The quasi frame and equations of non-lightlike curves in Minkowski $\mathbb{E}_1^3$ and $\mathbb{E}_1^4$

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Abstract. The quasi frame is an alternate frame to the Frenet-Serret frame but it is defined when the second derivative of the curve vanishes. It has the same behavior as a parallel transport frame but is easier in computation and has the same accuracy. In this paper, we investigate the quasi frame and equations of non-lightlike curves in 3-dimensional Minkowski space $\mathbb{E}_1^3$ and in 4-dimensional Minkowski space-time $\mathbb{E}_1^4$. Furthermore, we show the quasi frame can be considered as a generalization of Bishop frame in $\mathbb{E}_1^3$ and $\mathbb{E}_1^4$.
Keywords: Minkowski space, spacelike curve, timelike curve, Bishop frame.

1. Introduction

The Frenet frame was created to study the behavior of curves. The two curvatures $\{\kappa_i(s)\ |i = 1, 2\}$ in $\mathbb{E}^3$ (the three curvatures $\{\kappa_i(s)\ |i = 1, 2, 3\}$ in $\mathbb{E}^4$) play an effective role to identify the shape and size of the curve. The main disadvantage that appeared on this frame is when the second derivative in $\mathbb{E}^3$ (one of the curvatures $\{\kappa_i(s)\ |i = 1, 2, 3\}$ in $\mathbb{E}^4$) of a curve vanishes i.e. if the curve was

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a straight line or at an inflection point, the Frenet frame in these cases becomes undefined [1].

In 1975, R. Bishop created a frame that called an alternative frame or parallel transport frame that is well defined when the second derivative in $\mathbb{E}^3$ (one of the curvatures $\{\kappa_i(s) \mid i = 1, 2, 3\}$ in $\mathbb{E}^4$) is zero. This frame was known as Bishop frame [1]. The idea of Bishop in $\mathbb{E}^3$ ($\mathbb{E}^4$) based on the observation of a tangent vector field takes place in the same direction and the other vector fields take place in a plane perpendicular to the tangent vector field so, their derivatives take the same direction of the tangent vector field.

In 1983, Bishop and Hanson gave the advantages of a parallel transport frame [7] and regarded it as a developed frame of the Frenet frame. Many researchers have been using Bishop's concepts. In Euclidean space, see [3, 6]; in Minkowski space, see [2, 12]; In dual space, see [9] and this frame is developed to study of canal and tubular surfaces, see [8].

In 2015, C. Ekici and H. Tozak [4] defined a framing alternative to the Frenet-Serret frame called the quasi frame. The behavior of the quasi frame is similar to Bishop Frame but it is easier in computing, although both frames have similar accuracy. In 2020, M. Khalifa and R. A. Abdel-Baky used the quasi frame to study the skew ruled surfaces in Euclidean space[10].

In this paper, we investigate the quasi frame and equations of non-lightlike curves in 3-dimensional Minkowski space $\mathbb{E}_1^3$ and in 4-dimensional Minkowski space-time $\mathbb{E}_1^4$. Furthermore, we show the quasi frame can be considered as a generalization of Bishop frame in $\mathbb{E}_1^3$ and $\mathbb{E}_1^4$. This paper is organized as follows: In section 2, some basic definitions of the frame and equations of Frenet are presented in 3-dimensional Minkowski space $\mathbb{E}_1^3$ and 4-dimensional Minkowski space-time $\mathbb{E}_1^4$. In section 3, we investigate the quasi equations in 3-dimensional Minkowski space $\mathbb{E}_1^3$ in the three different cases of a non-lightlike curve by using the transformation matrix between the quasi and Frenet frames. In section 4, we investigate the quasi equations in 4-dimensional Minkowski space $\mathbb{E}_1^4$ in the four different cases of a non-lightlike curve by using the transformation matrices between the quasi and Frenet frames.

2. Preliminaries

The Minkowski space $\mathbb{E}_1^3$ is the space $\mathbb{R}^3$ with a metric g, where g is defined by $g = -dx_1^2 + dx_2^2 + dx_3^2$, where (x_1, x_2, x_3) is a coordinate system of $\mathbb{E}_1^3$. If $v \in \mathbb{E}_1^3$ then, the vector v is called a spacelike, a timelike or a lightlike(null), if $g(v, v) > 0$, $g(v, v) < 0$ or $g(v, v) = 0$ and $v \neq 0$, respectively. In particular, the vector $v = 0$ is a spacelike.

Let $\alpha(s)$ be any curve in Minkowski $\mathbb{E}_1^3$, then frenet equations are given by

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 \\ \epsilon_1\kappa_1 & 0 & \epsilon_2\kappa_2 \\ 0 & \epsilon_3\kappa_2 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}. \tag{1}$$

• If $\epsilon_1 = -1$ and $\epsilon_i = 1$ for $(i = 2, 3)$ then, the curve is spacelike with spacelike principal normal.

• If $\epsilon_i = 1$ for $(i = 1, 2, 3)$ then, the curve is spacelike with spacelike binormal.

• If $\epsilon_3 = -1$ and $\epsilon_i = 1$ for $(i = 1, 2)$ then, the curve is timelike.

The Minkowski space $\mathbb{E}_1^4$ is the space $\mathbb{R}^4$ with a metric g, where g is defined by
$g = -dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$, where (x_1, x_2, x_3, x_4) is a coordinate system of $\mathbb{E}_1^4$. If $v \in \mathbb{E}_1^4$ then, the vector v is called a spacelike, a timelike or a lightlike(null), if $g(v, v) > 0$, $g(v, v) < 0$ or $g(v, v) = 0$ and $v \neq 0$, respectively. In particular, the vector $v = 0$ is a spacelike.

Let $\alpha(s)$ be any curve in Minkowski $\mathbb{E}_1^4$ then, Frenet equations are given by

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}_1' \\ \mathbf{B}_2' \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ \epsilon_1\kappa_1 & 0 & \epsilon_2\kappa_2 & 0 \\ 0 & \epsilon_3\kappa_2 & 0 & \epsilon_4\kappa_3 \\ 0 & 0 & \epsilon_5\kappa_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}. \tag{2}$$

• If $\epsilon_1 = \epsilon_3 = -1$ *and* $\epsilon_i = 1$ for $(i = 2, 4, 5)$ then, the curve is spacelike with spacelike principal normal with spacelike principal first binormal.

• If $\epsilon_1 = \epsilon_5 = -1$ and $\epsilon_i = 1$ for $(i = 2, 3, 4)$ then, the curve is spacelike with spacelike principal normal with spacelike principal second binormal.

• If $\epsilon_5 = -1$ and $\epsilon_i = 1$ for $(i = 1, 2, 3, 4)$ then, the curve is spacelike with spacelike principal first and second binormals.

• If $\epsilon_3 = \epsilon_5 = -1$ and $\epsilon_i = 1$ for $(i = 1, 2, 4)$ then, the curve is timelike.

In $\mathbb{E}^3$, let $\alpha(s)$ be a curve, quasi frame depends on three orthonormal vectors, $\mathbf{T}(s)$ is the tangent vector, $\mathbf{N}_q(s)$ is the quasi normal and $\mathbf{B}_q(s)$ is the quasi binormal vector. The quasi frame $\{\mathbf{T}(s), \mathbf{N}_q(s), \mathbf{B}_q(s)\}$ is given by

$$\mathbf{T} = \frac{\alpha'}{\parallel \alpha' \parallel}, \quad \mathbf{N}_q = \frac{\mathbf{T} \times \mathbf{k}}{\parallel \mathbf{T} \times \mathbf{k} \parallel}, \quad \mathbf{B}_q = \mathbf{T} \times \mathbf{N}_q, \tag{3}$$

where $\mathbf{k}$ is the projection vector.

For our calculations, we have chosen $\mathbf{k} = (0, 0, 1)$ in this paper. In all cases that $\mathbf{T}$ and $\mathbf{k}$ are parallel then, the quasi frame is singular. Thus, in those cases $\mathbf{k}$ can be chosen as $\mathbf{k} = (0, 1, 0)$ or $\mathbf{k} = (1, 0, 0)$.

Let $\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}(s)\}$ be the Frenet-Serret frame vectors and $\theta(s)$ is an Euclidean angle between principal normal $\mathbf{N}(s)$ and quasi normal $\mathbf{N}_q(s)$ then, we obtain

$$\begin{aligned} \mathbf{N}_q &= cos\theta \ \mathbf{N} + sin\theta \ \mathbf{B}, \\ \mathbf{B}_q &= -sin\theta \ \mathbf{N} + cos\theta \ \mathbf{B}. \end{aligned} \tag{4}$$

• Let us consider a line curve parametrized by

$$\alpha(t) = (t, t, 0).$$

Easily, we see the Frenet frame is not suitable for this curve, while the quasi frame is given by

$$\begin{aligned}\mathbf{T}(t) &= (1/\sqrt{2}, 1/\sqrt{2}, 0),\\ \mathbf{N_q}(t) &= (1/\sqrt{2}, -1/\sqrt{2}, 0),\\ \mathbf{B_q}(t) &= (0, 0, 1).\end{aligned}$$

Which indicates that the quasi frame is better than Frenet frame.

• Let us consider a curve parametrized by

$$\alpha(t) = (t, t, t^9).$$

Easily, we get

$$\kappa(t) = \frac{72\sqrt{2}}{(2 + 82t^{16})^{3/2}}, \qquad \tau(t) = 0.$$

Since $\tau \equiv 0$ then, the angle between the Bishop frame and the Frenet frame is constant, therefore the Bishop frame is also not suitable for this curve, while the quasi frame is given by

$$\begin{aligned}\mathbf{T}(t) &= \frac{(1, 1, 9t^2)}{\sqrt{2 + 81t^{16}}},\\ \mathbf{N_q}(t) &= \frac{1}{2}(\sqrt{2}, -\sqrt{2}, 0),\\ \mathbf{B_q}(t) &= \frac{(9\sqrt{2}t^8, 9\sqrt{2}t^8, -2\sqrt{2})}{2\sqrt{2 + 81t^{16}}}.\end{aligned}$$

• Let us consider the curve parametrized by

$$\alpha(t) = (2t, t^2, t^3/3).$$

The quasi and Bishop frames of the curve have the same behavior but, the computing of the Bishop frame along the curve is difficult, although both of the frames have similar accuracy.

In $\mathbb{E}^4$, let $\alpha(s)$ be a curve, quasi frame depends on four orthonormal vectors, $\mathbf{T}(s)$ is the tangent vector, $\mathbf{N}_q(s)$ is the quasi normal, $\mathbf{B}_{1q}(s)$ is the quasi first binormal vector and $\mathbf{B}_{2q}(s)$ is the second binormal. The quasi frame $\{\mathbf{T}(s), \mathbf{N}_q(s), \mathbf{B}_{1q}(s), \mathbf{B}_{2q}(s)\}$ is given by

$$\begin{aligned}&\mathbf{T} = \frac{\alpha'}{\| \alpha' \|}, \quad \mathbf{N}_q = \frac{\mathbf{T} \times \mathbf{k}_1 \times \mathbf{k}_2}{\| \mathbf{T} \times \mathbf{k}_1 \times \mathbf{k}_2 \|},\\ &\mathbf{B}_{2q} = \zeta \frac{\mathbf{T} \times \mathbf{N}_q \times \alpha'''}{\| \mathbf{T} \times \mathbf{N}_q \times \alpha''' \|}, \quad \mathbf{B}_{1q} = \zeta \mathbf{B}_{2q} \times \mathbf{T} \times \mathbf{N}_q,\end{aligned} \tag{5}$$

where $\mathbf{k}_1$ and $\mathbf{k}_2$ are the projection vectors and ζ is ± 1 where the determinant of matrix is equal to 1.

For simplicity, we choose $\mathbf{k}_1 = (0,0,0,1)$ and $\mathbf{k}_2 = (0,0,1,0)$ in our calculations. However, the quasi frame is singular when $\mathbf{T}$ and $\mathbf{k}_1$ or $\mathbf{T}$ and $\mathbf{k}_2$ or $\mathbf{k}_1$ and $\mathbf{k}_2$ are parallel and in those cases we may change our projection vectors.

Let $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}_1(s), \mathbf{B}_2(s)\}$ are the Frenet-Serret frame vectors, where $\mathbf{T}(s)$, $\mathbf{N}(s)$, $\mathbf{B}_1(s)$ and $\mathbf{B}_2(s)$ are tangent, principal normal, first and second binormal vector fields, respectively and $\theta(s)$ is an Euclidean angle between principal normal $\mathbf{N}(s)$ and quasi normal $\mathbf{N}_q(s)$ then, we obtain

$$\begin{aligned} \mathbf{N}_q &= \cos\theta\ \cos\psi\ \mathbf{N} + (-\cos\phi\ \sin\psi + \sin\theta\ \sin\phi\ \cos\psi)\mathbf{B}_1 \\ &\quad + (\sin\phi\ \sin\psi + \cos\phi\ \sin\theta\ \cos\psi)\mathbf{B}_2 \\ \mathbf{B}_{1q} &= \cos\theta\ \sin\psi\ \mathbf{N} + (\cos\phi\ \cos\psi + \sin\theta\ \sin\phi\ \sin\psi)\mathbf{B}_1 \\ &\quad + (-\sin\phi\ \cos\psi + \cos\phi\ \sin\theta\ \sin\psi)\mathbf{B}_2 \\ \mathbf{B}_{2q} &= \sin\theta\ N + \cos\theta\ \sin\phi\ \mathbf{B}_1 + \cos\theta\ \cos\phi\ \mathbf{B}_2. \end{aligned} \tag{6}$$

3. Quasi equations in $\mathbb{E}_1^3$

In this section, we investigate quasi equations in 3-dimensional Minkowski space $\mathbb{E}_1^3$ in the three different cases of a non-lightlike curve by using the transformation matrix between quasi and Frenet-Serret frames. Furthermore, we introduce the quasi curvatures in Minkowski 3-space.

Theorem 3.1. *If $\alpha(s)$ is a spacelike curve with a quasi spacelike normal vector filed $\boldsymbol{N}_q(s)$ and a quasi timelike binormal vector field $\boldsymbol{B}_q(s)$ then, the quasi equations are given by*

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}_q' \\ \boldsymbol{B}_q' \end{bmatrix} = \begin{bmatrix} 0 & K_1 & -K_2 \\ -K_1 & 0 & K_3 \\ -K_2 & K_3 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N}_q \\ \boldsymbol{B}_q \end{bmatrix}, \tag{7}$$

where $K_1 = \kappa_1 \cosh\theta$, $K_2 = \kappa_1 \sinh\theta$ and $K_3 = \kappa_2 + \theta'$.

Proof 3.1. Let the transformation matrix is given by

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta & \sinh\theta \\ 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}. \tag{8}$$

By using Equation (1), we obtain

$$\mathbf{T}' = \kappa_1\ \mathbf{N} = \kappa_1\ \cosh\theta\ \mathbf{N}_q - \kappa_1\ \sinh\theta\ \mathbf{B}_q, \tag{9}$$

Since $\mathbf{N}_q = \cosh\theta\ \mathbf{N} + \sinh\theta\ \mathbf{B}$, $\mathbf{B}_q = \sinh\theta\ \mathbf{T} + \cosh\theta\ \mathbf{B}$ and by differentiating with respect to arc length s, we get

$$\begin{aligned} \mathbf{N}_q' &= -\kappa_1\ \cosh\theta\ \mathbf{T} + (\kappa_2 + \theta')\ \mathbf{B}_q, \\ \mathbf{B}_q' &= -\kappa_1\ \sinh\theta\ \mathbf{T} + (\kappa_2 + \theta')\ \mathbf{N}_q. \end{aligned} \tag{10}$$

Therefore, the proof is completed.

Corollary 3.1. *If $\alpha(s)$ is a spacelike curve with a quasi spacelike normal vector filed $N_q(s)$ and a quasi timelike binormal vector field $B_q(s)$ then, quasi curvatures $\{K_i | i = 1, 2, 3\}$ can be determined by*

$$\begin{aligned} K_1 &= g(\boldsymbol{T}', \boldsymbol{N}_q) = -g(\boldsymbol{N}'_q, \boldsymbol{T}), \\ K_2 &= g(\boldsymbol{T}', \boldsymbol{B}_q) = -g(\boldsymbol{B}'_q, \boldsymbol{T}), \\ K_3 &= -g(\boldsymbol{N}'_q, \boldsymbol{B}_q) = g(\boldsymbol{B}'_q, \boldsymbol{N}_q). \end{aligned} \tag{11}$$

Corollary 3.2. *If we put ($\kappa_2 = -\acute{\theta}$) in Equation (7), we get the same results as Bishop frame.*

The next two theorems can be proved analogously so, we omit their proofs.

Theorem 3.2. *If $\alpha(s)$ be a curve is a spacelike curve with a quasi timelike normal vector filed $\boldsymbol{N}_q(s)$ and a quasi spacelike binormal vector field $\boldsymbol{B}_q(s)$ then, quasi equations is given by*

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}'_q \\ \boldsymbol{B}'_q \end{bmatrix} = \begin{bmatrix} 0 & K_1 & -K_2 \\ K_1 & 0 & K_3 \\ K_2 & K_3 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N}_q \\ \boldsymbol{B}_q \end{bmatrix}, \tag{12}$$

where $K_1 = \kappa_1 \ \cosh\theta$, $K_2 = \kappa_1 \ \sinh\theta$ and $K_3 = \kappa_2 + \theta'$.

Corollary 3.3. *If $\alpha(s)$ be a curve is a spacelike curve with a quasi timelike normal vector filed $\boldsymbol{N}_q(s)$ and a quasi spacelike binormal vector field $\boldsymbol{B}_q(s)$ then, quasi curvatures $\{K_i | i = 1, 2, 3\}$ can be determined by*

$$\begin{aligned} K_1 &= -g(\boldsymbol{T}', \boldsymbol{N}_q) = g(\boldsymbol{N}'_q, \boldsymbol{T}), \\ K_2 &= -g(\boldsymbol{T}', \boldsymbol{B}_q) = g(\boldsymbol{B}'_q, \boldsymbol{T}), \\ K_3 &= g(\boldsymbol{N}'_q, \boldsymbol{B}_q) = -g(\boldsymbol{B}'_q, \boldsymbol{N}_q). \end{aligned} \tag{13}$$

Corollary 3.4. *If we put ($\kappa_2 = -\acute{\theta}$) in Equation (12), we get the same results as Bishop frame.*

Theorem 3.3. *If $\alpha(s)$ be a timelike curve with a quasi spacelike normal vector filed $\boldsymbol{N}_q(s)$ and a quasi spacelike binormal vector $\boldsymbol{B}_q(s)$ then, quasi equations is given by*

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}'_q \\ \boldsymbol{B}'_q \end{bmatrix} = \begin{bmatrix} 0 & K_1 & K_2 \\ K_1 & 0 & K_3 \\ K_2 & -K_3 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N}_q \\ \boldsymbol{B}_q \end{bmatrix}, \tag{14}$$

where $K_1 = \kappa_1 \cos\theta$, $K_2 = -\kappa_1 \sin\theta$ and $K_3 = \kappa_2 + \theta'$.

Note that: The transformation matrix is given by

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N}_q \\ \mathbf{B}_q \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix}. \tag{15}$$

Corollary 3.5. *If $\alpha(s)$ be a timelike curve with a quasi spacelike normal vector filed $\boldsymbol{N}_q(s)$ and a quasi spacelike binormal vector $\boldsymbol{B}_q(s)$ then, quasi curvatures $\{K_i | i = 1, 2, 3\}$ can be determined by*

$$
\begin{aligned}
K_1 &= g(\boldsymbol{T}', \boldsymbol{N}_q) = -g(\boldsymbol{N}'_q, \boldsymbol{T}),\\
K_2 &= g(\boldsymbol{T}', \boldsymbol{B}_q) = -g(\boldsymbol{B}'_q, \boldsymbol{T}),\\
K_3 &= g(\boldsymbol{N}'_q, \boldsymbol{B}_q) = -g(\boldsymbol{B}'_q, \boldsymbol{N}_q).
\end{aligned} \tag{16}
$$

Corollary 3.6. *If we put $(\kappa_2 = -\acute{\theta})$ in equations (14), we get the same results as Bishop frame.*

4. Quasi equations in $\mathbb{E}_1^4$

In this section, we investigate quasi equations in 4-dimensional Minkowski space $\mathbb{E}_1^4$ in the four different cases of a non-lightlike curve by using the transformation matrices between quasi and Frenet-Serret frames.

Theorem 4.1. *If $\alpha(s)$ be a timelike curve with a quasi spacelike normal vector filed $\boldsymbol{N}_q(s)$ with a quasi spacelike first binormal vector field $\boldsymbol{B}_{1q}(s)$ and a quasi spacelike second binormal vector field $\boldsymbol{B}_{2q}(s)$ then, quasi equations is given by*

$$
\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}'_q \\ \boldsymbol{B}'_{1q} \\ \boldsymbol{B}'_{2q} \end{bmatrix} = \begin{bmatrix} 0 & K_1 & K_2 & K_3 \\ K_1 & 0 & K_4 & K_5 \\ K_2 & -K_4 & 0 & K_6 \\ K_3 & -K_5 & -K_6 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N}_q \\ \boldsymbol{B}_{1q} \\ \boldsymbol{B}_{2q} \end{bmatrix}, \tag{17}
$$

where

$$
\begin{aligned}
K_1 &= \kappa_1 \ \cos\phi \ cos\psi,\\
K_2 &= \kappa_1 \ (-\cos\psi \ \sin\theta \ \sin\phi + \cos\theta \ \sin\psi),\\
K_3 &= \kappa_1 \ (\cos\theta \ \cos\psi \ \sin\phi + \sin\theta \ sin\psi),\\
K_4 &= \sin\theta \ (\kappa_3 \ \sin\psi + \phi') + \cos\theta \ \Big(\kappa_3 \ \cos\psi \ \sin\phi + \cos\phi \ (\kappa_2 - \psi')\Big),\\
K_5 &= -\cos\theta \ \cos^2\phi \ (\kappa_3 \ \sin\psi + \phi') + \cos\phi \ \sin\theta \ (\kappa_2 - \psi')\\
&\quad + \sin\phi \ \Big(\kappa_3 \ \cos\psi \ \sin\theta - \cos\theta \ \sin\phi \ (\kappa_3 \ \sin\psi + \phi')\Big),\\
K_6 &= -\sin^2\theta \ \Big[-\kappa_3 \ \cos\phi \ \cos\psi + \cos^2\psi \Big(\theta' + \sin\phi \ (\kappa_2 - \psi')\Big)\\
&\quad + \sin^2\psi \Big(\theta' + \sin\phi(\kappa_2 - \psi')\Big)\Big] - \cos^2\theta \Big[-\kappa_3 \ \cos\phi \ \cos\psi + \cos^2\phi \ \theta'\\
&\quad + \cos^2\psi \ \sin\phi \Big(\kappa_2 + \sin\phi \ \theta' - \psi'\Big) + \sin\phi \ \sin^2\psi \ \Big(\kappa_2 + \sin\phi \ \theta' - \psi'\Big)\Big].
\end{aligned}
$$

Proof 4.1. We have three possible simple rotations. The first rotation exists on the spacelike plane spanned by the spacelike Frenet first binormal $\mathbf{B}_1$ and the spacelike Frenet second binormal $\mathbf{B}_2$ with angle θ. The second rotation exists on the spacelike plane spanned by the spacelike Frenet principal normal $\mathbf{N}$ and the spacelike Frenet second binormal $\mathbf{B}_2$ with angle ϕ. The third rotation exists on the spacelike plane spanned by the spacelike Frenet principal normal $\mathbf{N}$ and the spacelike Frenet first binormal $\mathbf{B}_1$ with angle ψ so, the transformation matrix is given by

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & 0 & -\sin\phi \\ 0 & 0 & 1 & 0 \\ 0 & \sin\phi & 0 & \cos\phi \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi & 0 \\ 0 & \sin\psi & \cos\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{18}$$

so,

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N}_q \\ \mathbf{B}_{1q} \\ \mathbf{B}_{2q} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi\cos\psi & -\cos\phi\sin\psi & -\sin\phi \\ 0 & -\cos\psi\sin\theta\sin\phi+\cos\theta\sin\psi & \cos\theta\cos\psi+\sin\theta\sin\phi\sin\psi & -\cos\phi\sin\theta \\ 0 & \cos\theta\cos\psi\sin\phi+\sin\theta\sin\psi & \cos\psi\sin\theta-\cos\theta\sin\phi\sin\psi & \cos\theta\cos\phi \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}. \tag{19}$$

By using Equations (2), we obtain

$$\begin{aligned}
\mathbf{N}_q' =& \Big[\kappa_1 \ \cos\phi \ \cos\psi\Big]\mathbf{T} \\
&+ \Big[\kappa_2 \ \cos\phi \ \sin\psi - \cos\psi \ \sin\phi \ \phi' - \cos\phi \ \sin\psi \ \psi'\Big]\mathbf{N} \\
&+ \Big[\kappa_2 \ \cos\phi \ \cos\psi + \kappa_3 \ \sin\phi + \sin\phi \ \sin\psi \ \phi' - \cos\phi \ \cos\psi \ \psi'\Big]\mathbf{B}_1 \\
&+ \Big[-\kappa_3 \ \cos\phi \ \sin\psi + (-\cos\phi \ \phi')\Big]\mathbf{B}_2, \\
\mathbf{B}_{1q}' =& \Big[\kappa_1(-\cos\psi\sin\theta\sin\phi + \cos\theta\sin\psi)\Big]\mathbf{T} \\
&+ \Big[-\kappa_2(\cos\theta\cos\psi + \sin\theta\sin\phi\sin\psi) - \sin\theta\sin\psi\theta' \\
&- \cos\psi(\cos\theta\sin\phi\theta' + \cos\phi\sin\theta\phi') + (\cos\theta\cos\psi + \sin\theta\sin\phi\sin\psi)\psi'\Big]\mathbf{N} \\
&+ \Big[\kappa_2(-\cos\psi\sin\theta\sin\phi + \cos\theta\sin\psi) + \kappa_3\cos\phi\sin\theta - \cos\psi\sin\theta\theta' \\
&+ \sin\psi(\cos\theta\sin\phi\theta' + \cos\phi\sin\theta\phi') + (\cos\psi\sin\theta\sin\phi - \cos\theta\sin\psi)\psi'\Big]\mathbf{B}_1
\end{aligned} \tag{20}$$

$$
\begin{aligned}
&+\Big[\kappa_3(\cos\theta\cos\psi+\sin\theta\sin\phi\sin\psi)-\cos\theta\cos\phi\theta'+\sin\theta\sin\phi\phi'\Big]\mathbf{B}_2,\\
\mathbf{B}'_{2q} = &\Big[\kappa_1(\cos\theta\cos\psi\sin\phi+\sin\theta\sin\psi)\Big]\mathbf{T}\\
&+\Big[-\kappa_2(\cos\psi\sin\theta-\cos\theta\sin\phi\sin\psi)+\cos\theta\sin\psi\theta'\\
&+\cos\psi(-\sin\theta\sin\phi\theta'+\cos\theta\cos\phi\phi')+\cos\psi\sin\theta-\cos\theta\sin\phi\sin\psi)\psi'\Big]\mathbf{N}\\
&+\Big[\kappa_2(\cos\theta\cos\psi\sin\phi+\sin\theta\sin\psi)-\kappa_3\cos\theta\cos\phi+\cos\theta\cos\psi\theta'\\
&-\cos\theta\cos\phi\sin\psi\phi'-\sin\theta\sin\psi\psi'-\sin\phi(-\sin\theta\sin\psi\theta'+\cos\theta\cos\psi\psi')\Big]\mathbf{B}_1\\
&+\Big[\kappa_3(\cos\psi\sin\theta-\cos\theta\sin\phi\sin\psi)-\cos\phi\sin\theta\theta'-\cos\theta\sin\phi\phi'\Big]\mathbf{B}_2.
\end{aligned}
$$

Therefore,

$$
\begin{bmatrix}\mathbf{T}'\\ \mathbf{N}'_q\\ \mathbf{B}'_{1q}\\ \mathbf{B}'_{2q}\end{bmatrix}=\begin{bmatrix}0 & K_1 & K_2 & K_3\\ K_1 & 0 & K_4 & K_5\\ K_2 & -K_4 & 0 & K_6\\ K_3 & -K_5 & -K_6 & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}_q\\ \mathbf{B}_{1q}\\ \mathbf{B}_{2q}\end{bmatrix}.
$$

Corollary 4.1. *If $\alpha(s)$ be a timelike curve with a quasi spacelike normal vector filed $\boldsymbol{N}_q(s)$ with a quasi spacelike first binormal vector field $\boldsymbol{B}_{1q}(s)$ and a quasi spacelike second binormal vector field $\boldsymbol{B}_{2q}(s)$ then, quasi curvatures $\{K_i | i = 1,2,3,4,5,6\}$ can be determined by*

$$
\begin{aligned}
&K_1 = g(\boldsymbol{T}', \boldsymbol{N}_q) = -g(\boldsymbol{N}'_q, \boldsymbol{T}),\\
&K_2 = g(\boldsymbol{T}', \boldsymbol{B}_{1q}) = -g(\boldsymbol{B}'_{1q}, \boldsymbol{T}),\\
&K_3 = g(\boldsymbol{T}', \boldsymbol{B}_{2q}) = -g(\boldsymbol{B}'_{2q}, \boldsymbol{T}),\\
&K_4 = g(\boldsymbol{N}'_q, \boldsymbol{B}_{1q}) = -g(\boldsymbol{B}'_{1q}, \boldsymbol{N}_q)\\
&, K_5 = g(\boldsymbol{N}'_q, \boldsymbol{B}_{2q}) = -g(\boldsymbol{B}'_{2q}, \boldsymbol{N}_q)\\
&K_6 = g(\boldsymbol{B}'_{1q}, \boldsymbol{B}_{2q}) = -g(\boldsymbol{B}'_{2q}, \boldsymbol{B}_{1q}).
\end{aligned} \tag{21}
$$

Corollary 4.2. *If we put $\kappa_2 = \psi' + \phi'\ tan\phi\ cot\psi$ and $\kappa_3 = -\frac{\phi'}{sin\psi}$ in equations (17), we can easily find ($K_4 = 0 = K_5 = K_6$) and hence, we have the same result as Bishop frame.*

The next three theorems can be proved analogously so, we omit their proofs.

Theorem 4.2. *If $\alpha(s)$ is a spacelike curve with a quasi spacelike normal vector filed $\boldsymbol{N}_q(s)$ with a quasi spacelike first binormal vector field $\boldsymbol{B}_{1q}(s)$ and a quasi timelike second binormal vector field $\boldsymbol{B}_{2q}(s)$ then, quasi equations are given by*

$$
\begin{bmatrix}\boldsymbol{T}'\\ \boldsymbol{N}'_q\\ \boldsymbol{B}'_{1q}\\ \boldsymbol{B}'_{2q}\end{bmatrix}=\begin{bmatrix}0 & K_1 & K_2 & K_3\\ -K_1 & 0 & K_4 & K_5\\ -K_2 & -K_4 & 0 & K_6\\ K_3 & K_5 & K_6 & 0\end{bmatrix}\begin{bmatrix}\boldsymbol{T}\\ \boldsymbol{N}_q\\ \boldsymbol{B}_{1q}\\ \boldsymbol{B}_{2q}\end{bmatrix}, \tag{22}
$$

where

$$
\begin{aligned}
K_1 &= \kappa_1 \; \cos\psi \; \cosh\phi,\\
K_2 &= \kappa_1 \; (\cosh\theta \; \sin\psi + \cos\psi \; \sinh\theta \; \sinh\phi),\\
K_3 &= -\kappa_1 \; (\sin\psi \; \sinh\theta + \cos\psi \; \cosh\theta \; \sinh\phi),\\
K_4 &= \kappa_3 \; \cos\psi \; \cosh\theta \; \sinh\phi + \sinh\theta \; \Big[\cosh^2\phi \; (\kappa_3 \; \sin\psi - \phi')\\
&\quad + \sin\psi \; \sinh^2\phi(-\kappa_3 + \sin\psi \; \phi')\Big] + \cos^2\psi \; \Big[\sinh\theta \; \sinh^2\phi \; \phi'\\
&\quad + \cosh\theta \; \cosh\phi \; (\kappa_2 - \psi')\Big] + \cosh\theta \; \cosh\phi \; \sin^2\psi \; (\kappa_2 - \psi'),\\
K_5 &= -\kappa_3 \; \cos\psi \; \sinh\theta \; \sinh\phi + \cosh\theta \; \cosh^2\phi \; (-\kappa_3 \; \sin\psi + \phi')\\
&\quad + \cosh\theta \; \sin\psi \; \sinh^2\phi \; (\kappa_3 - \sin\psi \; \phi') - \cos^2\psi \; \Big[\cosh\theta \; \sinh^2\phi \; \phi'\\
&\quad + \cosh\phi \; \sinh\theta \; (\kappa_2 - \psi')\Big] + \cosh\phi \; \sin^2\psi \; \sinh\theta \; (-\kappa_2 + \psi'),\\
K_6 &= \kappa_3 \; \cos\psi \; \cosh\phi - \sin^2\psi \; \sinh^2\theta \; \Big(\theta' + \sinh\phi \; (\kappa_2 - \psi')\Big)\\
&\quad - \cos^2\psi \; \Big[\sinh^2\theta \; \Big(\theta' + \sinh\phi \; (\kappa_2 - \psi')\Big) + \cosh^2\theta \; \sinh\phi \; (-\kappa_2 + \sinh\phi \; \theta' + \psi')\Big]\\
&\quad + \cosh^2\theta \; \Big(\cosh^2\phi \; \theta' - \sin^2\psi \; \sinh\phi \; (-\kappa_2 + \sinh\phi \; \theta' + \psi')\Big).
\end{aligned}
$$

Note that: We have three possible simple rotations. The first rotation exists on the timelike plane spanned by the spacelike Frenet first binormal $\mathbf{B}_1$ and the timelike Frenet second binormal $\mathbf{B}_2$ with angle θ. The second rotation exists on the timelike plane spanned by the spacelike Frenet principal normal $\mathbf{N}$ and the timelike Frenet second binormal $\mathbf{B}_2$ with angle ϕ. The third rotation exists on the spacelike plane spanned by the spacelike Frenet principal normal $\mathbf{N}$ and the spacelike Frenet first binormal $\mathbf{B}_1$ with angle ψ so, the transformation matrix is given by

$$
\begin{aligned}
R = &\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \cosh\theta & \sinh\theta\\ 0 & 0 & \sinh\theta & \cosh\theta \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & \cosh\phi & 0 & \sinh\phi\\ 0 & 0 & 1 & 0\\ 0 & \sinh\phi & 0 & \cosh\phi \end{bmatrix}\\
&\cdot \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & \cos\psi & -\sin\psi & 0\\ 0 & \sin\psi & \cos\psi & 0\\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
\tag{23}
$$

so,

$$
\begin{bmatrix} \mathbf{T}\\ \mathbf{N}_q\\ \mathbf{B}_{1q}\\ \mathbf{B}_{2q} \end{bmatrix}
$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\psi\cosh\phi & -\cosh\phi\sin\psi & \sinh\phi \\ 0 & \cosh\theta\sin\psi + cos\psi\sinh\theta\sinh\phi & \cos\psi\cosh\theta - \sin\psi\sinh\theta\sin\phi & \cosh\phi\sinh\theta \\ 0 & \sin\psi\sinh\theta + \cos\psi\cosh\theta\sinh\phi & \cos\psi\sinh\theta - \cosh\theta\sin\psi\sinh\phi & \cosh\theta\cosh\phi \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}.$$

Corollary 4.3. *If $\alpha(s)$ is a spacelike curve with a quasi spacelike normal vector filed $\boldsymbol{N}_q(s)$ with a quasi spacelike first binormal vector field $\boldsymbol{B}_{1q}(s)$ and a quasi timelike second binormal vector field $\boldsymbol{B}_{2q}(s)$ then, quasi curvatures $\{K_i | i = 1,2,3,4,5,6\}$ can be determined by*

$$\begin{aligned} K_1 &= g(\boldsymbol{T}', \boldsymbol{N}_q) = -g(\boldsymbol{N}'_q, \boldsymbol{T}), \\ K_2 &= g(\boldsymbol{T}', \boldsymbol{B}_{1q}) = -g(\boldsymbol{B}'_{1q}, \boldsymbol{T}), \\ K_3 &= -g(\boldsymbol{T}', \boldsymbol{B}_{2q}) = g(\boldsymbol{B}'_{2q}, \boldsymbol{T}), \\ K_4 &= g(\boldsymbol{N}'_q, \boldsymbol{B}_{1q}) = -g(\boldsymbol{B}'_{1q}, \boldsymbol{N}_q), \\ K_5 &= -g(\boldsymbol{N}'_q, \boldsymbol{B}_{2q}) = g(\boldsymbol{B}'_{2q}, \boldsymbol{N}_q) \\ K_6 &= -g(\boldsymbol{B}'_{1q}, \boldsymbol{B}_{2q}) = g(\boldsymbol{B}'_{2q}, \boldsymbol{B}_{1q}). \end{aligned} \tag{24}$$

Corollary 4.4. *If we put $\kappa_2 = \psi' - \phi'\ tanh\phi\ cot\psi$ and $\kappa_3 = -\frac{\phi'}{sin\psi}$ in equations (22), we can easily find ($K_4 = 0 = K_5 = K_6$) and hence, we have the same result as Bishop frame.*

Theorem 4.3. *If $\alpha(s)$ is a spacelike curve with a quasi spacelike normal vector filed $\boldsymbol{N}_q(s)$ with a quasi timelike first binormal vector field $\boldsymbol{B}_{1q}(s)$ and a quasi spacelike second binormal vector field $\boldsymbol{B}_{2q}(s)$ then, quasi equations are given by*

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}'_q \\ \boldsymbol{B}_{1q'} \\ \boldsymbol{B}'_{2q} \end{bmatrix} = \begin{bmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & K_4 & K_5 \\ K_2 & K_4 & 0 & K_6 \\ -K_3 & -K_5 & K_6 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N}_q \\ \boldsymbol{B}_{1q} \\ \boldsymbol{B}_{2q} \end{bmatrix}, \tag{25}$$

where

$$\begin{aligned} K_1 &= \kappa_1\ \cos\phi\ \cosh\psi, \\ K_2 &= -\kappa_1\ (\cosh\psi\ \sin\phi\ \sinh\theta + \cosh\theta\ \sin\psi), \\ K_3 &= \kappa_1\ (\cosh\theta\ \cosh\psi\ \sin\phi + \sinh\theta\ \sinh\psi), \\ K_4 &= \sin\phi\ \Big[\kappa_3\ \cosh\theta\ \cosh\psi - sin\phi\ \sinh\theta\ (\kappa_3\ \sinh\psi - \phi')\Big] + cos\phi\cosh\theta\ (\kappa_2 + \psi') \\ &\quad + \cos^2\phi\ \sinh\theta\ (-\kappa_3\ \sinh\psi + \phi'), \\ K_5 &= \cos^2\phi\ \cosh\theta\ (\kappa_3\ \sinh\psi - \phi') + \sin\phi\ \Big((\cosh\psi\ \sinh\theta + \cosh\theta\ \sin\phi\ \sinh\psi)\ \kappa_3 \\ &\quad - \cosh\theta\ \sin\phi\ \phi'\Big) - \cos\phi\ \sinh\theta\ (\kappa_2 + \psi'), \\ K_6 &= \cosh^2\theta\ \Big[\kappa_3\ \cos\phi\ \cosh\psi + cos^2\phi\ \theta' + \cosh^2\psi\ sin\phi\ (\kappa_2 + \sin\phi\ \theta' + \psi') \\ &\quad - sin\phi\ \sinh^2\psi\ (\kappa_2 + \sin\phi\ \theta' + \psi')\Big] - \sinh^2\theta\ \Big[\kappa_3\ \cos\phi\ \cosh\psi \\ &\quad + \cosh^2\psi\ \Big(\theta' + sin\phi\ (\kappa_2 + \psi')\Big) - \sinh^2\psi\ \Big(\theta' + \sin\phi\ (\kappa_2 + \psi')\Big)\Big]. \end{aligned}$$

Note that: We have three possible simple rotations. The first rotation exists on the timelike plane spanned by the timelike Frenet first binormal $\mathbf{B}_1$ and the spacelike Frenet second binormal $\mathbf{B}_2$ with angle θ. The second rotation exists on the spacelike plane spanned by the spacelike Frenet principal normal $\mathbf{N}$ and the spacelike Frenet second binormal $\mathbf{B}_2$ with angle ϕ. The third rotation exists on the timelike plane spanned by the spacelike Frenet principal normal $\mathbf{N}$ and the timelike Frenet first binormal $\mathbf{B}_1$ with angle ψ so, the transformation matrix is given by

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh\theta & \sinh\theta \\ 0 & 0 & \sinh\theta & \cosh\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi & 0 & -\sin\phi \\ 0 & 0 & 1 & 0 \\ 0 & \sin\phi & 0 & \cos\phi \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\psi & \sinh\psi & 0 \\ 0 & \sinh\psi & \cosh\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{26}$$

so,

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N}_q \\ \mathbf{B}_{1q} \\ \mathbf{B}_{2q} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\phi\cosh\psi & \cos\phi\sinh\psi & -\sin\phi \\ 0 & \cosh\psi\sin\phi\sinh\theta + \cosh\theta\sinh\psi & \cosh\theta\cosh\psi + \sin\phi\sinh\theta\sinh\psi & \cos\phi\sinh\theta \\ 0 & \cosh\theta\cosh\psi\sin\phi + \sinh\theta\sinh\psi & \cosh\psi\sinh\theta + \cosh\theta sin\phi\sinh\psi & \cos\phi\cosh\theta \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}. \tag{27}$$

Corollary 4.5. *If $\alpha(s)$ is a spacelike curve with a quasi spacelike normal vector filed $\boldsymbol{N}_q(s)$ with a quasi timelike first binormal vector field $\boldsymbol{B}_{1q}(s)$ and a quasi spacelike second binormal vector field $\boldsymbol{B}_{2q}(s)$ then, quasi curvatures $\{K_i|i = 1,2,3,4,5,6\}$ can be determined by*

$$\begin{aligned} K_1 &= g(\boldsymbol{T}', \boldsymbol{N}_q) = -g(\boldsymbol{N}'_q, \boldsymbol{T}), \\ K_2 &= -g(\boldsymbol{T}', \boldsymbol{B}_{1q}) = g(\boldsymbol{B}'_{1q}, \boldsymbol{T}), \\ K_3 &= g(\boldsymbol{T}', \boldsymbol{B}_{2q}) = -g(\boldsymbol{B}'_{2q}, \boldsymbol{T}), \\ K_4 &= -g(\boldsymbol{N}'_q, \boldsymbol{B}_{1q}) = g(\boldsymbol{B}'_{1q}, \boldsymbol{N}_q), \\ K_5 &= g(\boldsymbol{N}'_q, \boldsymbol{B}_{2q}) = -g(\boldsymbol{B}'_{2q}, \boldsymbol{N}_q), \\ K_6 &= g(\boldsymbol{B}'_{1q}, \boldsymbol{B}_{2q}) = -g(\boldsymbol{B}'_{2q}, \boldsymbol{B}_{1q}). \end{aligned} \tag{28}$$

Corollary 4.6. *If we put $\kappa_2 = \psi' - \phi' \, tan\phi \, coth\psi$ and $\kappa_3 = \frac{\phi'}{\sinh\psi}$ in equation (25), we can easily find $(K_4 = 0 = K_5 = K_6)$ and hence, we have the same result as Bishop frame.*

Theorem 4.4. *If $\alpha(s)$ is a spacelike curve with a quasi timelike normal vector filed $\boldsymbol{N}_q(s)$ with a quasi spacelike first binormal vector field $\boldsymbol{B}_{1q}(s)$ and a quasi spacelike second binormal vector field $\boldsymbol{B}_{2q}(s)$, then quasi equations are given by*

$$\begin{bmatrix} \boldsymbol{T}' \\ \boldsymbol{N}_q' \\ \boldsymbol{B}_{1q}' \\ \boldsymbol{B}_{2q}' \end{bmatrix} = \begin{bmatrix} 0 & K_1 & K_2 & K_3 \\ K_1 & 0 & K_4 & K_5 \\ -K_2 & K_4 & 0 & K_6 \\ -K_3 & K_5 & -K_6 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{T} \\ \boldsymbol{N}_q \\ \boldsymbol{B}_{1q} \\ \boldsymbol{B}_{2q} \end{bmatrix}, \tag{29}$$

where

$$\begin{aligned}
K_1 &= \kappa_1 \ \cosh\phi \ \cosh\psi, \\
K_2 &= \kappa_1 \ (\cosh\psi \ \sin\theta \ \sinh\phi - \cos\theta \ \sinh\psi), \\
K_3 &= -\kappa_1 \ (\cos\theta \ \cosh\psi \ \sinh\phi + \sin\theta \ \sinh\psi), \\
K_4 &= -\sin\theta \ (\sinh\psi \ \kappa_3 + \phi') + \cos\theta \ \Big(- \cosh\psi \ \sinh\phi \ \kappa_3 + \cosh\phi \ (\kappa_2 + \psi')\Big), \\
K_5 &= \cos\theta \ \cosh^2\phi \ (\sinh\psi \ \kappa_3 + \phi') - \sinh\phi \ \Big(\cosh\psi \ \sin\theta \ \kappa_3 \\
&\quad + \cos\theta \ \sinh\phi \ (\sinh\psi \ \kappa_3 + \phi')\Big) + \cosh\phi \ \sin\theta(\kappa_2 + \psi'), \\
K_6 &= \cos^2\theta \ \Big[\cosh\phi \ \cosh\psi \ \kappa_3 - \cosh^2\phi \ \theta' + \cosh^2\psi \ \sinh\phi \ (-\kappa_2 + \sinh\phi \ \theta' \\
&\quad + -\psi')\sinh\phi \ \sinh^2\psi \ (\kappa_2 - \sinh\phi \ \theta' + \psi')\Big] + \sin^2\theta \ \Big[\cosh\phi \ \cosh\psi \ \kappa_3 \\
&\quad - \cosh^2\psi \ \Big(\theta' + \sinh\phi \ (\kappa_2 + \psi')\Big) + \sinh^2\psi \ \Big(\theta' + \sinh\psi \ (\kappa_2 + \psi')\Big)\Big].
\end{aligned}$$

Note that: We have three possible simple rotations. The first rotation exists on the spacelike plane spanned by the spacelike Frenet first binormal $\mathbf{B}_1$ and the spacelike Frenet second binormal $\mathbf{B}_2$ with angle θ. The second rotation exists on the timelike plane spanned by the timelike Frenet principal normal $\mathbf{N}$ and the spacelike Frenet second binormal $\mathbf{B}_2$ with angle ϕ. The third rotation exists on the timelike plane spanned by the timelike Frenet principal normal $\mathbf{N}$ and the spacelike Frenet first binormal $\mathbf{B}_1$ with angle ψ so, the transformation matrix is given by

$$R = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\phi & 0 & \sinh\phi \\ 0 & 0 & 1 & 0 \\ 0 & \sinh\phi & 0 & \cosh\phi \end{bmatrix}$$

$$\cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\psi & \sinh\psi & 0 \\ 0 & \sinh\psi & \cosh\psi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{31}$$

so,

$$\begin{bmatrix} \mathbf{T} \\ \mathbf{N}_q \\ \mathbf{B}_{1q} \\ \mathbf{B}_{2q} \end{bmatrix} \tag{32}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cosh\phi\cosh\psi & \cosh\phi\sinh\psi & \sinh\phi \\ 0 & \cos\theta\sinh\psi - \cosh\psi\sin\theta\sinh\phi & \cos\theta\cosh\psi - \sin\theta\sinh\phi\sinh\psi & -\cosh\phi sin\theta \\ 0 & \cos\theta\cosh\psi\sinh\phi + \sin\theta\sinh\psi & \cosh\psi\sin\theta + \cos\theta\sinh\phi\sinh\psi & \cos\theta\cosh\phi \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}.$$

Corollary 4.7. *If $\alpha(s)$ is a spacelike curve with a quasi timelike normal vector filed $\boldsymbol{N}_q(s)$ with a quasi spacelike first binormal vector field $\boldsymbol{B}_{1q}(s)$ and a quasi spacelike second binormal vector field $\boldsymbol{B}_{2q}(s)$ then, quasi curvatures $\{K_i | i = 1,2,3,4,5,6\}$ can be determined by*

$$\begin{aligned} K_1 &= -g(\boldsymbol{T}', \boldsymbol{N}_q) = g(\boldsymbol{N}'_q, \boldsymbol{T}), \\ K_2 &= g(\boldsymbol{T}', \boldsymbol{B}_{1q}) = -g(\boldsymbol{B}'_{1q}, \boldsymbol{T}), \\ K_3 &= g(\boldsymbol{T}', \boldsymbol{B}_{2q}) = -g(\boldsymbol{B}'_{2q}, \boldsymbol{T}), \\ K_4 &= g(\boldsymbol{N}'_q, \boldsymbol{B}_{1q}) = -g(\boldsymbol{B}'_{1q}, \boldsymbol{N}_q), \\ K_5 &= g(\boldsymbol{N}'_q, \boldsymbol{B}_{2q}) = -g(\boldsymbol{B}'_{2q}, \boldsymbol{N}_q), \\ K_6 &= g(\boldsymbol{B}'_{1q}, \boldsymbol{B}_{2q}) = -g(\boldsymbol{B}'_{2q}, \boldsymbol{B}_{1q}). \end{aligned}$$

Corollary 4.8. *If we put $\kappa_2 = -\psi' - \phi' tanh\phi coth\psi)$ and $\kappa_3 = -\frac{\phi'}{\sinh\psi}$ in equations* (29)*, we can easily find $(K_4 = 0 = K_5 = K_6)$ and hence, we have the same row result as Bishop frame.*

5. Conclusion

In this paper, we investigated the frame and equations of quasi for non-lightlike curves in 3-dimensional Minkowski space $\mathbb{E}^3_1$ and in 4-dimensional Minkowski space-time $\mathbb{E}^4_1$. Furthermore, we showed the quasi frame can be considered as a generalization of Bishop frame in $\mathbb{E}^3_1$ and $\mathbb{E}^4_1$.

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