p$-convex on $I$ and $a, b \in I$ with $a < b$. Then, one has the inequality*

$$f(\frac{a+b}{2}) \leq \int_a^b f(x) sech\left[p\left(x - \frac{a+b}{2}\right)\right] dx \leq \frac{f(a)+f(b)}{2}. \tag{7}$$

The aim of this paper is to establish new integral inequalities for product of two hyperbolic $p$-convex functions.

## 2. Results

**Theorem 2.1.** *If $f$, $g : I \to \mathbb{R}$ are two real-valued, non-negative and hyperbolic $p$-convex functions on $I$, then for any $a, b \in I$, we have*

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)g(x)dx \quad &\leq \quad \frac{M(a,b)}{2}\left[\frac{\coth[p(b-a)]}{p(b-a)} - csch^2[p(b-a)]\right] \\ &+ \quad \frac{N(a,b)}{2\sinh[p(b-a)]}\left[\coth[p(b-a) - \frac{1}{p(b-a)}\right], \end{aligned}$$

*where*

$$M(a,b) = f(a)g(a) + f(b)g(b), \quad N(a,b) = f(a)g(b) + f(b)g(a).$$

**Proof.** Since $f$ and $g$ are hyperbolic $p$-convex functions on $[a, b]$, then from (5), we have

$$f((1-t)a + tb) \leq \frac{\sinh[p(1-t)(b-a)]}{\sinh[p(b-a)]} f(a) + \frac{\sinh[pt(b-a)]}{\sinh[p(b-a)]} f(b), \tag{8}$$

and

$$g((1-t)a + tb) \leq \frac{\sinh[p(1-t)(b-a)]}{\sinh[p(b-a)]} g(a) + \frac{\sinh[pt(b-a)]}{\sinh[p(b-a)]} g(b). \tag{9}$$

By using (8) and (9), we have

$$
\begin{aligned}
f((1-t)a+tb)g((1-t)a+tb) &\le \frac{\sinh^2[p(1-t)(b-a)]}{\sinh^2[p(b-a)]} f(a)g(a) \\
&+\frac{\sinh[pt(b-a)]}{\sinh[p(b-a)]}\frac{\sinh[p(1-t)(b-a)]}{\sinh[p(b-a)]}[f(a)g(b)+f(b)g(a)] \\
&+\frac{\sinh^2[pt(b-a)]}{\sinh^2[p(b-a)]} f(b)g(b). \qquad (10)
\end{aligned}
$$

Integrating the both sides of (10) with respect to $t$ from 0 to 1, then we obtain

$$
\int_0^1 f((1-t)a+tb)g((1-t)a+tb)dt \le \qquad (11)
$$

$$
\begin{aligned}
&\le \frac{f(a)g(a)}{\sinh^2[p(b-a)]}\int_0^1 \sinh^2[p(1-t)(b-a)]dt \\
&+\frac{f(a)g(b)+f(b)g(a)}{\sinh^2[p(b-a)]}\int_0^1 \sinh[pt(b-a)]\sinh[p(1-t)(b-a)]dt \\
&+\frac{f(b)g(b)}{\sinh^2[p(b-a)]}\int_0^1 \sinh^2[pt(b-a)]dt. \qquad (12)
\end{aligned}
$$

By changing of variable $x=(1-t)a+tb$, we get

$$
\int_0^1 f((1-t)a+tb)g((1-t)a+tb)dt = \frac{1}{b-a}\int_a^b f(x)g(x)dx. \qquad (13)
$$

Moreover, it is easy observe that

$$
\begin{aligned}
\int_0^1 \sinh^2[p(1-t)(b-a)]dt &= \int_0^1 \sinh^2[pt(b-a)]dt \qquad (14) \\
&= \frac{1}{4p(b-a)}\sinh[2p(b-a)] - \frac{1}{2} \qquad (15)
\end{aligned}
$$

and

$$
\int_0^1 \sinh[pt(b-a)]\sinh[p(1-t)(b-a)]dt = \frac{1}{2}\left[\cosh p(b-a) - \frac{\sinh p(b-a)}{p(b-a)}\right]. \qquad (16)
$$

By substituting by the Equalities (13), (14) and (16) in (11), then we have

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b f(x)g(x)dx &\le \frac{f(a)g(a)+f(b)g(b)}{2\sinh^2[p(b-a)]}\left[\frac{\sinh[2p(b-a)]}{2p(b-a)} - 1\right] \\
&+\frac{f(a)g(b)+f(b)g(a)}{2\sinh^2[p(b-a)]}\left[\cosh[p(b-a)] - \frac{\sinh[p(b-a)]}{p(b-a)}\right].
\end{aligned}
$$

Hence

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(x)g(x)dx &\leq \frac{M(a,b)}{2}\left[\frac{\coth[p(b-a)]}{p(b-a)} - csch^2[p(b-a)]\right] \\ &+ \frac{N(a,b)}{2\sinh[p(b-a)]}\left[\coth[p(b-a) - \frac{1}{p(b-a)}\right].\end{aligned}$$

This completes the proofs. □

**Remark 2.1.** For $p \to 0$, we observe that

$$\lim_{p\to 0}\frac{1}{2}\left[\frac{\coth[p(b-a)]}{p(b-a)} - csch^2[p(b-a)]\right] = \frac{1}{3}, \tag{17}$$

and

$$\lim_{p\to 0}\frac{1}{2\sinh[p(b-a)]}\left[\coth[p(b-a) - \frac{1}{p(b-a)}\right] = \frac{1}{6}. \tag{18}$$

**Corollary 2.1.** *With the notations in Theorem 2.1, if $p \to 0$, then*

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \leq \frac{1}{3}M(a,b) + \frac{1}{2}N(a,b), \tag{19}$$

*which is analogous to inequality* (2) *in case of convex functions.*

**Theorem 2.2.** *If $f$, $g : I \to \mathbb{R}$ are two real-valued, non-negative and hyperbolic $p$-convex functions on $I$, then for any $a, b \in I$, we have*

$$\begin{aligned}&2\cosh^2\left[\frac{p(b-a)}{2}\right] f\left(\frac{a+b}{2}\right) g\left(\frac{a+b}{2}\right) \\ &-\frac{M(a,b)}{2\sinh p(b-a)}\left[\coth p(b-a) - \frac{1}{p(b-a)}\right] \\ &-\frac{N(a,b)}{2}\left[\frac{\coth p(b-a)}{p(b-a)} - csch^2 p(b-a)\right] \leq \frac{1}{b-a}\int_a^b f(x)g(x)dx,\end{aligned}$$

*where*

$$M(a,b) = f(a)g(a) + f(b)g(b), \quad N(a,b) = f(a)g(b) + f(b)g(a).$$

**Proof.** For $t \in [a,b]$, we can write

$$\frac{a+b}{2} = \frac{(1-t)a+tb}{2} + \frac{ta+(1-t)b}{2}.$$

Using the hyperbolic $p$-convexity of $f$ and $g$, we have

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) = f\left(\frac{(1-t)a+tb}{2}+\frac{ta+(1-t)b}{2}\right)\\
&\times g\left(\frac{(1-t)a+tb}{2}+\frac{ta+(1-t)b}{2}\right)\\
&\le \left[\frac{\sinh[\frac{p(b-a)}{2}]}{\sinh[p(b-a)]}f((1-t)a+tb)+\frac{\sinh[\frac{p(b-a)}{2}]}{\sinh[p(b-a)]}f(ta+(1-t)b)\right]\\
&\times \left[\frac{\sinh[\frac{p(b-a)}{2}]}{\sinh[p(b-a)]}g((1-t)a+tb)+\frac{\sinh[\frac{p(b-a)}{2}]}{\sinh[p(b-a)]}g(ta+(1-t)b)\right]\\
&= \frac{\sinh^2[\frac{p(b-a)}{2}]}{\sinh^2[p(b-a)]}[f((1-t)a+tb)g((1-t)a+tb)+f(ta+(1-t)b)g(ta+(1-t)b)]\\
&+ \frac{\sinh^2[\frac{p(b-a)}{2}]}{\sinh^2[p(b-a)]}[f((1-t)a+tb)g(ta+(1-t)b)+f(ta+(1-t)b)g((1-t)a+tb)].
\end{aligned}$$

For the second expression in the last equality, by using again the hyperbolic $p$-convexity of $f$ and $g$, we obtain

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \le \frac{\sinh^2[\frac{p(b-a)}{2}]}{\sinh^2[p(b-a)]}\Big[f((1-t)a+tb)g((1-t)a+tb)\\
&+f(ta+(1-t)b)g(ta+(1-t)b)\Big]\\
&+\frac{\sinh^2[\frac{p(b-a)}{2}]}{\sinh^2[p(b-a)]}\left\{\left[\frac{\sinh[p(1-t)(b-a)]}{\sinh[p(b-a)]}f(a)+\frac{\sinh[pt(b-a)]}{\sinh[p(b-a)]}f(b)\right]\right.\\
&\times\left[\frac{\sinh[pt(b-a)]}{\sinh[p(b-a)]}g(a)+\frac{\sinh[p(1-t)(b-a)]}{\sinh[p(b-a)]}g(b)\right]\\
&+\left[\frac{\sinh[pt(b-a)]}{\sinh[p(b-a)]}f(a)+\frac{\sinh[p(1-t)(b-a)]}{\sinh[p(b-a)]}f(b)\right]\\
&\left.\times\left[\frac{\sinh[p(1-t)(b-a)]}{\sinh[p(b-a)]}g(a)+\frac{\sinh[pt(b-a)]}{\sinh[p(b-a)]}g(b)\right]\right\}. \qquad (20)
\end{aligned}$$

Hence,

$$\begin{aligned}
&f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \le \frac{\sinh^2[\frac{p(b-a)}{2}]}{\sinh^2[p(b-a)]}\Big[f((1-t)a+tb)g((1-t)a+tb)\\
&+f(ta+(1-t)b)g(ta+(1-t)b)\Big]\\
&+\frac{\sinh^2[\frac{p(b-a)}{2}]}{\sinh^2[p(b-a)]}\left\{\frac{2M(a,b)}{\sinh^2[p(b-a)]}\sinh[pt(b-a)]\sinh[p(1-t)(b-a)]\right.\\
&\left.+\frac{N(a,b)}{\sinh^2[p(b-a)]}[\sinh^2[pt(b-a)]+\sinh^2[p(1-t)(b-a)]]\right\}. \qquad (21)
\end{aligned}$$

Integrating the both sides of (21) with respect to $t$ from 0 to 1, and by using the equalities (13), (14) and (16), we get

$$
\begin{aligned}
f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) &\leq \frac{\sinh^2[\frac{p(b-a)}{2}]}{\sinh^2[p(b-a)]}\frac{2}{b-a}\int_a^b f(x)g(x)dx \\
&+ \frac{\sinh^2[\frac{p(b-a)}{2}]}{\sinh^2[p(b-a)]}\frac{M(a,b)}{\sinh^2[p(b-a)]}\left[\cosh p(b-a) - \frac{\sinh[p(b-a)]}{p(b-a)}\right] \\
&+ \frac{\sinh^2[\frac{p(b-a)}{2}]}{\sinh^2[p(b-a)]}\frac{N(a,b)}{\sinh^2[p(b-a)]}\left[\frac{\sinh[2p(b-a)]}{2p(b-a)} - 1\right].
\end{aligned} \tag{22}
$$

By multiplying the both sides of (22) by $\frac{\sinh^2[p(b-a)]}{2\sinh^2[\frac{p(b-a)}{2}]}$, then, we obtain the desired inequality

$$
\begin{aligned}
&2\cosh^2\left[\frac{p(b-a)}{2}\right]f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) \\
&-\frac{M(a,b)}{2\sinh p(b-a)}\left[\coth p(b-a) - \frac{1}{p(b-a)}\right] \\
&-\frac{N(a,b)}{2}\left[\frac{\coth p(b-a)}{p(b-a)} - csch^2 p(b-a)\right] \leq \frac{1}{b-a}\int_a^b f(x)g(x)dx. \qquad \square
\end{aligned}
$$

**Corollary 2.2.** *With the notations in Theorem 2.2, if $p \to 0$, then*

$$2f\left(\frac{a+b}{2}\right)g\left(\frac{a+b}{2}\right) - \frac{1}{6}M(a,b) - \frac{1}{3}N(a,b) \leq \frac{1}{b-a}\int_a^b f(x)g(x)dx,$$

*which is analogous to inequality* (3) *in case of convex functions.*

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